Dirac Fermion Hierarchy of Composite Fermi Liquids

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Composite Fermi liquids (CFLs) are compressible states that can occur for 2D interacting fermions confined in the lowest Landau level at certain Landau level fillings. They have been understood as Fermi seas formed by composite fermions which are bound states of electromagnetic fluxes and electrons due to Halperin, Lee and Read [Phys. Rev. B 47, 7312 (1993)]. At half filling, an explicitly particle hole symmetric theory based on Dirac fermions [Phys. Rev. X 5, 031027 (2015)] was proposed by Son as an alternative low energy description. In this work, we investigated the Berry curvature of CFL model wave functions at filling fraction one quarter, and observed that it is uniformly distributed over Fermi sea except at the center where an additional π phase was found. Motivated by this, we propose an effective theory which generalizes Son's half filling theory, by internal gauge flux attachment, to *all* filling fractions that fermionic CFLs can occur. The numerical results support the idea of internal gauge flux attachment.

Composite Fermi liquids (CFLs) are gapless states that can occur at certain Landau level fillings ν. They were first explained by Halperin, Lee, and Read (HLR) [1] as Fermi seas (FSs) of electromagnetic flux attached composite fermions. While its success in explaining CFL's metallic feature, it is not obvious how HLR theory is consistent with the particle hole (PH) symmetry [2–4], which is an exact symmetry in a half filled Landau level if Landau level mixing is negligible. Recently an emergent Dirac fermion (DF) theory was proposed by Son [5] at half filling. With ψ as DF field, γ^μ as Gamma matrix, a_μ and A_μ as internal and external gauge fields respectively, Son's DF action is:

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\left(\partial_\mu - ia_\mu\right)\psi - \frac{1}{2}\frac{1}{2\pi}adA + \frac{1}{2}\frac{1}{4\pi}AdA. \quad (1)$$

where $e, \hbar$ are set to be 1, so magnetic length is $l_B^{-2}{=}B_A$, with $B_A{=}\epsilon^{ab}\partial_a A_b$ as the external magnetic field strength and $\epsilon^{xy}{=}{-}\epsilon^{yx}{=}1$ as the anti-symmetric symbol. Greek and Latin letters label space-time and spatial coordinates respectively. Higher order terms in Eq. (1) are omitted for simplicity. Son's DF theory is explicitly PH symmetric because PH acts in a way akin to time reversal on DFs. It also suggests intriguing dualities between Dirac and non-relativistic fermions in two dimensions [5–13].

Son's half filled DF theory predicts a π Berry phase, which in fact is a π Berry curvature singularity located at FS center [as DF is a two-component spinor], acquired by the composite fermion [DF] when transported around the Fermi surface. This π Berry phase and Berry curvature have been observed from numerics [14–17]. The FS Berry phase Φ_{FS} [18] has been argued to be closely tied to electron Hall conductivity σ_H [σ_H^{CF}: composite fermion Hall conductivity]: as pointed out by F. D. M. Haldane in the theory of anomalous Hall effect [19, 20], the non-quantized part of Hall conductivity is determined by the FS Berry phase, see Eq. (2). Variants of HLR theory with an emphasis on FS Berry phase have been studied in [11, 21, 22].

$$\sigma_H^{CF} = -\sigma_H = \frac{e^2}{h}\frac{\Phi_{FS}}{2\pi}, \quad \Phi_{FS} = -2\pi\nu. \quad (2)$$

In principle, CFLs can occur as long as the HLR flux attached particles are fermions; whether or not they occur depends on the details of interaction. When the underlying physical particles are fermions, the filling fractions of CFLs can be grouped into two classes: $\nu{=}1/2m$ if FSs are formed by composite fermions [we denote them as fCFLs], and $\nu{=}1{-}1/2m$ if by composite holes [anti-fCFLs]. In this work, we studied the Berry curvature of $\nu{=}1/4$ model wave function (MWF) as a case study, and proposed an effective theory for fCFLs and anti-fCFLs at *all* filling fractions that they can occur. This theory can be viewed as generalizing Son's DF theory by attaching each DF with $\pm|2m-2|$ internal gauge flux quanta. PH conjugate states are realized by attaching same amount but an opposite direction of fluxes. As a result at long wavelength, CFLs for fermions can be considered as descending from the $\nu{=}1/2$ PH symmetric states, in analogy to Jain's hierarchy [23] which interpreted incompressible quantum Hall fluids as descending from integer quantum Hall effects.

