This is the accepted manuscript made available via CHORUS. The article has been published as:

Cellular-Automaton Decoders with Provable Thresholds for Topological Codes

Aleksander Kubica and John Preskill

Phys. Rev. Lett. **123**, 020501 — Published 8 July 2019

DOI: 10.1103/PhysRevLett.123.020501

Cellular-automaton decoders with provable thresholds for topological codes

Aleksander Kubica[1,2] and John Preskill[3,4]

[1] *Perimeter Institute for Theoretical Physics, Waterloo, ON N2L 2Y5, Canada*
[2] *Institute for Quantum Computing, University of Waterloo, Waterloo, ON N2L 3G1, Canada*
[3] *Institute for Quantum Information & Matter, California Institute of Technology, Pasadena, CA 91125, USA*
[4] *Walter Burke Institute for Theoretical Physics, California Institute of Technology, Pasadena, CA 91125, USA*

We propose a new cellular automaton (CA), the Sweep Rule, which generalizes Toom's rule to any locally Euclidean lattice. We use the Sweep Rule to design a local decoder for the toric code in $d \geq 3$ dimensions, the Sweep Decoder, and rigorously establish a lower bound on its performance. We also numerically estimate the Sweep Decoder threshold for the three-dimensional toric code on the cubic and body-centered cubic lattices for phenomenological phase-flip noise. Our results lead to new CA decoders with provable error-correction thresholds for other topological quantum codes including the color code.

To fault-tolerantly operate a scalable universal quantum computer, one protects logical information using a quantum error-correcting code, and removes errors without disturbing the encoded information [1, 2]. This can be achieved with stabilizer codes [3]. Each stabilizer generator is measured, yielding an outcome ± 1, and a classical decoding algorithm then computes the recovery operator. Unfortunately, optimal decoding of generic stabilizer codes is computationally hard [4, 5]. Thus, to render this task tractable one should restrict attention to codes with some structure.

Topological stabilizer codes [6–12], such as the toric and color codes, are highly structured due to the geometric locality of their stabilizer generators. Namely, any stabilizer returning a -1 measurement outcome indicates the presence of errors in its neighborhood. By exploiting this syndrome pattern, many efficient decoders with high error-correction thresholds have been proposed [13–26]. However, most of these decoders use global classical information about the measurement outcomes and thus require communication between distant parts of the system. In any realistic setting, new faults appear during the time needed to collect and process global syndrome data [19, 27]. Thus, to avoid error accumulation we desire fast decoders, which ideally use only local information.

A very promising class of topological quantum code decoders is based on cellular automata (CA) [28–30]. CA decoders are very efficient because they naturally incorporate parallelization and can be implemented on dedicated hardware without any non-local communication. As initially suggested in Ref. [13], a simple CA, called Toom's rule [31–33], can successfully protect quantum information encoded into the 4D toric code on a hypercubic lattice. Moreover, recent numerical simulations [34–36] indicate that heuristic decoders based on Toom's rule have non-zero error-correction thresholds for toric codes in more than two dimensions.

In this article we address the fundamental question whether using a CA is a viable error-correction strategy for topological quantum codes. First, we propose a new CA, the Sweep Rule, which generalizes Toom's rule to any locally Euclidean lattice in $d \geq 2$ dimensions. The Sweep Rule shrinks $(k-1)$-dimensional domain walls for any $k = 2, \ldots, d$. Then, we use the Sweep Rule to design a new local decoder of the toric code in $d \geq 3$ dimensions, the Sweep Decoder, and rigorously prove a lower bound on its performance for perfect syndrome extraction. Finally, we numerically demonstrate that the Sweep Rule suppresses errors when measurements are noisy. In particular, we estimate the sustainable threshold error rate $p^{\mathrm{bcc}}_{\mathrm{sus}} = 0.99 \pm 0.02\%$ of the Sweep Decoder for phase-flip errors and imperfect syndrome measurements in the 3D toric code on the body-centered cubic (bcc) lattice; see Fig. 1. Our decoder works reliably against Pauli X or Z errors if the corresponding syndrome is at least one-dimensional, i.e., not point-like, and the error rate is below the threshold; thus it can protect topological quantum memories in $d \geq 4$ dimensions. Our results lead to new decoders for the color code in $d \geq 3$ dimensions; see [37] or the accompanying article [38].

