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## Continuous real-time tracking of a quantum phase below the standard quantum limit

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We propose a scheme for continuously measuring the evolving quantum phase of a collective spin composed of N pseudospins. Quantum non-demolition measurements of a lossy cavity mode interacting with an atomic ensemble are used to directly probe the phase of the collective atomic spin without converting it into a population difference. Unlike traditional Ramsey measurement sequences, our scheme allows for real-time tracking of time-varying signals. As a bonus, spin-squeezed states develop naturally, providing real-time phase estimation significantly more precise than the standard quantum limit of  $\Delta \phi_{SOL} = 1/\sqrt{N}$  radians.

Quantum systems have become robust platforms for metrology and tests of fundamental physics. Many applications rely on the dynamics of pseudospin-1/2 systems with two long-lived quantum states,  $|\uparrow\rangle$  and  $|\downarrow\rangle$ . After preparing an equal superposition of these two states, a physical interaction is studied by investigating its effect on the relative phase  $\phi(t)$ , with the state of each spin evolving in time as  $|\psi(t)\rangle = (|\downarrow\rangle + e^{i\phi(t)}|\uparrow\rangle)/\sqrt{2}$ . We propose a novel scheme that enables continuous tracking of this relative phase. Our scheme continuously and directly measures the real-time phase  $\phi(t)$  unlike the widely used Ramsey sequence [1-12], which indirectly measures the net accumulated phase  $\phi(T)$  during an interrogation time T. The typically destructive readout in a Ramsey sequence requires multiple state resets, rotations and repetitions of the sequence to infer the phase at different times from a population difference. In contrast, a single run of our protocol yields a continuous time series of phase measurements. Therefore, our scheme enables real-time tracking of time-varying signals that are not reproducible.

As an added benefit, our scheme yields continuous phase estimates with precision well beyond the standard quantum limit (SQL) of  $\Delta\phi_{SQL} = 1/\sqrt{N}$  radians that limits readout precision with N unentangled spins. In comparison to several proposals and experiments [13–19] that have demonstrated squeezed states with precision beyond the SQL, our scheme enjoys the advantage that the squeezing is produced, the phase accumulated, and the readout performed, all in the same spin quadrature.

Recent experiments have demonstrated phase tracking of a spin using quantum non-demolition (QND) measurements via a Faraday rotation angle [20]. In contrast, our proposal is based on interfering Raman transitions in a cavity and enables an intuitive interpretation of phase tracking in terms of elementary atom-cavity interactions that nearly balance one another. Our scheme directly reveals a phasor precessing in the equatorial plane of a Bloch sphere, in the spirit of the "hand on a clock" analogy at the core of quantum metrology.

We represent the collective angular momentum of N atomic spins by a classical Bloch vector of length N/2



FIG. 1. Schematic and working principle. (a) Two lasers drive a collection of atoms to interact with a cavity mode. The relative phase  $\phi(t)$  can be continuously tracked by homodyne detection of the field leaking out. (b) Cavity-assisted Raman transitions: The red (blue) pathway leads to the emission of a cavity photon accompanied by a spin flip  $|\downarrow\rangle \rightarrow |\uparrow\rangle$  ( $|\uparrow\rangle \rightarrow |\downarrow\rangle$ ). (c) Hierarchy of frequencies. (d) Classical Bloch vector picture: The red and blue pathways set up balanced, opposing superradiance pathways that lead to a coherent cancellation of the intracavity field when the Bloch vector (green) is along the y-axis ( $\phi = 0$ ). When the Bloch vector has a small x-component ( $\phi \neq 0$ ), the intracavity field from the two pathways add constructively, giving rise to non-zero output field.

with components  $J_x$ ,  $J_y$ ,  $J_z$  (Fig. 1(d, left)). With all spins initially in the same equal superposition state, the Bloch vector lies in the equatorial plane along a direction that we define as the *y*-axis. As the phase evolves, the Bloch vector acquires a small *x*-component,  $J_x = \frac{N}{2} \sin \phi(t) \approx \frac{N}{2} \phi(t)$ , for small deflections, and we propose a straightforward extension to large deflections in the conclusion. We arrange atom-cavity interactions wherein a cavity field quadrature is sourced by  $J_x$ . Continuous homodyne detection of this quadrature amounts to real-time, continuous, QND measurement of  $\phi(t)$ .