Based on the numerical study on $\nu{=}1/4$ MWFs, we found:

- FS contains $-2\pi\nu$ Berry phase in agreement with Eq. (2).
- The $-2\pi\nu$ FS Berry phase consists of a $-\pi$ peak located at FS center, and $-2\pi\left(\nu - \frac{1}{2}\right)$ phase uniformly distributed over FS.

Motivated by this, we conjectured an effective field theory, dubbed *flux-attached DF theory*, as follows [24]:

$$\begin{aligned}\mathcal{L} = {} & i\bar{\psi}\gamma^\mu\left(\partial_\mu - ia_\mu\right)\psi - \frac{1}{2m}\frac{1}{2\pi}adA \\ & -\eta\left(\frac{1}{2} - \frac{1}{2m}\right)\frac{1}{4\pi}ada + \left(\frac{1}{2} - \frac{\eta}{2}\frac{m-1}{m}\right)\frac{1}{4\pi}AdA.\end{aligned} \quad (3)$$

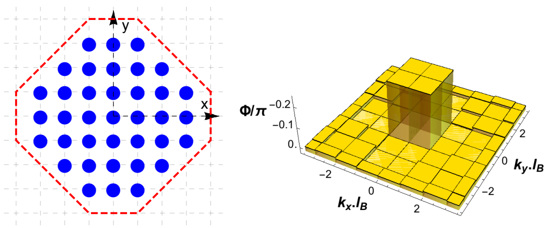

FIG. 1. Berry curvature distribution [right] obtained from $\Psi^{1/4}_{n=1}$ model wave function by a linear regression on a FS [left] consisting of N=37 dipoles. The red dashed line represents the path, which can be interpreted as FS boundary, along which *anti-clock-wisely* transporting a single composite fermion has $-2\pi\nu$ Berry phase. The area enclosed by the red dashed line contains 46 grids. The Berry curvature has a peak of around $-0.25+0.011{=}-0.239$ [in units of π], while the rest values are around $-(2/4{-}1)/46{=}0.011$. It suggests an interesting Berry curvature distribution for CFLs at a generic filling fraction in the thermodynamic limit: a $-\pi$ singularity at center and $-(2\pi\nu{-}\pi)$ uniformly distributed over FS.

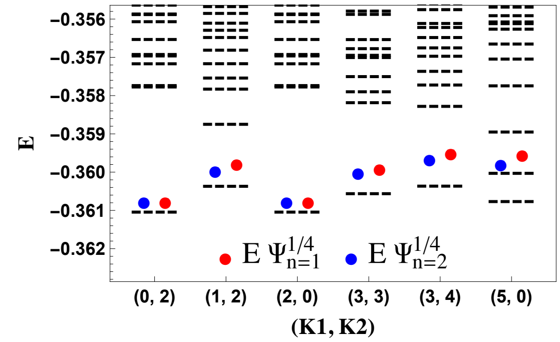

FIG. 2. Variational energies [red dots for $\Psi^{1/4}_{n=1}$, blue dots for $\Psi^{1/4}_{n=2}$] and exactly diagonalized Coulomb energies [dashed lines] as a function of many body momentum [35] (K_1, K_2) for N=10 electrons for the ν=1/4 filled LLL on a square torus. Energies are plotted in units of $e^2/\epsilon l_B$. For each K_1, K_2 is chosen to match the momentum of the lowest energy state. Due to inversion symmetry, only $K_1 \in [0, 5]$ are plotted.

Extra prescription: an extra $-(2\pi\nu-\pi)$ Berry phase uniformly distributed over the FS.

Eq. (3) represents a Fermi liquid theory at generic filling fractions: ν=1/2m [η=+1, fCFL] and its PH conjugation ν=1−1/2m [η=−1, anti-fCFL] where m is a positive integer. We conjecture that DFs are massless particles even at $m{\neq}1$ as constrained by FS Berry phase. Eq. (3) is determined by Luttinger theorem, Hall conductivity and FS Berry phase. We will argue that Berry curvature obtained from HLR motivated wave functions agrees with the prediction out of flux-attached DF theory. Further testings of Eq. (3) can be obtained from studying response functions, which we decide to show somewhere else.