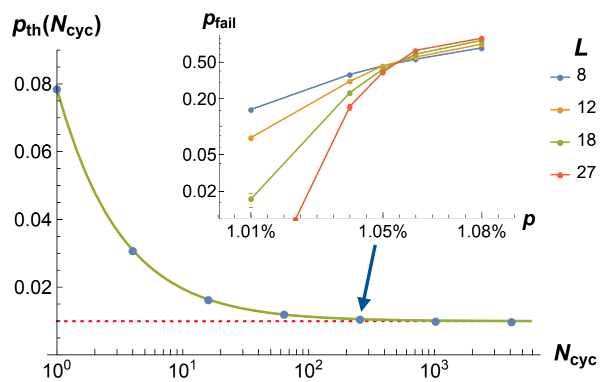

FIG. 1. (Inset) The failure probability $p_{\mathrm{fail}}(p, L)$ of the Sweep Decoder for the 3D toric code on the bcc lattice $\mathcal{L}$ after $N_{\mathrm{cyc}} = 2^8$ correction cycles, where p is the phase-flip error rate and L is the linear size of $\mathcal{L}$. We estimate the threshold $p_{\mathrm{th}}(N_{\mathrm{cyc}}) \approx 1.055\%$ from the crossing point of different curves. (Main) We find the sustainable threshold $p^{\mathrm{bcc}}_{\mathrm{sus}} = 0.99 \pm 0.02\%$ by fitting the numerical ansatz from Eq. (10) to the data.

Limitations of Toom's rule.—Consider the square lattice with a classical ± 1 spin placed on every face and encode one bit of information by setting all spins to

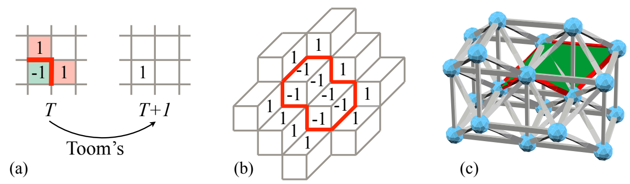

FIG. 2. (a) At time T the spin $s_C^{(T)} = -1$ (green face) differs from its neighbors to the east $s_E^{(T)} = 1$ and north $s_N^{(T)} = 1$ (red faces). According to Eq. (1), Toom's rule sets $s_C^{(T+1)} = 1$. (b) A 2D lattice built of three types of parallelograms. A domain wall (red) cannot be removed by repeated application of a naive generalization of Toom's rule. (c) The 3D toric code on the bcc lattice [39] has qubits on faces and X-stabilizers associated with edges. Any configuration of Z errors (green) results in a 1D loop-like X-syndrome (red).

be either $+1$ or -1. We want to protect the encoded bit against random spin flips, $\pm 1 \mapsto \mp 1$. This can be achieved with a CA, which flips certain spins based on *locally* available information. A simple example is the deterministic Toom's rule which sets the spin $s_C^{(T+1)}$ at time $T+1$ to

$$s_C^{(T+1)} = \operatorname{sgn}\left(s_C^{(T)} + s_E^{(T)} + s_N^{(T)}\right), \tag{1}$$

where $\operatorname{sgn}(\cdot)$ is the sign function, $s_E^{(T)}$ and $s_N^{(T)}$ are the neighboring spins on faces to the east and north at time T; see Fig. 2(a) The update can be simultaneously applied to all the spins in the square lattice.

