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A fundamental approach for the characterization and quantification of all kinds of resources is to study the conversion between different resource objects under certain constraints. Here we analyze, from a non-resource-specific standpoint, the optimal efficiency of resource formation and distillation tasks with only a single copy of the given quantum state, thereby establishing a unified framework of one-shot quantum resource manipulation. We find general bounds on the optimal rates characterized by resource measures based on the smooth max/min-relative entropies and hypothesis testing relative entropy, as well as the free robustness measure, providing them with general operational meanings in terms of optimal state conversion. Our results encompass a wide class of resource theories via the theory-dependent coefficients we introduce, and the discussions are solidified by important examples, such as entanglement, coherence, superposition, magic states, asymmetry, and thermal non-equilibrium.

Introduction.—The manipulation and characterization of resources are ubiquitous subjects of concern. In recent years, substantial research effort originated from the quantum information community has been devoted to a framework known as resource theory, which significantly advances the study of quantum physics and quantum technologies (see Ref. [1] for a recent overview). The framework centers around the task of quantifying the value of certain resource features (e.g. quantum entanglement) in various scenarios, in order to rigorously understand the essence of these resources and how to best utilize them. Resource theory is particularly interesting and powerful because of its versatility — similar methodologies are successfully applied to a plethora of important resource entities, such as entanglement [2, 3], coherence [4–6], superposition [7], magic states [8, 9], asymmetry [10, 11], purity [12, 13], thermal non-equilibrium [14–16], non-Gaussianity [17–19]. Therefore, a research line of fundamental importance is to investigate the unified, non-resource-specific aspects of resource theory and how they fit into different contexts [20–38].

In this work, we establish such a general scheme for operationally quantifying the resource content of quantum states through their value in fundamental "resource trading" tasks. More specifically, we are interested in the optimal rate of forming a quantum state using some standard resource states that serve as the "currency", and conversely that of using the given state to distill standard states, under typical free operations. Many specific forms of such tasks are of independent interest; for example, the task of entanglement formation induces the entanglement cost, an important entanglement measure [39, 40], and the tasks of entanglement distillation [39, 41, 42] and magic state distillation [43] play key roles in quantum information and computation. Here we focus on the practical scenario where only one copy (or finite copies) of the state is available (i.e. the one-shot setting), and some amount of error is allowed. Unlike the asymptotic theory (the limit of infinite i.i.d. copies) [21], only a few resource-specific results about entanglement [44], coherence [45–47], and (generalized) quantum thermodynamics [14, 48–51] (and magic states in a very recent work [52]) are known. Here we consider two important classes of free operations easily characterized by the theory of resource destroying (RD) maps [25]: the maximal free operations (e.g. non-entangling operations for entanglement, maximal incoherent operations (MIO) for coherence, Gibbs-preserving maps for thermodynamics), and the commuting operations (e.g. dephasing-covariant incoherent operations (DIO) for coherence [53], isotropic channels for discord when restricted to local operations and qudit systems [54]), which induce general distance monotones without optimization. We prove highly generic limits to the optimal rates of standard one-shot formation and distillation tasks under the above free operations, and show that they can be nearly achieved in many cases. These general bounds take unified and simple forms in terms of resource monotones based on the smoothed max-relative entropy or the free (also called "standard") robustness for formation, and the smoothed min-relative entropy or the hypothesis testing relative entropy for distillation, divided by a certain modification coefficient that encodes the resource value of the standard states. To put it another way, the results endow these resource monotones with operational meanings in terms of "normalized" one-shot resource conversion tasks, providing a general operational interpretation to the min-relative entropy measure and supplementing those of the max-relative entropy and free robustness measures recently unveiled via erasure [30] and discrimination tasks [29, 35]. In particular, we find that taking maximum resource states as the currency not only makes the most sense out of formation/distillation tasks conceptually, but also leads to nice mathematical structures of the results. For example, we show that several key resource measures (and therefore the corresponding modification coefficients) of *golden states* (a notion of max-resource states we introduce) collapse to the same value in generic convex theories, which leads to nearly tight bounds. Our results generalize the existing resource-specific ones, and we shall also elucidate the results by suitable new examples.