Model Wave Function.— In the following, we examine CFL MWFs at ν=1/4 as a case study. The MWFs were proposed based on the ideas of HLR's flux attachment [25, 26]. Key ingredients of the MWFs at ν=1/2m include flux attachment represented by the Jastrow factor and the lowest Landau level (LLL) projection P_{LLL} operator,

$$\Psi\left(\{\boldsymbol{k}\},\{z\}\right) = P_{LLL}\{\det_{ia} e^{i\boldsymbol{k}_a\cdot\boldsymbol{r}_i}\prod_{i<j}^{N}(z_i-z_j)^{2m}\}. \quad (4)$$

where $\{\boldsymbol{k}\}$ are distinct and clustered to form a *compact FS*. Holomorphic determinant MWFs are obtained after approximating [27–29] P_{LLL} by creating dipoles $\{d_i\}$, in accordance with the dipole-momentum locking which is a fundamental property of composite fermions in a LLL. With $\sigma(z)$ as the modified Weierstrass sigma function [30, 31], $\{\alpha_k\}$ as the center of mass zeros which set the topological sector, MWF at ν=1/2m reads [32, 33],

$$\Psi_n^{\frac{1}{2m}}(\{\boldsymbol{d}\},\{\alpha\},\{z\}) = \det_{ia} M_{ia}\prod_{i<j}^{N}\sigma^{2(m-n)}(z_i-z_j) \quad (5)$$
$$\times\prod_{k}^{2m}\sigma\left(\sum_{i}^{N}z_i-\alpha_k\right)\prod_{i}^{N}e^{-\frac{1}{2}z_iz_i^*}.$$

where $M_{ia}{=}e^{\frac{n}{2m}z_id_a^*}\prod_{k\neq i}^{N}\sigma^n\left(z_i-z_k-d_a+\bar{d}\right)$. The $\bar{d}$ is a free parameter, *i.e.* changing $\bar{d}$ only renormalizes the MWF [15]. n represents a scheme of flux attachment: $2n$ out of the total $2m$ flux quanta are shifted from electron's position to form a dipole. Such as momentum quantization, dipoles $\{d_i\}$ are quantized by periodic boundary condition [34, 35] to take discrete values $d \in \{\mathbb{L}/(nN)\}$ where $\mathbb{L}$ is the 2D periodic lattice defining the torus.

We adopt the lattice Monte Carlo method [16] to study the Berry phase. We consider ν=1/4 MWFs with $m{\geq}n$ [see Appendix]: $\Psi^{1/4}_{n=1}$ and $\Psi^{1/4}_{n=2}$. They are found to have large overlaps with each other for all dipole configurations, *e.g.* $|\langle\Psi^{1/4}_{n=1}|\Psi^{1/4}_{n=2}\rangle| \geq 97\%$ for N=69 dipoles. This means that observables computed from either of them are almost identical.

In a half filled LLL, Coulomb interaction low energy states were found to have a remarkably large overlap [15] with the cluster-like ansatz of Eq. (5). At one quarter, second quantizing a MWF to compute overlap becomes difficult for large system sizes. Instead, as shown in FIG. 2, we present the energy spectrum of LLL Coulomb interaction and the variational energy of MWF for N=10 electrons on a square torus. The variational energies of MWFs and exact diagonalization energies are close, but slightly worse compared to one half states. As pointed out in [26], at half filling in the lowest two Landau levels, varying short range interactions induce a first-order phase transition from striped phase to a strongly paired

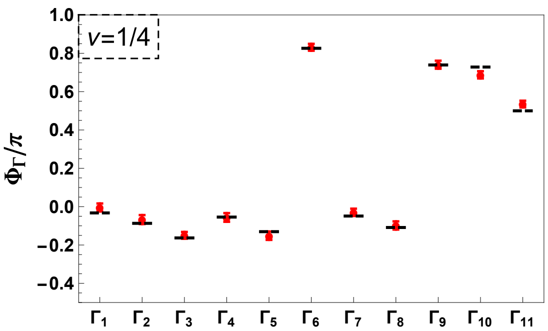

FIG. 3. Comparison of the Berry phases Φ_Γ associated with various clock-wised paths $\{\Gamma\}$=$\Gamma_1,...,\Gamma_6$ on a FS of N=37 dipoles [see FIG. 1 for FS] computed from $\Psi^{1/4}_{n=1}$ [red dots] and from the formula Φ_Γ=$\delta_\Gamma\cdot\pi$+$(2\pi\nu-\pi)A_\Gamma/A_{FS}$ [black lines], where δ_Γ is the winding number of Γ relative to the FS center, A_Γ and A_{FS} are the $\boldsymbol{k}$-space areas enclosed by the path Γ and FS area respectively. See appendix for details about paths and more examples including N=69 FS and ν=1/3.