We can rephrase Toom's rule as a conditional spin update determined by the local configuration of the 1D domain wall, i.e., the set of all edges of the lattice separating faces with spins of different value. Let $\epsilon^{(T)}$ and $\sigma^{(T)}$ denote the set of faces with -1 spins and the corresponding domain wall at time $T = 1, 2, \ldots$. We write $\sigma^{(T)} = \partial_2 \epsilon^{(T)}$ to capture the fact that $\sigma^{(T)}$ is the boundary of $\epsilon^{(T)}$ containing all the edges bounding faces in $\epsilon^{(T)}$. Then, Toom's rule flips a spin on some face f, i.e., $s_f^{(T+1)} = -s_f^{(T)}$, iff the east and north edges of f belong to $\sigma^{(T)}$; see Fig. 2(a). If we know $\sigma^{(T)}$ and the set of all spins flipped between time T and $T+1$, which we denote by $\varrho^{(T)}$, then the domain wall at time $T+1$ is

$$\sigma^{(T+1)} = \sigma^{(T)} + \partial_2 \varrho^{(T)} \tag{2}$$

with addition modulo 2. Note that this update does not require the knowledge of the actual spin values but only the locations of flipped spins. Thus, it may be viewed as a local rule governing the dynamics of the domain wall. Moreover, if the domain wall disappears by time T, i.e., $\sigma^{(T)} = 0$, then $\varrho = \sum_{i=1}^{T-1} \varrho^{(i)}$ can serve as an estimate [40] of $\epsilon^{(1)}$ with the boundary $\partial_2 \varrho$ matching the initial domain wall $\sigma^{(1)}$. As we will see later, correcting errors in the toric code in $d \geq 3$ dimensions can also be rephrased as estimating $\epsilon^{(1)}$ given its boundary $\sigma^{(1)}$, by exploiting the domain-wall structure of the syndrome.

This version of Toom's rule works for the square lattice, but it is not obvious how to generalize it to other 2D lattices, or to higher dimensions. To illustrate the difficulty, consider the 2D lattice in Fig. 2(b). If one uses a simple update rule *"flip a spin iff east and north edges of the face belong to the domain wall"*, then there exist spin configurations with domain walls which cannot be removed by repeated application of this rule. For such error syndromes, the Toom's rule decoder fails to correct the erroneous spins. To define a workable version of Toom's rule, the lattice must have suitable properties, which we now specify.

Causal lattices.—We consider a lattice $\mathcal{L}$, which is a triangulation (possibly without any symmetries) of the Euclidean space $\mathbb{R}^2$. We denote by $\Delta_i(\mathcal{L})$ the set of all i-simplices of $\mathcal{L}$. In particular, $\Delta_0(\mathcal{L})$, $\Delta_1(\mathcal{L})$ and $\Delta_2(\mathcal{L})$ correspond to vertices, edges and triangular faces of $\mathcal{L}$. We assume that each $\Delta_i(\mathcal{L})$ contains countably many elements and define the sweep direction as a unit vector $\vec{t} \in \mathbb{R}^2$ not perpendicular to any edge of $\mathcal{L}$.

We define a path $(u : w)$ between two vertices u and w of the lattice $\mathcal{L}$ to be a collection of edges $(u, v_1), \ldots, (v_n, w) \in \Delta_1(\mathcal{L})$, where $v_i \in \Delta_0(\mathcal{L})$. If the sign of the inner product $\vec{t} \cdot (v_i, v_{i+1})$ is the same for all edges in the path $(u : w)$, then we call the path causal and denote it by $(u \updownarrow w)$. We remark that any pair of the vertices of $\mathcal{L}$ is connected by a path but there might not exist a causal path between them; see Fig. 3(a). Finally, we define the causal distance

$$d_{\updownarrow}(u, w) = \min_{(u \updownarrow w)} |(u \updownarrow w)| \tag{3}$$

to be the length of the shortest causal path between u and w; if there is no causal path, then $d_{\updownarrow}(u, w) = \infty$.

We observe that the sweep direction $\vec{t}$ induces a binary relation $\preceq$ over the set of vertices $\Delta_0(\mathcal{L})$. We say that u precedes w, i.e., $u \preceq w$ for $u, w \in \Delta_0(\mathcal{L})$, iff $u = w$ or there exists a causal path $(u \updownarrow w)$ and $\vec{t} \cdot (v_i, v_{i+1}) > 0$ for any edge $(v_i, v_{i+1}) \in (u \updownarrow w)$. Equivalently, we write $w \succeq u$ and say that w succeeds u. Abusing the notation, we write $\kappa \succeq w$ if all vertices $\Delta_0(\kappa)$ of a k-simplex $\kappa \in \Delta_k(\mathcal{L})$ succeed w, i.e., $u \succeq w$ for all $u \in \Delta_0(\kappa)$; similarly for $\kappa \preceq w$.