Preliminaries.—Let $\mathcal{H}_d$ be the Hilbert space of dimension $d < \infty$, and $\mathcal{D}(\mathcal{H}_d)$ be the set of density operators acting on $\mathcal{H}_d$. Also let $\mathcal{F}(\mathcal{H}_d) \subseteq \mathcal{D}(\mathcal{H}_d)$ be the set of free states in the resource theory under consideration (the brackets are dropped onwards when the Hilbert space is clear from the context). We assume that the set of free states is topologically closed, so that the maxima or minima over it are well-defined.

We first formally define several information-theoretic quantities and resource measures. Let ρ, σ be density operators [55]. The Uhlmann fidelity of ρ and σ is given by $f(\rho,\sigma) := \left(\mathrm{Tr}\sqrt{\sqrt{\sigma}\rho\sqrt{\sigma}}\right)^2 = \left\|\sqrt{\rho}\sqrt{\sigma}\right\|_1^2$. The free fidelity of ρ, which measures the maximum overlap with free states, is defined as $\mathfrak{f}(\rho) := \max_{\sigma\in\mathcal{F}} f(\rho,\sigma)$. The *max-relative entropy* and *min-relative entropy* between ρ and σ are respectively given by [56]

$$D_{\max}(\rho\|\sigma) := \log\min\{\lambda : \rho \leq \lambda\sigma\},$$

which is well-defined when $\mathrm{supp}(\rho) \subseteq \mathrm{supp}(\sigma)$, and

$$D_{\min}(\rho\|\sigma) := -\log\mathrm{Tr}\{\Pi_\rho\sigma\}$$

where Π_ρ denotes the projector onto $\mathrm{supp}(\rho)$, which is well-defined when $\mathrm{supp}(\rho)\cap\mathrm{supp}(\sigma)\backslash\{0\}$ is non-empty. They roughly represent two ends of the spectrum of quantum Rényi relative entropy (see Appendix A [79] for more rigorous statements). To account for finite accuracy, the smoothed versions are needed. Let $\mathcal{B}^\epsilon(\rho) := \{\rho' : f(\rho',\rho) \geq 1-\epsilon\}$. The smoothed max- (min-) relative entropy between ρ and σ is then given by minimizing (maximizing) over this ϵ-vicinity of ρ:

$$D^\epsilon_{\max}(\rho\|\sigma) := \min_{\rho'\in\mathcal{B}^\epsilon(\rho)} D_{\max}(\rho'\|\sigma),$$
$$D^\epsilon_{\min}(\rho\|\sigma) := \max_{\rho'\in\mathcal{B}^\epsilon(\rho)} D_{\min}(\rho'\|\sigma).$$

For the min-relative entropy we also consider a slightly different type of smoothing known as the operator-smoothing:

$$D^\epsilon_H(\rho\|\sigma) := \max_{0\leq P\leq I, \mathrm{Tr}\{P\rho\}\geq 1-\epsilon}(-\log\mathrm{Tr}\{P\sigma\}).$$

We use the notation D^ϵ_H since this is equivalent to the hypothesis testing relative entropy defined in Ref. [57].

One can then define corresponding resource measures by the minimum divergence with free states:

$$\mathfrak{D}_{\max(\min)}(\rho) := \min_{\sigma\in\mathcal{F}} D_{\max(\min)}(\rho\|\sigma).$$

Due to the data processing inequalities for $D_{\max}$ [58], $D_{\min}$ [59–61] and the purified distance $P(\rho,\sigma) = \sqrt{1-f(\rho,\sigma)}$ [62], it holds that $\mathfrak{D}_{\max,\min}$ are monotonically non-increasing ($\mathfrak{f}$ is non-decreasing) under all free operations. The smoothed versions of these resource measures are simply defined by replacing the divergences with smoothed ones. Another important type of resource measure is the *free robustness/log-robustness*:

$$R(\rho) := \min\{s \geq 0 : \exists\sigma\in\mathcal{F}, \frac{1}{1+s}\rho + \frac{s}{1+s}\sigma \in \mathcal{F}\},$$
$$LR(\rho) := \log(1+R(\rho)).$$

The smoothed versions are similarly given by minimizing over $\mathcal{B}^\epsilon(\rho)$. By definition, if $\mathcal{F}$ is an affine set, i.e. any state expressed by an affine combination of free states is free (in e.g. coherence, asymmetry theories), then any resource state $\rho \notin \mathcal{F}$ does not have finite free robustness, although infinite free robustness does not necessarily indicate that $\mathcal{F}$ is affine (see Appendix B [79]). We formally introduce the following condition for $\mathcal{F}$ for convenience of later discussions:

Condition (FFR). All states have finite free robustness, i.e. $R(\rho) < \infty, \forall\rho$.