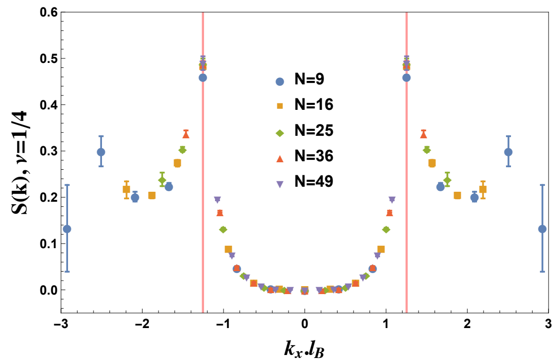

FIG. 4. Guiding center structure factor $S(k)$ as a function of k_x along k_y=0 axes computed from $\Psi^{1/4}_{n=1}$. The plot is obtained after a finite size scaling for MWFs of $\sqrt{N}\times\sqrt{N}$ square FSs where N is the number of electrons. The red lines are $2k_F.l_B$=$\sqrt{2\pi\nu}$, a value of twice of the Fermi wave vector obtained by applying Luttinger theorem on a square FS. The fact that $S(k)$ plots fit into one curve and the numerical singularities match with the analytical value implies that Luttinger theorem is true for CFLs.

Moore-Read state, followed by a possible crossover to a weak paring phase. The exact diagonalization states we obtained at Coulomb point at one quarter, presumably, are weakly paired states; tuning $v_{1,3}$ pseudo-potentials might help improve the overlaps. See Appendix for comments for the $(5,0)$ sector.

Berry Curvature.— We next turn to the numerical investigation of the Berry curvature shown in FIG. 1. Since only MWFs with compact dipole configurations are identified as CFLs [15–17], we neither take off composite fermions deep inside the FS nor excite them too far away from the Fermi surface. Instead, Berry phases were computed on clock-wised paths close to the Fermi surface, after which the Berry curvatures were mapped out by a linear regression [36]. The Berry curvature distribution was found as described in FIG. 1. See FIG. 3 for a consistency check which indicates FIG. 1 makes sense.

Same results were found even for bosonic CFL ν=1/3 states [37, 38]. From wave function point of view, the Berry curvature feature can be argued as follows [39]: the determinant (of two fluxes) and the Jastrow factor are implementing respectively the π and the uniform part of the Berry phase. We believe that the Berry curvature feature we observed on ν=1/4 and ν=1/3 model states applies to other filling fractions as well since MWFs at different Landau level fillings essentially differ only by a different power of Jastrow factors. See Appendix for more discussions.

Effective Action.— The presence of π Berry curvature singularity at $\nu\neq 1/2$ strongly suggests the emergence of DFs at low energy at generic filling fractions. In this section, we justify our proposed flux-attached DF theory Eq. (3) by starting from a Dirac type effective action with undetermined coefficients, Eq. (6). We will then fix $C_{1,2,3}$ by physical requirements: Luttinger theorem, FS Berry phase and Hall conductivity, and argue the Berry curvature distribution is consistent with the prediction from flux-attached DF picture.

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\left(\partial_\mu - ia_\mu\right)\psi - \frac{C_1}{2\pi}adA - \frac{C_2}{4\pi}ada + \frac{C_3}{4\pi}AdA. \quad (6)$$

Some knowledge about the FS can be obtained before an interpretation [whether non-relativistic HLR fermions or relativistic DFs] of the particles that compose the FS is assigned. The first requirement for being a Fermi liquid is no net magnetic fields on the FS: $\langle B_\psi\rangle\equiv\langle\epsilon^{ab}\partial_a a_b\rangle$=0 where $\langle...\rangle$ denotes mean field expectation value. Second, in Landau Fermi liquid theory, the FS volume is determined by the charge density, known as Luttinger theorem. It has been conjectured [5, 14, 40] that CFLs satisfy Luttinger theorem too, *i.e.* composite Fermi wave vector is determined by electrons' filling factor. In FIG. 4, we investigated Luttinger theorem for CFLs by computing the guiding center structure factor $S(k)$ of a ν=1/4 MWF. $S(k)$ is defined as $\frac{1}{2N_\phi}\langle\{\delta\rho(k),\delta\rho(-k)\}\rangle$ where $\delta\rho(k)\equiv\rho(k)-\langle\rho(k)\rangle$ represents the fluctuation of density relative to the ground state mean value and obeys the Girvin-MacDonald-Plazman algebra [41]. As a hallmark of CFL, there are peaks in $S(k)$ and the peak positions are tied to k=$2k_F$, twice the Fermi wave vector. The measured k_F agrees with the value predicted from Luttinger theorem, suggesting that Luttinger theorem applies to CFLs [5, 14, 40]. We will then assume Luttinger theorem for CFLs and use it as a constraint to derive the effective action.