We can view the partial order $\preceq$ between vertices of the lattice as a causality relation between points in the $(1+1)$D spacetime with $\vec{t}$ corresponding to the time [41] direction; see Fig. 3(a)(b). We define the future $\uparrow(v)$ and past $\downarrow(v)$ of a vertex $v \in \Delta_0(\mathcal{L})$ as the collection of all simplices of $\mathcal{L}$ succeeding and preceding v, namely

$$\uparrow(v) = \bigcup_{k=0}^{d} \{\kappa \in \Delta_k(\mathcal{L}) | \kappa \succeq v\}, \tag{4}$$

$$\downarrow(v) = \bigcup_{k=0}^{d} \{\kappa \in \Delta_k(\mathcal{L}) | \kappa \preceq v\}. \tag{5}$$

Every finite subset of vertices $V \subseteq \Delta_0(\mathcal{L})$ has a unique supremum, the vertex $\sup V$, where $\sup V$ lies in the future of each $u \in V$, and furthermore $\sup V$ lies in the past

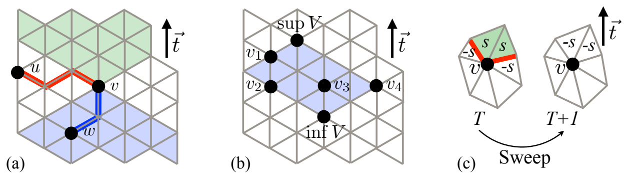

FIG. 3. (a) Vertices u and v are connected by a path $(u : v)$ (red), but there is no causal path between them; v and w are connected by a causal path $(v \updownarrow w)$ (blue). We shaded in green and blue the future $\uparrow(v)$ and past $\downarrow(v)$ of v. (b) The causal diamond $\Diamond(V)$ (blue) of a subset of vertices $V = \{v_1, v_2, v_3, v_4\}$ is defined as the intersection of the future of the infimum of V with the past of the supremum of V. (c) The Sweep Rule is defined for every vertex and locally updates ± 1 spins on neighboring faces. Since the vertex v is trailing (see below), spins on two green faces will be flipped.

of each vertex w which is in the future of each $v \in V$. The infinum $\inf V$ is defined analogously. Lastly, we define the causal diamond $\Diamond(V)$ as the intersection of the future of $\inf V$ and the past of $\sup V$, i.e.,

$$\Diamond(V) = \uparrow(\inf V) \cap \downarrow(\sup V). \tag{6}$$

This discussion of causal structure generalizes to lattices embedded in a torus; however, caution is needed since the partial order is well-defined only within contractible regions. For higher-dimensional lattices we make certain assumptions about their causal structure, such as the existence of unique infimum and supremum of V. To avoid technicalities, we call lattices satisfying those assumptions causal; see Appendix A. Note that causal lattices are sufficient to define the Sweep Rule and prove a non-zero threshold of the Sweep Decoder.

Sweep Rule.—Let $\mathcal{L}$ be a 2D causal lattice with ± 1 spins on triangular faces and $\epsilon \subseteq \Delta_2(\mathcal{L})$ denote the set of all faces with -1 spins. The corresponding domain wall σ can be found as the boundary $\partial_2 \epsilon$. Let v be a vertex of $\mathcal{L}$ and denote by $\sigma|_v$ the restriction of the domain wall σ to the edges incident to v. We say that v is trailing if $\sigma|_v$ is non-empty and belongs to the future of v, namely $\sigma|_v \subset \uparrow(v)$; see Fig. 4. We propose a new local spin update rule defined for every vertex v of $\mathcal{L}$.