We consider N atoms trapped at the antinodes of a cavity with resonance frequency  $\omega_c$  and decay rate  $\kappa$ , as shown in Fig. 1(a). The states  $|\downarrow\rangle$  and  $|\uparrow\rangle$  have an energy separation  $\hbar\omega_0 \gg \hbar\kappa$  and form a pseudospin-1/2 system described by the Pauli spin operators  $\hat{\sigma}_i$ , i = x, y, z, with raising (lowering) operators  $\hat{\sigma}_+$  ( $\hat{\sigma}_-$ ). The N atoms form a collective spin with total angular momentum components  $\hat{J}_x, \hat{J}_y, \hat{J}_z$ , with  $\hat{J}_i = \sum_{j=1}^N \hat{\sigma}_i^j/2$ . We assume the dipole-allowed transitions  $|\downarrow\rangle \leftrightarrow |e\rangle$  and  $|\uparrow\rangle \leftrightarrow |e\rangle$  with frequencies  $\omega_{\downarrow e}$  and  $\omega_{\uparrow e}$  to be respectively driven using lasers with frequencies  $\omega_1$  and  $\omega_2$  in a far-detuned regime with detuning  $\Delta \gg \omega_0, \kappa$ , allowing for the adiabatic elimination of  $|e\rangle$  [21]. The two drive lasers differ by a frequency  $2\omega_0$  (Fig. 1(c)) and do not by themselves drive  $|\downarrow\rangle \leftrightarrow |\uparrow\rangle$  Raman transitions; however, they are symmetrically detuned by  $\omega_0$  from  $\omega_c$  and participate in cavityassisted Raman transitions as illustrated in Fig. 1(b) [22]. When the Rabi frequencies of the two drive lasers are balanced, i.e.  $\Omega_1 = \Omega_2 = \Omega_0$ , the atom-cavity Hamiltonian, to leading order in  $1/\Delta$ , is simply the sum of a Jaynes-Cummings and an anti-Jaynes-Cummings interaction and is given by [22]

$$\hat{H}_{\text{QND}} = \frac{\hbar\Omega_{\text{QND}}}{2}\hat{X}\hat{J}_x.$$
 (1)

Here  $\hat{X} = (\hat{a} + \hat{a}^{\dagger})/\sqrt{2}$  is the amplitude quadrature, with  $\hat{a}, \hat{a}^{\dagger}$  the annihilation and creation operators for the cavity mode, and  $\hat{Y} = (\hat{a} - \hat{a}^{\dagger})/\sqrt{2}i$  is the conjugate phase quadrature such that  $[\hat{X}, \hat{Y}] = i$ . The atom-cavity interaction strength is  $\Omega_{\text{QND}} = \sqrt{2}\Omega_0 g_0/\Delta$  with  $g_0$  the single atom-cavity vacuum Rabi frequency. If the two drive lasers have initial phases  $\psi_1$  and  $\psi_2$ , the cavity quadrature  $(\hat{a}^{\dagger}e^{i(\psi_1+\psi_2)/2} + \text{H.c.})$  is coupled to the spin component  $(\hat{J}_+e^{i(\psi_1-\psi_2)/2} + \text{H.c.})$ , where  $\hat{J}_+ = \hat{J}_x + i\hat{J}_y$ . Here we assume  $\psi_1 = \psi_2 = 0$  without loss of generality.

Classically, the intracavity fields established by the two balanced drives exactly cancel when  $J_x = 0$  (Fig. 1(d)). However, even with  $\langle \hat{J}_x \rangle = 0$ ,  $\langle \hat{J}_x^2 \rangle \neq 0$ , i.e. quantum fluctuations source the Y quadrature of the cavity field. In the regime  $\kappa^2 \gg N\Omega_{\text{QND}}^2$ ,  $\hat{Y}$  is slaved to  $\hat{J}_x$  as

$$\hat{Y}(t) \approx -\frac{\Omega_{\text{QND}}}{\kappa} \hat{J}_x(t) + \hat{\mathcal{F}}(t),$$
 (2)

where the noise operator  $\hat{\mathcal{F}}(t)$  arises from coupling of the cavity mode to external modes through the lossy mirror (Fig. 1(a)) [23, 24]. The field leaking out is to be monitored via balanced homodyne detection using a local oscillator at frequency  $(\omega_1 + \omega_2)/2$  with phase tuned to detect the output field quadrature that is sourced by the intracavity Y quadrature. The photocurrent thus recorded is a measurement of the Y quadrature which, from Eq. (2), amounts to measuring  $J_x$ .