By allowing σ to be any state (instead of a free state) in the definition of free robustness, one obtains the so-called generalized robustness/log-robustness, R_G/LR_G. It can be easily verified that $LR_G(\rho) = \mathfrak{D}_{\max}(\rho)$.

We next briefly overview the theory of *resource destroying (RD) maps* [25]. A map λ from states to states is an RD map if it satisfies the following conditions: i) mapping all non-free states to a free state, i.e. $\forall\rho\notin\mathcal{F}, \lambda(\rho)\in\mathcal{F}$; ii) preserving free states, i.e. $\forall\sigma\in\mathcal{F}, \lambda(\sigma) = \sigma$. Two types of RD maps are of particular importance: i) *Exact RD maps*, which output the closest free state as measured by the relative entropy. Simple forms are known in e.g. coherence, asymmetry and non-Gaussianity theories. (See Appendix C [79] for a detailed introduction.) ii) *RD channels*. They often induce desirable features, e.g. the image free state is continuous under variation of the input state due to data processing inequalities. Examples include the dephasing channel for coherence theory and the twirling channel for asymmetry theory. In Appendix D [79], we show that if any state takes finite free robustness, then there does not exist an RD channel in that theory.

An RD map λ induces typical classes of quantum channels via a collection of simple, general conditions. This work focuses on the following two important ones: i) the resource non-generating operations $\mathscr{F}_{\mathrm{NG}} := \{\mathcal{E} \mid \lambda\circ\mathcal{E}\circ$

$\lambda = \mathcal{E} \circ \lambda\}$ [63], which induces the maximal set of free operations in the sense that any other operation can create resource from a free state; ii) the commuting operations $\mathscr{F}_{\lambda,\mathrm{Comm}} = \{\mathcal{E} \mid \lambda \circ \mathcal{E} = \mathcal{E} \circ \lambda\}$. One can then construct simple resource measures $\delta_\lambda(\cdot) = \delta(\cdot, \lambda(\cdot))$ where δ is any contractive distance measure, which is monotonically non-increasing under the commuting operations [25]. Here we shall use $\mathfrak{D}_{\max(\min),\lambda}(\rho) := D_{\max(\min)}(\rho \| \lambda(\rho))$, with smoothed versions defined by minimizing (or maximizing) over $\mathcal{B}^\epsilon(\rho)$. The counterpart for free fidelity is similarly given by $\mathfrak{f}_\lambda(\rho) := f(\rho, \lambda(\rho))$.

We implicitly assume that the resource measures appearing throughout the paper are well-defined (the free robustness case is highlighted since it is of crucial importance in resource theories).

Resource currencies and modification coefficients.— Resource manipulation tasks are commonly defined relative to some standard or unit resource that serve as the "currency": the formation task is about preparing the target state with a supply of standard resource, while the distillation task is about producing standard resource from the given state. More generally, consider some family of states $\{\phi_d\}$ consisting of a state $\phi_d \in \mathcal{D}(\mathcal{H}_d)$ for each different $d \in \mathbb{D}$ where $\mathbb{D} \subseteq \mathbb{Z}_+$ is a set of valid dimensions as a definition of a resource currency (e.g. $\mathbb{D} = \{2^k\}, k \in \mathbb{Z}_+$ for multi-qubit theories), and call them *reference states*. Also let $d^{\downarrow(\uparrow)} \in \mathbb{D}$ be some dimension smaller (greater) than d (e.g. take $d^\uparrow = d + 1$ when $\mathbb{D} = \mathbb{Z}_+$). We then introduce the following *modification coefficients*, which will naturally emerge in the later discussions on one-shot rates:

$$m_f(\phi_d) := -\log \mathfrak{f}(\phi_d)/\log d, \tag{1}$$

$$m_{\max(\min)}(\phi_d) := \mathfrak{D}_{\max(\min)}(\phi_d)/\log d, \tag{2}$$

$$m_{LR}(\phi_d) := LR(\phi_d)/\log d. \tag{3}$$

Similarly, $m_{f,\lambda}$ and $m_{\max(\min),\lambda}$ are defined by using $\mathfrak{f}_\lambda$ and $\mathfrak{D}_{\max(\min),\lambda}$ for Eqs. (1) and (2).