Definition 1 (Sweep Rule)**.** *If a vertex v is trailing, then find a subset $\varphi(v)$ of neighboring faces of v in the future $\uparrow(v)$ with boundary locally matching the domain wall, i.e., $(\partial_2 \varphi(v))|_v = \sigma|_v$, and flip spins on faces in $\varphi(v)$.*

This Rule is deterministic and there is a unique $\varphi(v)$, which one can find in constant time. The spin update results in the domain wall being locally pushed away from any trailing vertex v; see Fig. 4. Note that nothing happens if a vertex is not trailing. We can, however, consider a very similar CA, the Greedy Sweep Rule, which always tries to push the domain wall away from v in the sweep direction $\vec{t}$, irrespective of v being trailing; see Appendix B.

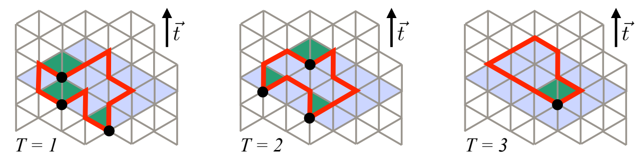

FIG. 4. For each trailing vertex v (black) at time $T = 1, 2, 3$ the Sweep Rule finds a subset $\varphi(v)$ of neighbouring faces (green) in the future $\uparrow(v)$, whose boundary $\partial_2 \varphi(v)$ locally matches the domain wall $\sigma^{(T)}$ (red), i.e., $(\partial_2 \varphi(v))|_v = \sigma^{(T)}|_v$. Flipping spins in $\varphi(v)$ pushes $\sigma^{(T)}$ away from v in the sweep direction $\vec{t}$. Note that $\varphi(v)$ and $\sigma^{(T)}$ are always in the causal diamond $\Diamond\left(\sigma^{(1)}\right)$ (blue) of the initial domain wall $\sigma^{(1)}$.

Lemma 2 (Sweep Rule Properties)**.** Let σ be a domain wall in the causal lattice $\mathcal{L}$. If the Sweep Rule is simultaneously applied to every vertex of $\mathcal{L}$ at time steps $T = 1, 2, \ldots$, then

1. (Support) the domain wall $\sigma^{(T)}$ at time T stays within the causal diamond $\Diamond(\sigma) = \Diamond(\Delta_0(\sigma))$, i.e.,
$$\sigma^{(T)} \subset \Diamond(\sigma), \tag{7}$$
2. (Propagation) the causal distance $d_{\updownarrow}(v, \sigma) = \min_{u \in \Delta_0(\sigma)} d_{\updownarrow}(v, u)$ between σ and any vertex v of $\sigma^{(T)}$ is at most T, i.e.,
$$d_{\updownarrow}(v, \sigma) \leq T, \tag{8}$$
3. (Removal) the domain wall is removed by time T^*, i.e., $\sigma^{(T)} = 0$ for all $T > T^*$, where
$$T^* = \max_{(\inf \sigma \updownarrow \sup \sigma)} |(\inf \sigma \updownarrow \sup \sigma)|. \tag{9}$$

See Appendix C for a proof.

The Sweep Rule may also be defined for vertices of a d-dimensional causal lattice $\mathcal{L}$ with spins placed on k-simplices $\Delta_k(\mathcal{L})$, where $k = 2, \ldots, d$. However, for $k \neq d$ the local choice of spins to flip $\varphi(v)$ may not be unique (this does not happen in 2D). Thus, we consider a family of rules corresponding to different ways of choosing $\varphi(v)$ in such a way that, roughly speaking, the local causal structure of the domain wall is preserved after flipping spins on k-simplices in $\varphi(v)$; see Appendix B.

Sweep Decoder.—We may use the d-dimensional version of the Sweep Rule to decode the toric code on the d-dimensional causal lattice $\mathcal{L}$. Recall that the toric code of type $k = 1, \ldots, d-1$ is defined by placing qubits on k-simplices of $\mathcal{L}$, and associating X- and Z-stabilizers with $(k-1)$- and $(k+1)$-simplices. Then, Z-stabilizers, Z-logical operators and X-syndromes correspond to, respectively, the elements of $\operatorname{im} \partial_{k+1}$, $\ker \partial_k$ and $\operatorname{im} \partial_k$, where ∂_i denotes the i-boundary operator; see Appendix A. If $\epsilon \subseteq \Delta_k(\mathcal{L})$ is the set of qubits affected by Z errors, then the corresponding X-syndrome is $\sigma = \partial_k \epsilon$. Thus, for $k \geq 2$, decoding of Z errors can be phrased as the already discussed problem of estimating locations of -1 spins given the corresponding domain wall. Note that for $k \le d-2$ decoding of X errors is analogous but in the dual lattice $\mathcal{L}^*$ with the Z-syndrome forming a $(d-k-1)$-dimensional domain wall.