Measurement back-action in the  $J_z$  quadrature arises because of the indistinguishability of the two pathways that give rise to the intracavity field (Fig. 1(b)): The field leaking out is consistent with equal probability amplitudes for tipping the Bloch vector above or below the equator and therefore increases the spread in  $J_z$  without affecting its mean value.

The drive lasers also lead to undesirable, off-resonant free-space scattering processes with total rate  $\gamma_{\rm sc}$  that degrade atomic coherence. We consider three such singleatom decoherence mechanisms [22]: (a) dephasing with probability  $r_d$ : random rotation about the z-axis, (b) spontaneous Raman spin flips :  $|\downarrow\rangle \rightarrow |\uparrow\rangle (r_{\downarrow\uparrow})$  and  $|\uparrow\rangle \rightarrow |\downarrow\rangle (r_{\uparrow\downarrow})$ , and (c) atom loss  $(r_l)$ : the atom decays to a state  $|s\rangle$  outside the  $|\downarrow\rangle - |\uparrow\rangle$  manifold and no longer interacts with the cavity mode. The probabilities are related by  $r_d + r_{\downarrow\uparrow} + r_{\uparrow\downarrow} + r_l = 1$ .

Under continuous measurement, the dynamics of the density matrix  $\rho$  of the atom-cavity system is governed by the stochastic master equation [25–27]:

$$\dot{\rho} = -i/\hbar[\hat{H}_{\text{QND}},\rho] + \kappa \mathcal{D}[\hat{a}]\rho + \gamma_{\text{sc}} \sum_{j=1}^{N} \mathcal{L}_{1}^{j}\rho + \sqrt{\eta\kappa}\xi(t) \left(i\rho\hat{a}^{\dagger} - i\hat{a}\rho - \sqrt{2}\langle\hat{Y}\rangle\rho\right), \qquad (3)$$

with decoherence effects bundled in  $\mathcal{L}_1^j \rho$ , given by

$$\mathcal{L}_{1}^{j}\rho = r_{\downarrow\uparrow}\mathcal{D}[\hat{\sigma}_{+}^{j}]\rho + r_{\uparrow\downarrow}\mathcal{D}[\hat{\sigma}_{-}^{j}]\rho + \frac{r_{d}}{4}\mathcal{D}[\hat{\sigma}_{z}^{j}]\rho + \frac{r_{l}}{2}\left(\mathcal{D}\left[|s\rangle_{j}\left\langle\downarrow\right|_{j}\right]\rho + \mathcal{D}\left[|s\rangle_{j}\left\langle\uparrow\right|_{j}\right]\rho\right), \quad (4)$$

with  $\mathcal{D}[\hat{O}]\rho = \hat{O}\rho\hat{O}^{\dagger} - \hat{O}^{\dagger}\hat{O}\rho/2 - \rho\hat{O}^{\dagger}\hat{O}/2$ , the Lindblad dissipator. In Eq. (3),  $\eta$  is the detection efficiency, and  $\xi(t)$  is a white-noise process satisfying  $\overline{\xi(t)} = 0$  and  $\overline{\xi(t)}\xi(t') = \delta(t-t')$ . The measured photocurrent i(t) is

$$i(t) = \mathcal{G}e|\alpha_{\rm LO}| \left(\eta \sqrt{2\kappa} \langle \hat{Y} \rangle + \sqrt{\eta} \xi(t)\right), \tag{5}$$

with detector gain  $\mathcal{G}$ , electronic charge e, and local oscillator photon flux  $|\alpha_{\rm LO}|^2$  with units of photons/time.

With no decoherence, measuring for very long times will result in preparing states arbitrarily close to Dicke states in the  $J_x$  basis. However, decoherence restricts the maximum achievable squeezing well before the state begins to wrap around the Bloch sphere. This enables a Gaussian approximation where we only track the dynamics of the means and covariances of all operators and pairs of operators of the atom-cavity system. The 5 operators  $\hat{X}, \hat{Y}, \hat{J}_x, \hat{J}_y$  and  $\hat{J}_z$  result in a total of 20 dynamical equations [22].