It is common to consider certain notions of "maximum" resource states as the reference states (so that the formation and distillation tasks essentially achieve the effect of dilution and concentration of resource respectively), although one can in principle choose more general classes of states. The modification coefficients of max-resource states may encode key features of the resource theory, such as the "size" of the set of free states. For example, compare the qubit coherence and magic state theories: in the Bloch representation, the incoherent states only from a zero-measure axis, while the stabilizer (non-magic) states form an octahedron which occupies a significant chunk of the Bloch sphere [43]; Loosely speaking, the maximum magic state is thus much "closer" to stabilizer states, which leads to a smaller modification coefficient, as compared with the case of coherence.

Now, we point out the remarkable fact that there is a family of pure max-resource states such that the different types of modification coefficients may collapse to the same value, in a generic class of theories. To this end, we introduce the following condition.

Condition (CH). The set of free states $\mathcal{F}$ is formed by a convex hull of pure free states.

This property is quite lenient and holds for many theories such as entanglement, coherence, superposition, magic states. We then obtain the following:

Theorem 1. *Suppose the resource theory satisfies Condition (CH). Then, for any d, there exists a pure state $\hat{\Phi}_d \in \mathcal{D}(\mathcal{H}_d)$ such that $m_f(\hat{\Phi}_d) = m_{\min}(\hat{\Phi}_d) = m_{\max}(\hat{\Phi}_d) := g_d$ where $\hat{\Phi}_d$ achieves the maxima of $m_f, m_{\min}, m_{\max}$. Furthermore, if $\tilde{\lambda}$ is an exact RD map, then $m_{f,\tilde{\lambda}}(\hat{\Phi}_d) = m_{\min,\tilde{\lambda}}(\hat{\Phi}_d) = m_{\max,\tilde{\lambda}}(\hat{\Phi}_d) = g_d$.*

See Appendices E and F [79] for proofs and discussions concerning this result. We call such $\hat{\Phi}_d$ a *golden state* and g_d the *golden coefficient* for dimension d.

Now we briefly discuss a few important examples of golden states and coefficients. The coherence theory comes with golden states $|\hat{\Phi}_d\rangle = \frac{1}{\sqrt{d}} \sum_j |j\rangle$, and the complete dephasing channel is an exact RD map; Theorem 1 fully applies, and $g_d = 1$ for all $d \in \mathbb{D} = \mathbb{Z}_+$. For entanglement theory, golden states can take the form $|\hat{\Phi}_d\rangle = \frac{1}{d^{1/4}} \sum_{j=1}^{\sqrt{d}} |j\rangle|j\rangle$ where d is the dimension of the bipartite system with local dimension $\sqrt{d}$, and $g_d = 1/2$ for all $d \in \mathbb{D} = \{k^2 | k \in \mathbb{Z}_+\}$. Note that the simple forms of one-shot entanglement/coherence manipulation results [44–47] rely heavily on this specific property of the golden coefficients being constant for any valid dimension. The theory of magic states has golden state $\hat{\Phi}_2 = \frac{1}{2}(I + (X+Y+Z)/\sqrt{3})$ with $g_2 = \log(3-\sqrt{3}) \approx 0.34$ [31] for a single-qubit system, where X, Y, Z are Pauli matrices. Another interesting case is the theory of quantum thermodynamics, where the only free state is the Gibbs state and Condition (CH) is not satisfied. But it can be shown that golden states with the same maximal resource and collapsing properties still exist and the g_d can be easily calculated (see Appendix E [79]). In particular, the infinite-temperature case (i.e. the purity theory) has $g_d = 1$ for all d (where every pure state is a golden state).

