Continuous Topological Phase Transition between Two 1D Antiferromagnetic Spin-1 Boson Superfluids with the Same Symmetry

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A continuous topological phase transition between two 1D anti-ferromagnetic spin-1 boson superfluids with the same symmetry

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Spin-1 bosons on a 1-dimensional chain, at incommensurate filling with anti-ferromagnetic spin interaction between neighboring bosons, may form a spin-1 boson condensed state that contains both gapless charge and spin excitations. We argue that the spin-1 boson condensed state is unstable, and will become one of two superfluids by opening a spin gap. One superfluid must have spin-1 ground state on a ring if it contains an odd number of bosons and has no degenerate states at the chain end. The other superfluid has spin-0 ground state on a ring for any numbers of bosons and has a spin-$\frac{1}{2}$ degeneracy at the chain end. The two superfluids have the same symmetry and only differ by a spin-$SO(3)$ symmetry protected topological order. Although Landau theory forbids a continuous phase transition between two phases with the same symmetry, the phase transition between the two superfluids can be generically continuous, which is described by a conformal field theory (CFT) $su(2)_2 \oplus u(1)_4 \oplus \overline{su(2)}_2 \oplus \overline{u(1)}_4$. Such a CFT has a spin fractionalization: spin-1 excitation can decay into a spin-$\frac{1}{2}$ right mover and a spin-$\frac{1}{2}$ left mover. We determine the critical theory by solving the partition function based on emergent symmetries and modular invariance condition of CFTs.

Introduction: 1+1D spin-1 charge-2 boson system at incommensurate filling has $SO(3) \times U(1) \times U_t(1)$ symmetry, where $U_t(1)$ is the translation symmetry. Besides the boson condensing state that has gapless spin-1 charge-2 excitations, anti-ferromagnetic spin interactions can lead to superfluid states. The superfluids have fully gapped spin excitations[1] and gapless charge excitations. (For integral filling, see Ref. 2.) Under the protection of $SO(3) \times U(1) \times U_t(1)$, there can be two such kinds of superfluids:
(1) two spin-1 bosons form a spin-singlet bound state and the spin-0 charge-4 bosons form a superfluid, which is referred as non-topological superfluid (nTSF).
(2) In some sense, the spin-1 bosons fractionalize into charge-1 spin-$\frac{1}{2}$'s. The spin-$\frac{1}{2}$'s form singlet pairs and then the spin-0 charge-2 pairs form a superfluid. Or more precisely, the spin part is analogous to the AKLT state in a spin-1 chain[3,4]. The resulting superfluid is referred as topological superfluid (TSF).
For a system on a ring with N charge-2 spin-1 bosons, when N is even, both the nTSF and TSF has a spin-0 ground state. However, when N is odd, the ground state of the nTSF has spin-1, (resembling the dimerized phase in spin-1 antiferromagnetic chain,) while that of the TSF is still a spin-0 state. Moreover, for a system with open ends and N is even, only the TSF has emergent a spin-$\frac{1}{2}$ at each end.

Conventional Landau theory forbids two phases with the same symmetry to have a continuous phase transition between them. In this paper, we show that there is a consistent theory to describe a continuous phase transition between the TSF and nTSF. (Some examples of topological phase transitions that do not change the symmetry can be found in Ref. 5–11.) The critical point is described by a conformal field theory (CFT) built from $u(1)_M$ and $su(2)_k$ current algebras[12]. We find the phase boundary of the two superfluids may allow more than one CFT that share the same lattice symmetry. Among them, only one set of CFTs, namely $u(1)_M \oplus su(2)_2 \oplus \overline{u(1)}_M \oplus \overline{su(2)}_2$ that are consistent candidate of critical theory with a single symmetric relevant operator (and smallest central charge). We argue that this theory describes the spin-1 boson condensed state with anti-ferromagnetic interaction. When perturbing the state with the only symmetric relevant operator, the system flows to either one of the above two superfluid phases. Meanwhile, we hope to provide a new way to find consistent symmetric critical theories based on modular invariance of CFTs and anomaly matching condition of symmetry charges. We also proposed a bosonic model that realizes the topological superfluid phase.