We average the simulated photocurrent (Eq. (5)) in a window  $[T_i, T_f]$  to obtain an estimate as

$$J_x^{(\mathrm{m})} = -\frac{\kappa}{\Omega_{\mathrm{QND}}} Y^{(\mathrm{m})} = \frac{-(\mathcal{G}e|\alpha_{\mathrm{LO}}|)^{-1}}{\eta\sqrt{C\gamma_{\mathrm{sc}}}(T_{\mathrm{f}} - T_{\mathrm{i}})} \int_{T_{\mathrm{i}}}^{T_{\mathrm{f}}} i(t)dt,$$
(6)

where  $C = 2\Omega_{\text{QND}}^2/\kappa\gamma_{\text{sc}}$  is the dimensionless atomcavity cooperativity [22]. The phase is estimated as  $\phi^{(\text{m})} = (J_x^{(\text{m})}/(N/2))/\mathcal{V}(t)$ , where the visibility  $\mathcal{V}(t)$  [28] accounts for the shortening of the Bloch vector, evaluated either at the window center or end depending on where the phase is estimated. While we use a simple averaging procedure for clarity, optimal filters, such as Kalman filters, can be applied for superior phase tracking [29–31].

The precision of the phase estimate in a window is determined by the window duration. A characteristic time,  $T_0 = (\eta C \gamma_{\rm sc})^{-1}/(N/4)$ , is the time required to average down the photon shot-noise ( $\xi(t)$  term, Eq. (5)) in estimating  $J_x^{(m)}$  (Eq. (6)) to the standard quantum limit  $\Delta J_{x,\rm SQL}^2 = N/4$ .



FIG. 2. Real-time continuous tracking of a time-varying phase. (a) A single experimental run: A squeezed state is prepared during  $[-50T_0, 0]$ , with the initial measured phase  $\phi_0^{(m)}$ (blue triangle) varying in each run. Subsequently, a phase modulation  $\phi^{(a)}(t) = 15 \mod \times \sin(t/40T_0)$  (black line) is applied e.g. using a time-varying magnetic field. The blue, filled (red, hollow) markers are estimates  $\phi_{\rm SSS}^{(m)}$  ( $\phi_{\rm CSS}^{(m)}$ ) of the phase using the measured photocurrent in windows of duration  $8T_0$  that account for (do not account for) the initial offset  $\phi_0^{(m)}$ . The gray shaded region indicates the 1- $\sigma$  SQL tolerance for this applied signal. Representative Bloch spheres for  $t \leq 0$ indicate the state before and after the state preparation stage. For t > 0, Bloch spheres indicate the deflection of the spin as a result of the phase modulation (black dots on the spheres indicate the zero phase reference), as well as the equivalent spin state used for the respective estimates  $\phi_{\text{CSS}}^{(m)}$ ,  $\phi_{\text{SSS}}^{(m)}$ . (b) Histogram of phase errors  $\phi_{\rm SSS}^{(m)} - \phi^{(a)}$  (blue) and  $\phi_{\rm CSS}^{(m)} - \phi^{(a)}$ (red) over 2048 runs in one particular measurement window  $[48T_0, 56T_0]$ . (c) Single-run precision gain of the estimates  $\phi_{\rm SSS}^{\rm (m)}$  relative to the SQL at different window centers t. Here,  $\Delta \tilde{\phi}_{\rm SSS}^2$  is the variance of Gaussian fits to histograms such as the blue histogram in (b). Decoherence results in decreased gain over time.

For our numerical experiments, we use  $N = 10^5$  atoms

identically coupled to a cavity mode with C = 0.1. We work in a bad-cavity regime such that  $NC\gamma_{\rm sc} = 0.2\kappa$ , achievable by arranging for  $\Omega_{\rm QND} = 10^{-3}\kappa$ . We adopt a "symmetric loss" model wherein the three decoherence mechanisms degrade the atomic coherence at equal rates, and spin-flips in either direction occur with equal probability. This implies  $r_d = 1/3$ ,  $r_{\downarrow\uparrow} = r_{\uparrow\downarrow} = 1/6$  and  $r_l = 1/3$ . Our results are not very sensitive to the specific choice of relative rates. The loss in visibility only depends on the total decoherence rate, while the measurement of  $J_x$  marginally improves if the atom loss channel is dominant (see Eq. (8)). Finally, the detection efficiency is assumed to be  $\eta = 0.4$  [18].