Optimal rates of one-shot resource manipulation.— Before stating the results, we define another useful condition defined for the set of free states $\mathcal{F}$ and pure reference states $\{\Phi_d\}$, which we call Condition (CT).

Condition (CT). For any given $d \in \mathbb{D}$, $\mathrm{Tr}\{\Phi_d \sigma\}$ is constant for any $\sigma \in \mathcal{F}$ (equivalently, Φ_d belongs to the dual set of $\mathcal{F}$ as introduced in Ref. [26]).

For instance, for the theory of coherence, it can be easily verified that $\mathrm{Tr}\{\hat{\Phi}_d \sigma\} = 1/d$ for any incoherent state σ, so that the Condition (CT) is satisfied when the golden states are chosen as reference states. In Appendix G [79], we provide a new example based on a multi-qubit superposition theory. A diagram that illustrates the classification of resource theories relevant to this work can be found in Appendix H [79].

For state ρ, given reference states $\{\phi_d\}$, the optimal rate of one-shot formation task with ϵ error tolerance i.e. the *one-shot ϵ-formation cost*, under the set of operations $\mathscr{F}$, is defined to be the minimum size of reference state that achieves the task:

$$\Omega^\epsilon_{C,\mathscr{F}}(\rho \leftarrow \{\phi_d\}) := \log\min\{d \in \mathbb{D} : \exists \mathcal{E} \in \mathscr{F}, \mathcal{E}(\phi_d) \in \mathcal{B}^\epsilon(\rho)\}.$$

Below let $\mathfrak{R}$ be some resource measure and m be some type of modification coefficient that will be specified. The following theorem establishes general bounds for the one-shot ϵ-formation cost under the two aforementioned classes of free operations (proofs in Appendices I, J [79]):

Theorem 2. *For reference states $\{\phi_d\}$, let $d_0 = \min\{d \in \mathbb{D} : \mathfrak{R}(\phi_d) \geq \mathfrak{R}^\epsilon(\rho)\}$. Then*

$$\Omega^\epsilon_{C,\mathscr{F}}(\rho \leftarrow \{\phi_d\}) \geq \frac{\mathfrak{R}^\epsilon(\rho)}{m(\phi_{d_0})}. \quad (4)$$

for i) $\mathscr{F} = \mathscr{F}_{\rm NG}$, $\mathfrak{R} = \mathfrak{D}_{\max}$, $m = m_{\max}$ for any $\mathcal{F}$; ii) $\mathscr{F} = \mathscr{F}_{\rm NG}$, $\mathfrak{R} = LR$, $m = m_{LR}$ for $\mathcal{F}$ satisfying Condition (FFR); iii) $\mathscr{F} = \mathscr{F}_{\lambda,\rm Comm}$, $\mathfrak{R} = \mathfrak{D}_{\max,\lambda}$, $m = m_{\max,\lambda}$ for any $\mathcal{F}$ and λ.

On the other hand, for pure reference states $\{\Phi_d\}$, let $d'_0 = \min\{d \in \mathbb{D} : -\log\mathfrak{f}(\Phi_d) \geq \mathfrak{R}^\epsilon(\rho)\}$. Then

$$\Omega^\epsilon_{C,\mathscr{F}}(\rho \leftarrow \{\Phi_d\}) < \frac{\mathfrak{R}^\epsilon(\rho)}{m_f(\Phi_{d_0'^\downarrow})} + \log\frac{d'_0}{d_0'^\downarrow}. \quad (5)$$

for i) $\mathscr{F} = \mathscr{F}_{\rm NG}$, $\mathfrak{R} = \mathfrak{D}_{\max}$ for $\mathcal{F}$ satisfying Condition (CT); ii) $\mathscr{F} = \mathscr{F}_{\rm NG}$, $\mathfrak{R} = LR$ for convex $\mathcal{F}$ satisfying Condition (FFR); iii) $\mathscr{F} = \mathscr{F}_{\lambda,\rm Comm}$, $\mathfrak{R} = \mathfrak{D}_{\max,\lambda}$ for $\mathcal{F}$ satisfying Condition (CT) and any λ.