Algorithm: Sweep Decoder

Input: X-syndrome $\sigma \in \operatorname{im} \partial_k$, $k = 2, \ldots, d-1$
Output: k-dimensional correction $\varrho \subseteq \Delta_k(\mathcal{L})$
initialize $T = 1$, $\sigma^{(1)} = \sigma$
unless $T > T_{\max}$ or $\sigma^{(T)} = 0$ repeat:

1. apply the Sweep Rule simultaneously to every vertex of $\mathcal{L}$ to get $\varrho^{(T)}$
2. find $\sigma^{(T+1)} = \sigma^{(T)} + \partial_k \varrho^{(T)}$
3. update time step $T \leftarrow T + 1$

if $T \le T_{\max}$ [42], then $\varrho = \sum_{i=1}^{T-1} \varrho^{(i)}$, otherwise $\varrho =$ `FAIL`
return ϱ

This *Sweep Decoder* may fail for either one of two reasons. First, it might not terminate within time $T_{\max}$, which results in $\varrho =$ `FAIL`. Second, the correction ϱ combined with the initial error ϵ may implement a non-trivial logical operator, i.e., $\varrho + \epsilon \notin \operatorname{im} \partial_{k+1}$. However, the Sweep Decoder a has non-zero error-correction threshold — if the Z error rate is below threshold, then the failure probability rapidly approaches zero as the code distance grows. We establish this fact by deriving a lower bound $p^*_{\mathrm{th}} > 0$ on the threshold.

Theorem 3 (Threshold)**.** *Consider a family of causal lattices $\mathcal{L}$ of growing linear size L on the d-dimensional torus, and define the toric code of type $k = 2, \ldots, d-1$ on $\mathcal{L}$. There exists a constant $p^*_{\mathrm{th}} > 0$, such that for any phase-flip error rate $p < p^*_{\mathrm{th}}$ the failure probability of the Sweep Decoder for perfect syndrome extraction goes to zero as $L \to \infty$.*

In Appendix D we present a rigorous proof of Theorem 3 based on renormalization group ideas [17, 28, 43]; here we only outline the proof strategy.

Proof. First, we decompose each error configuration into recursively defined "connected components," where a "level-n" connected component has a linear size growing exponentially with n. The probability of a level-n connected component is doubly-exponentially small in p/p^*_{th}. The connected components are well isolated from other errors; therefore, using Lemma 2 and some modest assumptions about the lattice family, we can show that a connected component with linear size small compared to L will be successfully removed by repeated application of the Sweep Rule. Therefore, the Sweep Decoder fails only if the error configuration contains a level-n connected component with size comparable to L, which is very improbable for large L and $p < p^*_{\mathrm{th}}$. □

Numerical simulations.—In Theorem 3 we assumed that the Sweep Rule is applied flawlessly, but in a realistic scenario the Rule itself is noisy. We have numerically investigated the performance of the Sweep Decoder for the 3D toric code on the bcc lattice with qubits on faces and a phenomenological noise model. Each correction cycle consists of one time step of the Sweep Decoder, adding new Pauli Z errors on qubits with probability p and extracting syndrome bits, which are flipped with probability p. Using Monte Carlo simulations we find the threshold $p_{\mathrm{th}}(N_{\mathrm{cyc}})$ for a fixed number N_{cyc} of noisy correction cycles followed by perfect syndrome extraction and full decoding. Note that $p_{\mathrm{th}}(1)$ is the threshold for perfect syndrome extraction. We are, however, interested in the so-called sustainable threshold $p^{\mathrm{bcc}}_{\mathrm{sus}} = \lim_{N_{\mathrm{cyc}} \to \infty} p_{\mathrm{th}}(N_{\mathrm{cyc}})$ [27, 44]. We observe that the threshold $p_{\mathrm{th}}(N_{\mathrm{cyc}})$ is very well approximated by the numerical ansatz