Topological superfluid: We propose to find the TSF phase in the doped t-J model composed of hard-core spin-1 bosons. The Hamiltonian of one such model could be

$$H = -t\sum_{j,\sigma}(b^\dagger_{j,\sigma}b_{j+1,\sigma} + h.c.)] + U\sum_j n_j(n_j-1) + \sum_j \left[J_1 \vec{S}_j\cdot\vec{S}_{j+1} - J_2(\vec{S}_j\cdot\vec{S}_{j+1})^2\right] \qquad (1)$$

where $\sigma = \pm 1, 0$ is the spin index, $\vec{S}_j$ is the spin on site j and $n_j = \sum_{\sigma=\pm1,0} b^\dagger_{j,\sigma}b_{j,\sigma}$ is the boson density on site j, which we restrict to be 0 or 1. When the boson number N and site number N_L are equal $N = N_L$, the above system is in the AKLT state for small J_2 and is in the dimmer state for a larger $J_2 > J_1 > 0$. After we add some small doping, $N < N_L$ and $\frac{N_L}{N} \approx 1$, we expect the AKLT state to become the TSF state and the dimmer state to become the nTSF state.[13]

Partition functions of $U(1)\times SO(3)$ symmetric critical theories: In general, it is difficult to find the critical theory of strongly interacting models such as (1) beyond Landau paradigm. In this paper, we provide a procedure to determine the low energy theories at critical point, by identifying partition functions, based on current algebra and modular invariance of a conformal field theory. Physical data of a symmetric critical point is all low energy excitations, their energies as a function of momentum, together with their charges under the symmetry. The 1+1d critical point has the nice feature that the low energy gapless excitations can be described by a conformal field theory. When the critical point is between phases under symmetry protection, the CFT has extended current algebra. One famous example with explicit Lagrangians is Wess-Zumino-Witten (WZW) models.[14,15] Traditional treatments of the critical theory start with first mapping the spin model to a free fermion model and then non-abelian bosonlizing local operators. However, CFTs with the same current algebras actually can have more than one IR solution of CFTs. Their distinction is clear in the operator content, but rather subtle in the Lagrangians.[12] Here, we take a new route by directly constructing partition functions of the critical theory based on symmetries, anomalies and constraints on local operators. The complete physical data are encapsulated in the partition function of a CFT.

The partition function of a CFT defined on a Euclidean spacetime torus is given by

$$Z(\tau) = \mathrm{Tr} e^{-\mathrm{Im}(\tau)H + \mathrm{i}\,\mathrm{Re}(\tau)K} \tag{2}$$

where H is the Hamiltonian, K is the total momentum operator. τ is a formal parameter describing the shape of the torus. $\mathrm{Re}(\tau)$ and $\mathrm{Im}(\tau)$ can be viewed as the periodicity in space and time direction respectively. All fields in $1+1$d CFT can be factorized into right- and right-moving fields, $f(z = x^1 - \mathrm{i}x^0)$ and left-moving fields $\overline{f}(\overline{z} = x^1 + \mathrm{i}x^0)$, where x^1 and x^0 are space and time coordinate. The critical point under consideration is also invariant under $G = SO(3) \times U(1)$ onsite symmetry. In the low energy, G can act on $f(z), \overline{f}(\overline{z})$ independently. The symmetry is thus enlarged to $G \times \overline{G}$. All operators in CFTs with this enlarged symmetries can be constructed from the currents and representations of $su(2)_k$ and $u(1)_M$ current algebras (which we explain below, also see Supplemental Material at [URL] and Ref. 12 for further details), where the integer number k and M are called levels. That levels are an integer is a property of rational CFTs. IR fixed points of WZW models with non-Abelian group are believed to be rational based on one-loop renormalization group computation.[14] Levels and symmetry charges are restricted by $U(1) \times U_t(1)$ mixed anomaly, a character of boson condensed state.