By combining the above results with Theorem 1, we can reduce the modification coefficients to golden ones and obtain roughly matching bounds:

Corollary 3. *For golden states $\{\hat{\Phi}_d\}$, suppose Conditions (CH) and (CT) are satisfied, and let $d_0 = \min\{d \in \mathbb{D} : g_d \log d \geq \mathfrak{R}^\epsilon(\rho)\}$. Then*

$$\frac{\mathfrak{R}^\epsilon(\rho)}{g_{d_0}} \leq \Omega^\epsilon_{C,\mathscr{F}}(\rho \leftarrow \{\hat{\Phi}_d\}) < \frac{\mathfrak{R}^\epsilon(\rho)}{g_{d_0^\downarrow}} + \log\frac{d_0}{d_0^\downarrow}. \quad (6)$$

for i) $\mathscr{F} = \mathscr{F}_{\rm NG}$, $\mathfrak{R} = \mathfrak{D}_{\max}$; ii) $\mathscr{F} = \mathscr{F}_{\tilde{\lambda},\rm Comm}$, $\mathfrak{R} = \mathfrak{D}_{\max,\tilde{\lambda}}$ for exact RD map $\tilde{\lambda}$.

The constructions used for showing the achievable formation costs provide interesting implications to the existence of *root states*, max-resource states in the strongest sense, from which any state defined on the same Hilbert space can be obtained by some free operation:

Corollary 4. *For any $\mathcal{F}(\mathcal{H}_d)$ such that the maxima of $m \in \{m_f, m_{\min}, m_{\max}\}$ coincide at some pure (golden) state $\hat{\Phi}_d$ (e.g. $\mathcal{F}(\mathcal{H}_d)$ satisfying Condition (CH)), $\hat{\Phi}_d$ serves as a root state if $\mathcal{F}(\mathcal{H}_d)$ further satisfies either of the following: i) Condition (CT), ii) Condition (FFR) and $m_{\max}(\Phi_d) = m_{LR}(\Phi_d)$ for any pure state $\Phi_d \in \mathcal{D}(\mathcal{H}_d)$.*

We provide the proof, as well as an extensive discussion on root states, in Appendix K [79]. This in particular implies that if there exist no root states, then the free and generalized robustness measures do not coincide at pure states in general. (see Ref. [64] for a related discussion for the theory of multipartite entanglement).

As for distillation, we consider the standard version with error tolerance on the output. The optimal rate, namely the *one-shot ϵ-distillation yield*, under free operations $\mathscr{F}$, is defined to be the maximum size of the target reference state:

$$\Omega^\epsilon_{D,\mathscr{F}}(\rho \rightarrow \{\phi_d\}) := \log\max\{d \in \mathbb{D} : \exists \mathcal{E} \in \mathscr{F}, \mathcal{E}(\rho) \in \mathcal{B}^\epsilon(\phi_d)\}.$$

We first provide the following bounds for the one-shot ϵ-distillation yield under resource non-generating operations (proofs and additional results in Appendix L [79]):

Theorem 5. *For pure reference states $\{\Phi_d\}$, let $d_0 = \max\{d \in \mathbb{D} : -\log\mathfrak{f}(\Phi_d) \leq \mathfrak{D}^\epsilon_H(\rho)\}$. Then for any $\mathcal{F}$,*

$$\Omega^\epsilon_{D,\mathscr{F}_{\rm NG}}(\rho \rightarrow \{\Phi_d\}) \leq \frac{\mathfrak{D}^\epsilon_H(\rho)}{m_f(\Phi_{d_0})}. \quad (7)$$

Suppose further that $\mathcal{F}$ satisfies Condition (FFR). For reference states $\{\phi_d\}$, let $d_0 = \max\{d \in \mathbb{D} : LR(\phi_d) \leq \mathfrak{D}^\epsilon_H(\rho)\}$. Then

$$\Omega^\epsilon_{D,\mathscr{F}_{\rm NG}}(\rho \rightarrow \{\phi_d\}) > \frac{\mathfrak{D}^\epsilon_H(\rho)}{m_{LR}(\phi_{d_0^\uparrow})} - \log\frac{d_0^\uparrow}{d_0}. \quad (8)$$

For commuting operations, we find the following upper bound (proof in Appendix M [79]):