The PH conjugate of a CFL is supposed to have same FS size [40, 42–44]. In HLR picture, this is because FSs of fCFLs and anti-fCFLs are formed by composite-fermions composite-holes respectively whose Fermi level are the same. DF theory interprets the FS as formed by DFs, which fill DF bands up to the Fermi level determined by electrons' filling factor through Luttinger theorem. As a result, DF density must be,

$$\rho_\psi = \frac{1}{2m}\frac{1}{2\pi l_B^2}, \quad \nu = \begin{cases} 1/2m \text{ or,} \\ 1-1/2m. \end{cases} \tag{7}$$

Taking variation of a_0 and A_0 for Eq. (6), the DF density and electron density at mean field level are found to be: $\langle\rho_\psi\rangle = C_1\frac{B_A}{2\pi}$ and $\langle\rho_A\rangle = C_3\frac{B_A}{2\pi}$. Hence, Luttinger theorem and Hall conductivity determine C_1 to be 1/2m and C_3 to be $\frac{1}{2} - \frac{\eta}{2}\frac{m-1}{m}$ at mean field where $\eta=+1$ for fCFLs and $=-1$ for anti-fCFLs. Based on the observation of a π peak concentrated at FS center, we conjecture DF is massless. Unlike Son's theory, the presence of Chern-Simons (CS) term $\frac{C_2}{4\pi}ada$ has a non-universal contribution to σ_H [45], and induces nonzero $U(1)_A$ charge to DFs [21, 38]. The effects of CS term are canceled provided FS carries $2\pi C_2$ Berry phase; in other words, CS term assigns the FS with extra $2\pi C_2$ Berry phase [46]. This $2\pi C_2$ phase, together with the $-\pi$ Berry curvature singularity located at FS center, comprises the total $-2\pi\nu$ FS berry phase as observed. Based on the fact that Berry phase is odd under PH transformation, we set $C_2 = \eta\left(\frac{1}{2} - \frac{1}{2m}\right)$. Thus we determined Eq. (3) by Luttinger theorem, Hall conductivity and FS Berry phase. See Appendix for more discussions.

We will then argue for the Berry curvature distribution presented in FIG. 1 based on a flux-attached DF picture. The CS term induces $U(1)_A$ charge to DFs, making the flux-attached DF theory not in the LLL [21, 38]; same issue is present in HLR theory at all fillings [21]. In spite of not being a LLL theory, the effect of LLL projection is well known: as a fundamental property of a dipolar electron in magnetic field, the dipole vector $\boldsymbol{d}$ is perpendicular and strength proportional to the kinetic momentum vector $\boldsymbol{k}$, i.e. the so-called *dipole-momentum locking*; LLL projection, generally speaking, shifts the flux attachment center away from the electron's location to create a dipole. In Son's DF theory, 2 fluxes turn an electron into DF, and DF's spin represents dipole. At $\nu=1/2m$, we expect DFs are attached with $(2m-2)$ residual flux quanta, which after LLL projection are shifted away from DF's location to form a dipole. To distinguish the $(2m-2)$-flux dipole from the total $2m$-flux dipole [which include spin], we dub the former as *residual dipoles*. Hence in flux-attached DF theory, DF has *residual dipole-momentum locking* in addition to being a spin half spinor.

The residual dipole-momentum locking of DF has a nontrivial impact on the Berry phase associated with transporting a composite fermion [dipolar DF] in the momentum space [$\boldsymbol{k}$-space]. The Berry curvature distribution is predicted to be: (I) $\boldsymbol{k}$-space uniform except at the FS center point $\boldsymbol{k}=0$, (II) where there is an additional π Berry phase. The argument goes as follows [47].

The dipole-momentum locking provides a nature mapping from the real space to the $\boldsymbol{k}$-space. The motion of a dipolar DF in $\boldsymbol{k}$-space induces the rotation of the residual dipoles in real space [48]. Since the real-space density is uniform, and since $\boldsymbol{k}$-space area is proportional to real space area, (I) is a manifestation of the real space Aharonov-Bohm effect. The contribution to the FS Berry phase Φ_{FS} from (I) should be $-(2\pi\nu - \pi)$ in accordance with the fact that it vanishes at half filling. Then, (II) origins from being massless spin-half DFs. Finite mass tilts the DF's spin away from the 2D plane hence mass term $|\mathcal{M}|$ represents how much the FS Berry phase Φ_{FS} deviates from $-2\pi\nu$. We thus conjecture that DF mass $|\mathcal{M}|=0$ which we emphasis is not protected by symmetry but instead constrained to take this value by the FS Berry phase Eq. (2).