A necessary condition for a CFT to describe a bosonic critical theory of a lattice model is that the Euclidean partition function must be invariant, under modular transformation of the torus,[16]

$$\mathcal{S} : Z(\tau) \to Z\left(-\frac{1}{\tau}\right), \quad \mathcal{T} : Z(\tau) \to Z(\tau+1). \tag{3}$$

Namely tori with shape τ, $-\frac{1}{\tau}$ and $\tau+1$ are all conformally equivalent and share identical (bosonic) partition functions.

Following the working assumption that CFTs with modular invariant partition functions always correspond

operators	spin	charge	$U_t(1)$	$h, \overline{h}$
$e^{\pm \mathrm{i}\varphi} V^{su2_2}_{1,m}$	1	± 2	$\pm\frac{1}{2}k_B$	$1, 0$
$e^{\pm \mathrm{i}\overline{\varphi}} \overline{V}^{su2_2}_{1,m}$	1	± 2	$\mp\frac{1}{2}k_B$	$0, 1$
$e^{\pm \mathrm{i}(\varphi+\overline{\varphi})}$	0	± 4	0	$\frac{1}{2}, \frac{1}{2}$
$e^{\pm \mathrm{i}(\varphi-\overline{\varphi})}$	0	0	$\pm k_B$	$\frac{1}{2}, \frac{1}{2}$
$e^{\pm \mathrm{i}\varphi} \overline{V}^{su2_2}_{1,m}$	1	± 2	$\pm\frac{1}{2}k_B$	$\frac{1}{2}, \frac{1}{2}$
$V^{su2_2}_{1,m} e^{\pm \mathrm{i}\overline{\varphi}}$	1	± 2	$\mp\frac{1}{2}k_B$	$\frac{1}{2}, \frac{1}{2}$
$V^{su2_2}_{1,m} \overline{V}^{su2_2}_{1,m'}$	0, 1, 2	0	0	$\frac{1}{2}, \frac{1}{2}$
$e^{\pm \mathrm{i}\frac{\varphi+\overline{\varphi}}{2}} V^{su2_2}_{\frac{1}{2},\pm\frac{1}{2}} \overline{V}^{su2_2}_{\frac{1}{2},\pm\frac{1}{2}}$	0, 1	± 2	0	$\frac{5}{16}, \frac{5}{16}$
$e^{\pm \mathrm{i}\frac{\varphi-\overline{\varphi}}{2}} V^{su2_2}_{\frac{1}{2},\pm\frac{1}{2}} \overline{V}^{su2_2}_{\frac{1}{2},\pm\frac{1}{2}}$	0, 1	0	$\pm\frac{1}{2}k_B$	$\frac{5}{16}, \frac{5}{16}$

TABLE I. The local operators (the primary fields) in the Z_A-CFT, which is a version of the $su(2)_2 \oplus u(1)_4 \oplus \overline{su(2)_2} \oplus \overline{u(1)}_4$ CFT. And their spin, $U(1)$ charge, and $U_t(1)$ charge.

to $1+1$d quantum (bosonic) models on a lattice, we construct UV completable CFTs. In particular, there is one set of CFTs, each has a single relevant symmetric direction. They describe critical points of lattice models with $U(1) \times SO(3) \times U_t(1)$ symmetry.

The simplest one has the following partition function

$$\begin{aligned} Z_A(\tau) &= \left|\chi_0^{u1_4}\chi_0^{su2_2} + \chi_2^{u1_4}\chi_1^{su2_2}\right|^2 \\ &+ \left|\chi_0^{u1_4}\chi_1^{su2_2} + \chi_2^{u1_4}\chi_0^{su2_2}\right|^2 + \left|\chi_1^{u1_4} + \chi_3^{u1_4}\right|^2 \left|\chi_{\frac{1}{2}}^{su2_2}\right|^2 \end{aligned} \tag{4}$$

Here, the CFT is generated by $su(2)_2 \oplus u(1)_4 \oplus \overline{su(2)}_2 \oplus \overline{u(1)}_4$ current algebra acting on right and left movers. In general for positive integral level k and M, the partition function $Z(\tau)$ of a rational (finite) CFT can be organized into a finite sum, $Z(\tau) = \sum_{\mu,\nu} M_{\mu\nu}\chi_\mu\overline{\chi}_\nu$. It means physical excitations are (finite number of) representations of both right and left current algebras, labeled by μ, ν respectively. χ_μ ($\overline{\chi}_\nu$) is a so-called character associated with the primary field of right(left) current algebras. The character χ_μ encodes the spectrum of excitations that are also the representation μ of current algebras. The multiplicity $M_{\mu\nu}$ must be a non-negative integer, representing the number of times the excitations (μ, ν) appear in the spectrum.