We now demonstrate the ability of our scheme to track in real-time, a phase modulation  $\phi^{(a)}(t)$  applied for t > 0(Fig. 2(a, black solid line)). At time  $t = -50T_0$ , the collective spin is initialized to a coherent spin state (CSS) along the y-axis whose initial phase is  $\phi_0 = 0$ . First, measuring the photocurrent in the state preparation window  $[-50T_0, 0]$  gives a phase estimate  $\phi_0^{(m)}$  (blue triangle). This estimate is obtained at the end of this window using the procedure described below Eq. (6). The value of  $\phi_0^{(m)}$ varies from trial-to-trial with a variance  $\Delta \phi_{\rm SQL}^2 = 1/N$ corresponding to the phase uncertainty of the initial CSS. The long state preparation window ensures strong averaging down of the photon shot-noise, leading to a state with reduced phase uncertainty around  $\phi_0^{(m)}$ , i.e. a spin squeezed state. For the subsequent real-time tracking, two choices for the initial phase reference could be used:  $\phi_0(=0) \text{ or } \phi_0^{(m)}$ .

During the time  $[0, 200T_0]$ , we average the photocurrent in windows of duration  $8T_0$  to extract a raw phase estimate  $\phi^{(m)}(j)$  for window  $j = 1, 2, \dots$  We construct two estimates for the phase at the window centers,  $\phi_{\text{CSS}}^{(\text{m})}(j) = \phi^{(\text{m})}(j) - \phi_0$  (hollow red squares), and  $\phi_{\text{SSS}}^{(\text{m})}(j) = \phi^{(\text{m})}(j) - \phi_0^{(\text{m})}$  (filled blue squares). The precision of these estimates is determined not just by the window duration over which the raw estimate is obtained, but also by the precision of the phase reference. To determine the single-run precision of these estimates, we run 2048 trials of the experiment and histogram the error in these estimates, an example of which is shown in Fig. 2(b) for the window  $[48T_0, 56T_0]$ . The estimates  $\phi_{\text{CSS}}^{(\text{m})}$  use the imprecise zero phase  $\phi_0$  of the initial CSS as reference, and lead to a broad error histogram (red). In contrast, the estimates  $\phi_{\rm SSS}^{(m)}$  lead to a narrow error histogram (blue) whose spread is instead dominated by the imprecision in obtaining the raw estimates  $\phi^{(m)}(j)$ over short windows (here,  $8T_0$ ), demonstrating the improved precision of the phase reference  $\phi_0^{(m)}$  over  $\phi_0$  [32]. In Fig. 2(c) we show that the variance  $\Delta \phi_{SSS}^2$  of the estimates  $\phi^{(\rm m)}_{\rm SSS}$  is significantly less than  $\Delta\phi^2_{\rm SQL}$  in all windows over the time we consider here, demonstrating the potential for real-time phase tracking with precision beyond the SQL.



FIG. 3. (a) A sudden jump in the phase with amplitude  $\phi_{\rm J} = 40$  mrad at  $T_{\rm J} = 50T_0$  is tracked in the same run using moving windows of durations  $T_{\rm W} = 2T_0$  (red) and  $T_{\rm W} = 20T_0$  (blue), showing the faster response of the shorter window. (b) Protocol to estimate  $\phi_{\rm J}$ . (c) Histograms, over 2048 runs, of  $\phi_{\rm J}^{(\rm m)}$  for  $T_{\rm W} = 2T_0$  (red) and  $T_{\rm W} = 40T_0$  (blue), demonstrating the higher precision of the longer window. For  $T_{\rm W} = 2T_0$ , W<sub>2</sub> was offset by a small time 0.2T<sub>0</sub> to allow transients on timescales of  $\kappa^{-1}$  to decay. (d) Gain in precision over a CSS in Ramsey mode as the duration of W<sub>1</sub> and W<sub>2</sub> is varied, for fixed  $C\gamma_{\rm sc}$  and different values of C. Analytic results (lines) calculated using Eqs. (7) and (8) are in excellent agreement with simulations (markers).