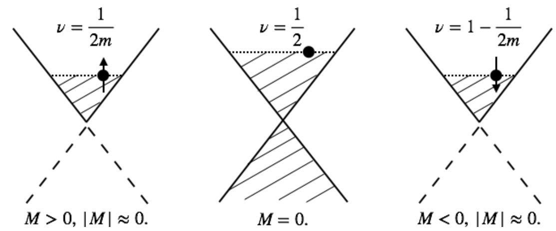

FIG. 5. Illustrations of FS, particles, band structure in Son's half filled DF theory [middle] and theory $\mathcal{L}_{\psi_+}$ in Eq. (8). PH acts like time reversal, thus flipping the fluxes [arrows] attached to the DFs [black dots]. FS sizes of PH conjugate states are the same, fixed by Luttinger theorem. The valance band, represented as dashed straight line, has been integrated out. The band mass is conjectured to be negligible.

In the end, we want to make a connection to Son's half filling theory. Due to the lack of PH symmetry when $\nu \neq 1/2$, Son's theory Eq. (1) acquires a mass $\mathcal{M}$. To describe low energy excitations around the Fermi surface, the lower band needs to be integrated out, inducing a sgn$\frac{(\mathcal{M})}{2}$ level CS term to the action: $\mathcal{L}_{\psi_+} = i\bar{\psi}'_+\gamma^\mu\left(\partial_\mu - ia'_\mu\right)\psi'_+ - \frac{1}{2}\frac{1}{2\pi}a'dA + \frac{1}{2}\frac{1}{4\pi}AdA + \frac{\text{sgn}\,(\mathcal{M})}{2}\frac{1}{4\pi}a'da'$, where ψ_+ describes the upper band fermion only. Fixing the electron density to be ν, the field strength of a'_μ is found to be $\langle B_{a'}\rangle \equiv \langle\epsilon^{ab}\partial_a a'_b\rangle = (1-2\nu)\frac{B_A}{2\pi}$. The upper band density $\langle\rho_{\psi'_+}\rangle = \frac{1}{2}\frac{B_A}{2\pi} - \frac{\text{sgn}\,(\mathcal{M})}{2}\frac{\langle B_{a'}\rangle}{2\pi}$ is then $\frac{1}{2m}\frac{B_A}{2\pi}$ for both fCFLs and anti-fCFLs, provided sign of mass is positive for fCFLs and negative for anti-fCFLs. The filling fraction for ψ'_+ is then: $\eta(2m-2)^{-1}$ which is again an even denominator fraction. Hence, upon a statistics preserving flux attachment transformation, composite fermions would perceive no magnetic fields on average. After carrying out the flux attachment singular gauge transformation, we arrive at [49],

$$\mathcal{L}_{\psi_+} = i\bar{\psi}_+\gamma^\mu\left(\partial_\mu - ia_\mu\right)\psi_+ + \frac{\text{sgn}(\mathcal{M})}{2m}\frac{1}{4\pi}ada \\ - \frac{1}{2m}\frac{1}{2\pi}adA + \left(\frac{1}{2} - \frac{\text{sgn}(\mathcal{M})}{2}\frac{m-1}{m}\right)\frac{1}{4\pi}AdA. \tag{8}$$

where ψ_+ is the upper band flux-attached Dirac fermion field which perceives no net magnetic field at mean field $\langle B_a\rangle\equiv\langle\epsilon^{ab}\partial_a a_b\rangle=0$. The FS of theory $\mathcal{L}_{\psi_+}$ can be viewed as formed by flux-attached DFs, whose density is determined through Luttinger theorem, and whose PH conjugate is attached with same amount but opposite fluxes, see FIG. 5. After adding back the valance band to cast the effective action into the standard notion and identifying η with $\mathrm{sgn}(\mathcal{M})$, we arrive at Eq. (3).

Note Added in Proof.— After the initial preprint [49] of this work was completed, Ref. (50) appeared which overlapped with this work and considered the same types of theories from a different but complementary perspective. Main reorganization and revision in this manuscript are as follows: ordering of numerical and theoretical parts was changed; connection to Son's theory was moved to the end; the paragraph following Eq. (7) was added.

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