The $u(1)_M$ and $su(2)_k$ CFTs are well-established.[12] Concisely, the character $\chi_m^{u1_M}$ of $u(1)_M$ CFT, where $0 \le m < M$ and $R^2 = M$, (see supplementary material for the exact form) corresponds to primary fields of $u(1)_M$ current algebra,

$$e^{\mathrm{i}\,(\frac{m}{R} + nR)\varphi(z)}. \tag{5}$$

The $u(1)_M$ primary fields represent the gapless charge excitations and carry the $U(1)$ charge quantum number. In addition, they also carry the $U_t(1)$ quantum number (or large momentum, explained later). The character

$\chi_j^{su2_k}$ of $su(2)_k$ CFT, where $j = 0, \frac{1}{2}, \cdots, \frac{k}{2}$, corresponds to the primary fields of $su(2)_k$ CFT, denoted as $V_{j,m}^{su2_k}$, where $m = -j, -j+1, \cdots, j$. They represent gapless spin excitations, carrying spin quantum number.

Z_A is modular invariant under $\mathcal{S}$ and $\mathcal{T}$ transformation (see Supplemental Material at [URL] for characters, $\mathcal{S}$ and $\mathcal{T}$ matrices for $u(1)_M$ and $su(2)_k$ current algebras). Now we show that Z_A-CFT in (4) describes the critical point between the two superfluids. The critical theory between the two superfluids is first a spin-1 charge-2 boson condensing state. All gapless excitations must carry integer spins and even charges. The minimal charge is 2 and the minimal spin is 1. Generically, CFTs constructed from $su(2)_k \oplus u(1)_M \oplus \overline{su(2)}_k \oplus \overline{u(1)}_M$ current algebra for a fixed level k and M are not unique, but can have many versions corresponding to different modular invariant partition functions. They are all candidates to describe charge-2 spin-1 boson condensing states. These states may even be transmuted into one and other under interactions (that are marginal and symmetric). However, some condensing states are more stable than others. First, the integral spin sector $su(2)_k$ with smallest possible k is most stable. Since $su(2)_k$ CFT with smaller k has smaller central charge $\frac{3k}{k+2}$ and is more stable under RG flow where the central charge can monotonically decrease or stay the same at best. Second, the spin sector is most naturally realized by $su(2)_k$ CFT with even k. In fact, if we gap the charge sector by ($U(1)$ or $U_t(1)$ symmetry breaking) perturbations, the charge sector is insulating, the spins effectively become a spin-1 chain. On a spin-integer chain, the phase transition respecting $SO(3)$ and translation symmetry can only be $su(2)_k$ CFTs with even k.[17,18,19] We conclude that $su(2)_2$ CFT for the spin sector is the most stable. Its primary fields can carry spin 0, $\frac{1}{2}$ and 1.

The charge sector is built from $u(1)_M$ CFTs. Charge excitations are described by $u(1)_M$ primary field (5). The purely left charge-2 spin-1 boson $e^{i a\phi} V_1^{su2_2}$ must have conformal spin $h - \overline{h} = 1$ and that fixes $e^{i a\varphi}$ to have scaling dimension $h = \frac{1}{2}$ and $a = 1$. (In this paper, we adopt the normalization such that $\langle\varphi(z)\varphi(z')\rangle \sim (z - z')^{-1}$ and similarly for left-moving field $\overline{\varphi}$.) The low energy field can always factorized into left and right movers, we allow charge-1 primary field in left (right) $u(1)_M$ current algebra, which would be $e^{i \frac{1}{2}\phi}$, therefore the minimal level we need is $M = 4$ by comparing with (5). See Supplemental Material at [URL] for theories with other consistent levels but same (orbifold) structure. $e^{i \frac{1}{2}\varphi}$ and $e^{i \frac{1}{2}\overline{\varphi}}$, each carries charge $q = 1$ so that there are local excitations $e^{i \frac{\varphi+\overline{\varphi}}{2}} V_{\frac{1}{2},l}^{su2_2} \overline{V}_{\frac{1}{2},l}^{su2_2}$ carrying the minimal charge 2 and spin 0 or 1.