$$p_{\mathrm{th}}(N_{\mathrm{cyc}}) \sim p^{\mathrm{bcc}}_{\mathrm{sus}}(1 - (1 - p_{\mathrm{th}}(1)/p^{\mathrm{bcc}}_{\mathrm{sus}}) N_{\mathrm{cyc}}^{-\gamma}), \qquad (10)$$

with the fitting parameters $p^{\mathrm{bcc}}_{\mathrm{sus}} = 0.99 \pm 0.02\%$ and $\gamma = 0.855 \pm 0.010$; see Fig. 1. These numerical results were actually obtained for a variant of the Sweep Decoder based on the Greedy Sweep Rule, which has a higher threshold than the decoder based on the Sweep Rule. In Appendix B we discuss the Greedy Sweep Rule, explain how it generalizes to locally Euclidean lattices, and use it to estimate the sustainable threshold of the 3D toric code on the cubic lattice $p^{\mathrm{cubic}}_{\mathrm{sus}} = 1.98 \pm 0.02\%$.

Discussion.—We have presented a new CA, the Sweep Rule, which generalizes Toom's rule to any locally Euclidean d-dimensional lattice. This Rule can be used to decode a topological quantum code whose error syndrome is at least one dimensional, including the color code; see [37, 38]. We proved that a decoder based on the Sweep Rule has a non-zero accuracy threshold for the toric code, and we numerically studied its performance against a phenomenological noise model.

Our results provide a rigorous justification for using CA error-correction strategies for topological quantum codes. We hope that our proof techniques will lead to new CA decoders with provable thresholds for codes on lattices with boundaries, hyperbolic lattices or other quantum low-density parity-check codes.

The Sweep Rule may be of independent interest for defining statistical-mechanical problems inspired by quantum information [45–47]. As for Toom's rule, one can consider a non-deterministic variant of the Sweep Rule and study the evolution of spins generated by this probabilistic CA. We conjecture that the resulting spin dynamics is non-ergodic and that the phase diagram contains regions with multiple coexisting stable phases, as established in 2D by Toom [31].

ACKNOWLEDGMENTS

A.K. thanks Nicolas Delfosse for invaluable feedback throughout the project, and Ben Brown and Mike Vasmer for stimulating discussions. A.K. acknowledges funding provided by the Simons Foundation through the "It from Qubit" Collaboration. Research at Perimeter Institute is supported by the Government of Canada through Industry Canada and by the Province of Ontario through the Ministry of Research and Innovation. J.P. acknowledges support from ARO, DOE, IARPA, NSF, and the Simons Foundation. The Institute for Quantum Information and Matter (IQIM) is an NSF Physics Frontiers Center.