Theorem 6. *For pure reference states $\{\Phi_d\}$ and RD channel Λ, let $d_0 = \max\{d \in \mathbb{D} : \mathfrak{f}_\Lambda(\Phi_d) \geq 2^{-\mathfrak{D}^\epsilon_{H,\Lambda}(\rho)} - 2\sqrt{\epsilon}\}$. Then for any $\mathcal{F}$,*

$$\Omega^\epsilon_{D,\mathscr{F}_{\Lambda,\rm Comm}}(\rho \rightarrow \{\Phi_d\}) \leq \frac{-\log(2^{-\mathfrak{D}^\epsilon_{H,\Lambda}(\rho)} - 2\sqrt{\epsilon})}{m_{f,\Lambda}(\Phi_{d_0})}. \quad (9)$$

For now we are only able to obtain lower bounds for some special notions of commuting operations (see Appendix M [79]).

Moreover, in Appendix N [79], we instead consider distillation with error tolerance on the input, for which a greater collection of bounds in slightly different forms (using e.g. state-smoothing $\mathfrak{D}^\epsilon_{\min}$ or continuity bounds) can be established.

These results allow us to obtain nontrivial bounds for resource trading in specific theories by computing the modification coefficients (which can be efficiently done in many cases [9, 65–70]). For example, the golden coefficients of coherence, entanglement and purity theories induce bounds directly given by the smooth resource measures without modification, which is consistent with previous results [44–46, 71]. As a more informative example, we briefly remark on the theory of magic states. It can be inferred from recent results in [72] that $m_f(\Phi) = m_{\min}(\Phi) = m_{\max}(\Phi)$ holds for the so-called "Clifford magic states" Φ, and $m(\Phi_2^{\otimes m}) = m(\Phi_2)$ where $m \in \{m_f, m_{\min}, m_{\max}\}$ for any qubit pure state Φ_2 [73] (meanwhile, m_{LR} is generically larger and non-constant). This in particular is relevant to the conventional magic state distillation where the reference states are copies of $|T\rangle := (|0\rangle + e^{i\pi/4}|1\rangle)/\sqrt{2}$. By using $m(T^{\otimes m}) = \log(4 - 2\sqrt{2}) \approx 0.23$ as can be easily verified and the known values of $m_{LR}(T^{\otimes m})$ [9, 74], one can obtain several bounds for manipulating multiple T-states/gates under all stabilizer-preserving operations, which complements the recent results in Ref. [52] for a slightly different setup. We leave extended discussions on the implications to magic states and quantum computation for follow-up works.

We also note that the resource measures considered in this work often admit efficient SDP formulation [9, 69] as well as analytical expressions [9, 65–68, 70], which make our bounds of practical use in many important circumstances.

Concluding remarks.—This work establishes general bounds that relate the optimal rates of typical one-shot resource formation and distillation tasks to resource monotones based on one-shot divergences and log-robustness, without specifying the resource theory. We introduce the modification coefficients to take into account the resourcefulness of the currency, and find that they exhibit the remarkable collapsing property for a simple notion of max-resource states. We examined two important classes of free operations, namely the resource non-generating operations and operations that commute with the RD map.

Our results not only provide nontrivial and practically useful bounds for these tasks, but also characterize the resourcefulness of quantum states defined in general resource theories in terms of direct one-shot resource conversion, providing general operational meanings to the resource measures discussed in this work. They are potentially applicable to a large class of theories beyond the specific ones studied earlier (e.g. entanglement, coherence, thermal non-equilibrium), allowing one to obtain nontrivial bounds for optimal resource manipulation in specific contexts. Our results also complement the studies on the complete set of monotones [26, 35, 75–77], which provide the necessary and sufficient conditions for state transformations between two states under free operations. A complete set of monotones generally consists of infinite number of resource monotones [23], which makes the computation impractical. Therefore, the simpler expressions obtained in this work would give clearer insights into resource manipulation tasks.

For future work, it would be intriguing to further investigate the achievability of these fundamental limits (especially for distillation), apply this framework to specific contexts such as magic states and superposition to gain new insights into these theories, explore the connections and implications to the asymptotic theory, and extend the ideas to resource theory settings beyond quantum states, in accordance with [22–24, 28, 33–38, 78].

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