The critical point is also invariant under $U_t(1)$ translation symmetry. The $U_t(1)$ quantum number can only be carried by $u(1)_M$ primary fields, to be consistent with fusion rules among primary fields. Boson condensed states have a mixed $U(1)$ and $U_t(1)$ anomaly. This property is useful to restrict the levels of $u(1)_M$ CFTs and fix the $U_t(1)$ charge in each critical theories. We first define the $U_t(1)$ "charge". In the gapless bosonic system in 1d, there are low energy excitations carrying small momentum ~ 0, representing for example phonon modes. There are also low energy excitations with large momentum. They are center of mass motions of all bosons that can be generated by Galilean boost. Each boson is excited with a momentum kick of the order $\frac{2\pi}{L}$, and the total momentum change is of the order $N\frac{2\pi}{L} \equiv k_B = \frac{2\pi}{a}\nu$, proportional to the boson density, where N is total boson number, a is lattice spacing and $\nu = \frac{N}{N_L}$ is the boson filling number. We denote momentum of the center of mass motion as the $U_t(1)$ "charge" q_t, which is always of order k_B. Excitations with momentum $\ll k_B$ has $q_t = 0$. The center of mass motion can also be viewed as an excitation that changes the boundary condition of the field in $u(1)_M$ CFT from a periodic one to $\phi(L) = \phi(0) + \frac{L}{N} q_t$, which is sometimes called a large gauge transformation. (See Supplemental Material at [URL] for a further physical discussion.)

A boson condensed state has a characteristic feature. Modes with the center of mass momentum can be excited when pumping $U(1)$ flux through the ring. To measure this mixed anomaly in charge-2 spin-1 boson condensing state, we turn on a $U(1)$-flux $\Phi = 2\pi$ through the ring. The process boosts each boson that carries $U(1)$ charge $q = 2$ by a momentum $\frac{q\Phi}{L} = \frac{4\pi}{L}$, thus in total changes q_t by $2k_B$. Low energy theories shall carry this amount of anomaly. The $U_t(1)$ charge of local operators is determined in accord with $U(1)$ charge via matching the mixed anomaly. The anomaly of the operator $e^{\pm i(\varphi - \overline{\varphi})}$, carrying total charge $q = 0$ and non-trivial q_t, is most manifest to compute. The right-moving part $e^{i\varphi}$ has scaling dimension $(h, \overline{h}) = (\frac{1}{2}, 0)$ and thus represents a chiral fermion. Its left charges are $q = 2$ and q_t to be determined. The right-moving part $e^{i\varphi}$ has $U(1)$ charge q and q_t (and similarly, the left-moving part $e^{-i\overline{\varphi}}$ has $-q$ and q_t charges) . Adding 2π-flux of $U(1)$ would create q right-moving fermions and change $U_t(1)$ charge by qq_t; and annihilate q left-moving fermions and change $U_t(1)$ charge by qq_t, as illustrated in Fig.1. The total change of $U_t(1)$ charge is $2qq_t$. That it equals $2k_B$ fixes $q_t = \frac{1}{2}k_B$. It follows that in total, the operator $O_{e,b}(z, \overline{z}) = e^{i\left(\frac{e}{\sqrt{M}} + \frac{b}{2}\sqrt{M}\right)\varphi + i\left(\frac{e}{\sqrt{M}} - \frac{b}{2}\sqrt{M}\right)\overline{\varphi}}$ has charges

$$q = 2e, \quad q_t = bk_B \tag{6}$$

The partition function Z_A determines all allowed local excitations, operators corresponding to primary fields, organized in Table. I. The local operators carry spin, charge, and translation $U_t(1)$ charge. The spin quantum number is the same as the spin j of primary fields in $su(2)_k$ CFT. All local operators in Table I carry even charges. However, there are excitations with fractionalized $U_t(1)$ charge $\pm\frac{1}{2}k_B$. Furthermore, the excitation carries spin-0, and charge-0 (left charge ± 2 and right charge ∓ 2. Such an excitation is indeed created by adding π-flux of $U(1)$ through the ring.