[1] P. Shor, in *Proceedings of 37th Conference on Foundations of Computer Science* (IEEE Comput. Soc. Press, 1996) pp. 56–65.
[2] J. Preskill, Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences **454**, 385 (1998).
[3] D. Gottesman, Physical Review A **54**, 1862 (1996).
[4] M. H. Hsieh and F. Le Gall, Physical Review A **83**, 052331 (2011).
[5] P. Iyer and D. Poulin, IEEE Transactions on Information Theory **61**, 5209 (2015).
[6] A. Y. Kitaev, Annals Phys. **303**, 2 (2003).
[7] S. B. Bravyi and A. Y. Kitaev, arXiv:quant-ph/9811052 (1998).
[8] H. Bombin and M. A. Martin-Delgado, Physical Review Letters **97**, 180501 (2006).
[9] J. Haah, Physical Review A **83**, 042330 (2011).
[10] H. Bombin, in *Topological Codes*, edited by D. A. Lidar and T. A. Brun (Cambridge University Press, 2013).
[11] B. J. Brown, D. Loss, J. K. Pachos, C. N. Self, and J. R. Wootton, Reviews of Modern Physics **88**, 045005 (2016).
[12] A. Kubica, B. Yoshida, and F. Pastawski, New Journal of Physics **17**, 083026 (2015).
[13] E. Dennis, A. Kitaev, A. Landahl, and J. Preskill, Journal of Mathematical Physics **43**, 4452 (2002).
[14] D. S. Wang, A. G. Fowler, A. M. Stephens, and L. C. L. Hollenberg, Quantum Information and Computation **10**, 456 (2010).
[15] D. S. Wang, A. G. Fowler, and L. C. L. Hollenberg, Physical Review A **83**, 020302 (2011).
[16] G. Duclos-Cianci and D. Poulin, Physical Review Letters **104**, 050504 (2010).
[17] S. Bravyi and J. Haah, Physical Review Letters **111**, 200501 (2013).
[18] A. G. Fowler, A. C. Whiteside, and L. C. L. Hollenberg, Physical Review Letters **108**, 180501 (2012).
[19] A. G. Fowler, Physical Review Letters **109**, 180502 (2012).
[20] H. Anwar, B. J. Brown, E. T. Campbell, and D. E. Browne, New Journal of Physics **16**, 063038 (2014).
[21] N. Delfosse, Physical Review A **89**, 012317 (2014).
[22] H. Bombin, Physical Review X **5**, 031043 (2015).
[23] N. Delfosse and G. Zémor, arXiv:1703.01517 (2017).
[24] N. Delfosse and N. H. Nickerson, arXiv:1709.06218 (2017).
[25] N. H. Nickerson and B. J. Brown, arXiv:1712.00502 (2017).
[26] N. Maskara, A. Kubica, and T. Jochym-O'Connor, arXiv:1802.08680 (2018).
[27] B. M. Terhal, Rev. Mod. Phys. **87**, 307 (2015).
[28] J. Harrington, *Analysis of quantum error-correcting codes: symplectic lattice codes and toric codes*, Ph.D. thesis, California Institute of Technology (2004).
[29] M. Herold, E. T. Campbell, J. Eisert, and M. J. Kastoryano, npj Quantum Information **1**, 15010 (2015).
[30] G. Dauphinais and D. Poulin, Communications in Mathematical Physics **355**, 519 (2017).
[31] A. L. Toom, in *Multicomponent Systems*, Vol. 6 (1980) pp. 549–575.
[32] C. H. Bennett and G. Grinstein, Physical Review Letters **55**, 657 (1985).
[33] G. Grinstein, IBM Journal of Research and Development **48**, 5 (2004).
[34] F. Pastawski, L. Clemente, and J. I. Cirac, Physical Review A **83**, 012304 (2011).
[35] N. P. Breuckmann, K. Duivenvoorden, D. Michels, and B. M. Terhal, Quantum Information and Computation **17**, 0181 (2017).
[36] N. P. Breuckmann and X. Ni, Quantum **2**, 68 (2018).
[37] A. Kubica, *The ABCs of the color code: A study of topological quantum codes as toy models for fault-tolerant quantum computation and quantum phases of matter*, Ph.D. thesis, California Institute of Technology (2018).
[38] A. Kubica and N. Delfosse, *High threshold color-code decoders from toric-code decoders* (in preparation).
[39] The figure was created using vZome available at `http://vzome.com`.
[40] This strategy, however, is neither guaranteed to terminate nor to return $\varrho = \epsilon^{(1)}$.
[41] We warn the reader that later we use the time T to index how many times the CA rule is applied.
[42] $T_{\max}$ upper-bounds the runtime of the Sweep Decoder and is of the order of the linear size L of the lattice $\mathcal{L}$; see Appendix D 3.
[43] P. Gács and J. Reif, Journal of Computer and System Sciences **36**, 125 (1988).
[44] B. J. Brown, N. H. Nickerson, and D. E. Browne, Nature Communications **7**, 4 (2015).
[45] A. Kubica and B. Yoshida, arXiv:1402.0619 (2014).
[46] B. Yoshida and A. Kubica, arXiv:1404.6311 (2014).
[47] A. Kubica, M. E. Beverland, F. Brandão, J. Preskill, and K. M. Svore, Physical Review Letters **120**, 180501 (2018).