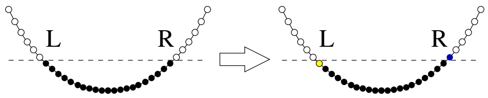

FIG. 1. For right-moving fermions with charge q and $U_t(1)$-charge q_t on a ring, adding a $2\pi/q$-flux of $U(1)$ through the ring will creat a fermion and add a $U_t(1)$-charge q_t. So the $U(1)\times U_t(1)$ mixed anomaly for the right-moving fermions is given by qq_t, *i.e.* 2π-flux of $U(1)$ will create a $U_t(1)$-charge qq_t. Similarly, for left-moving fermions with charge q and $U_t(1)$-charge $-q_t$, the $U(1)\times U_t(1)$ mixed anomaly is qq_t.

The CFT with Z_A is a representative of a series of $su(2)_2 \oplus u(1)_M \oplus \overline{su(2)}_2 \oplus \overline{u(1)}_M$ CFTs with $M = 0$ mod 4, which are all Z_2 orbifold CFTs. (See Supplemental Material at [URL] for their operator content and a formal description of orbifold theories.) We see that minimal charge is ± 2, minimal q_t charge is $\pm\frac{2}{M}$ and the mixed anomaly is $2k_B$. In each theory, there is only one relevant operator $\mathcal{O}$ symmetric under $SO(3)\times U(1)\times U_t(1)$ symmetry.

As shown in Table I, the single symmetric relevant operator is

$$\mathcal{O} = \sum_{l=x,y,z} V^{su2_2}_{1,l}\overline{V}^{su2_2}_{1,l} \tag{7}$$

that carries trivial quantum numbers and total scaling dimension $\Delta = h+\bar{h} = 1 < 2$. We would like to show that this perturbation drives the critical state to a superfluid state where all spin excitations are gapped.

Topological continuous phase transition between superfluids We consider the two phases connected by the critical theory Z_A perturbed by the relevant operator $\mathcal{O}$. First the perturbation is all within the spin sector $su(2)_2 \oplus \overline{su(2)}_2$ with central charge $c+\bar{c} = 3$. The spin sector can be described by three Majorana fermions that form the spin-1 representation of $SU(2)$. The relevant operator $\mathcal{O}$ corresponds to $SU(2)$ symmetric mass term of Majorana fermions with scaling dimension $(\frac{1}{2},\frac{1}{2})$. Adding the mass term can gap out all spin excitations. Therefore, the critical state described by $su(2)_2 \oplus u(1)_4 \oplus \overline{su(2)}_2 \oplus \overline{u(1)}_4$, a spin-1 charge-2 boson condensed state, is unstable against the gap openning for spin excitations.

At incommensurate filling, translation symmetry cannot be spontaneously broken. The gapped spin sector is therefore $SO(3)\times U_t(1)$ symmetric. Based on $1+1d$ SPT classification, the charge-2 spin-1 bosons can form two stable phases where charge operators have algebraic correlations, and all spin operators have exponential decaying correlations. One phase is the TSF and the other is the nTSF[1,2] introduced in the introduction.

The critical theory is also supported by its analog on the antiferromagnetic spin-1 chain. There, the critical point between the AKLT phase and the dimerized phases that spontaneously break translation symmetry are believed to be described by $su(2)_2 \oplus \overline{su(2)}_2$ CFT, and the transition is driven by $\mathcal{O}$ operator.[17,20] This result is confirmed by DMRG calculation[21]. We believe that in the superfluid case with all density configurations summed over, the spin part of the transition is still described by $su(2)_2 \oplus \overline{su(2)}_2$ CFT. Combining that the charge part is described by $u(1)_M \oplus \overline{u}(1)_M$. Z_A-CFT can be tested by DMRG computation in the model (1) at the critical point. From the minimal exponent of boson to be $\frac{5}{8}$ to demonstrate that this is a critical point described by a $u(1)\oplus su(2)_2$ CFT. One can find the minimal exponent of spin-1 charge-0 with nearly vanishing momentum to be 1, demonstrating that between TSF and trivial superfluid, the critical theory is $u(1)_4 \oplus su(2)_2$ CFT with partition function Z_A (as opposed to direct product of $u(1)_M$ and $su(2)_2$ theories. See Supplemental Material at [URL] for their partition functions denoted as Z_B and Z_C.)

operators	spin	charge	$U_t(1)$	$h,\bar{h}$
$\mathrm{e}^{\pm\,\mathrm{i}\, m\frac{\varphi+\bar{\varphi}}{2}}$	0	$\pm 2m$	0	$\frac{m^2}{8},\frac{m^2}{8}$
$\mathrm{e}^{\pm\,\mathrm{i}\,2\varphi}$	0	± 4	$\pm k_B$	$2,0$
$\mathrm{e}^{\pm\,\mathrm{i}\,2\bar{\varphi}}$	0	± 4	$\mp k_B$	$0,2$

TABLE II. The local operators in the $Z^{\mathrm{SF}}_{\mathrm{top}}$-CFT.

operators	spin	charge	$U_t(1)$	$h,\bar{h}$
$\mathrm{e}^{\pm\,\mathrm{i}\, m\frac{\varphi-\bar{\varphi}}{2}}$	0	0	$\pm\frac{1}{2}k_B$	$\frac{m^2}{8},\frac{m^2}{8}$
$\mathrm{e}^{\pm\,\mathrm{i}\,2\varphi}$	0	± 4	$\pm k_B$	$2,0$
$\mathrm{e}^{\pm\,\mathrm{i}\,2\bar{\varphi}}$	0	± 4	$\mp k_B$	$0,2$

TABLE III. The local operators in the $Z^{\mathrm{SF}}_{\mathrm{tri}}$-CFT.

Since superfluid phases are gapless phases, they can also be described by modular invariant CFTs. To obtain the partition functions of two superfluids, we start with the critical Z_A-CFT. We may assume that the pure $u(1)_4$ operators $\mathrm{e}^{\pm\,\mathrm{i}\,2\varphi}$ that are mutually local with the gapping operator in the spin sector remain. They appear in the CFTs that describe the two superfluid phases and carry same quantum numbers. This motivates us to use $u(1)_4$ CFT to describe two superfluids. There are two modular invariant partition functions for $u(1)_4$ CFT's. We believe they describe the two superfluid phases. The partition function describing the topological superfluid phase is

$$Z^{\mathrm{SF}}_{\mathrm{top}}(\tau) = \sum_{i=0}^{3} |\chi^{u1_4}_i(\tau)|^2 \tag{8}$$

Its operator content is given in Table II. The $U(1)\times U_t(1)$ mixed anomaly remains $2k_B$. The minimal charge is 2, as carried by the operator $\mathrm{e}^{\mathrm{i}\,\frac{1}{2}(\varphi+\bar{\varphi})}$. We stress that while the low energy theory of TSF can be described by $u(1)_4$ CFT, the phase itself requires the gapped spin-1 sector to reveal its nature of fractionalization. The nTSF is described by modular invariant partition function

$$Z^{\mathrm{SF}}_{\mathrm{tri}} = |\chi_0^{u1_4}|^2 + \chi_1^{u1_4}\overline{\chi}_3^{u1_4} + |\chi_2^{u1_4}|^2 + \chi_3^{u1_4}\overline{\chi}_1^{u1_4}. \quad (9)$$

Its operator content is given in Table III. The minimal charge is 4, as carried by the operator $\mathrm{e}^{\,\mathrm{i}\,(\varphi+\overline{\varphi})}$.

We have shown that regarding Z_A-CFT is consistent to describe the critical point of charge-2 spin-1 bosons with antiferromagnetic interaction and give a proposal based on a boson t-J model. (See Supplemental Material at [URL] for another construction from a spin-$\frac{1}{2}$ fermion model,[22]) and may be realized implemented in a spin-ladder model[23].

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