An advantage of our scheme is that the same photocurrent data from a single run can be analyzed using multiple methods to extract varying information. As a demonstration, we use varying window durations  $T_{\rm W}$  to extract precise timing and amplitude information from a sudden jump in phase (at  $T_{\rm J} = 50T_0$  in Fig. 3(a)). Starting with an initial CSS at t = 0, we continuously estimate the phase by averaging the photocurrent over moving windows of durations  $T_{\rm W} = 2T_0$  (red) and  $T_{\rm W} = 20T_0$  (blue). Clearly, the shorter window reproduces the time variation of the phase more precisely. To estimate the amplitude of the jump  $\phi_{\rm J}$ , we compute the difference  $\phi_{\rm J}^{(\rm m)}$  in the estimates  $\phi_{W_1}^{(m)}, \phi_{W_2}^{(m)}$  in the two windows  $W_1 \equiv [T_J - T_W, T_J]$ and  $W_2 \equiv [T_J, T_J + T_W]$  that border the jump time  $T_J$ (Fig. 3(b)) [33]. While the shorter window results in faster response, the longer window gives a more precise estimate of the jump amplitude (Fig. 3(c)).

Alternatively, the sudden phase jump in the protocol depicted in Fig. 3(b) can be replaced with a "dark" phase accumulation time of duration  $T_{\rm D}$  where no measurements are performed. The scheme can then be identified as a Ramsey-like sequence where a squeezed state is prepared in W<sub>1</sub>, phase accumulates in an interrogation time  $T_{\rm D}$ , and finally, phase is read out in W<sub>2</sub>, without ever converting the phase information into a population difference. In this Ramsey mode, the achievable gain in phase resolution using the prepared squeezed state compared to a CSS is

$$\frac{\Delta\phi_{\rm SQL}^2}{\Delta\phi^2} = \frac{\Delta J_{x,\rm SQL}^2}{\left(\Delta J_{x,\rm diff.}^{\rm (m)}\right)^2} \mathcal{V}^2,\tag{7}$$

where  $J_{x,\text{diff}}^{(m)} = J_{x,W_2}^{(m)} - J_{x,W_1}^{(m)}$  and  $\mathcal{V}$  is the visibility at the end of the first window [17, 18]. Fig. 3(d) plots the numerically extracted gain (markers) versus the window duration  $T_{\rm W}$  for different values of cooperativity C. Gaussian fits to histograms of  $J_{x,\text{diff.}}^{(m)}$  were used to extract values for  $(\Delta J_{x,\text{diff.}}^{(m)})^2$ . We find analytically that the normalized variance in the difference measurement varies with  $T_{\rm W}$  as [22]

$$\frac{(\Delta J_{x,\text{diff.}}^{(\text{m})})^2 (T_{\text{W}})}{\Delta J_{x,\text{SQL}}^2} = 2\frac{T_0}{T_{\text{W}}} + \frac{8\beta}{3\eta NC} \frac{T_{\text{W}}}{T_0}, \qquad (8)$$

where  $\beta = r_d + r_{\downarrow\uparrow} + r_{\uparrow\downarrow} + r_l/2$ , giving a minimum normalized variance of  $8\sqrt{\beta/3\eta NC}$  at  $T_{\rm W}^{\rm opt} = T_0\sqrt{3\eta NC/4\beta}$ . The expression for  $\beta$  shows that the normalized variance is not very sensitive to the relative probabilities of the decoherence mechanisms. For typical values of  $C \sim 0.1$ and  $N \sim 10^5$ , Fig. 3(d) shows that a gain upwards of 11 dB can be achieved. The  $(NC)^{-1/2}$  scaling of the minimum normalized variance in  $J_{x,\rm diff}^{\rm (m)}$  leads to an optimal phase resolution scaling as  $\Delta\phi \sim N^{-3/4}$  compared to  $\Delta\phi_{\rm SQL} = N^{-1/2}$  radians.

In conclusion, we have proposed and analyzed a scheme for continuous real-time tracking of a quantum phase with precision beyond the SQL. Interfering cavityassisted Raman transitions have been considered previously for deterministic squeezing schemes [34] and quantum simulations of the Dicke model [35–37]. The frequency arrangement of our drive lasers is also related to two-tone drive schemes for back-action evading measurements of mechanical oscillators [26, 38, 39] and for measuring the state of individual superconducting qubits [40, 41]. Furthermore, while Ramsev sequences only measure phase changes unambiguously in the interval  $[-\pi/2,\pi/2]$ , our scheme readily extends to tracking large excursions  $|\phi(t)| \gg \pi$ : The measured current i(t) can be used in a feedback loop [42-46] to adjust the differential phase offset  $\psi_1 - \psi_2$  of the drive lasers such that i(t) is continuously driven back to zero. The feedback loop continuously adjusts the spin component probed by the cavity mode such that it is always perpendicular to the mean spin direction, while mapping the phase  $\phi(t)$ onto the feedback signal as  $\phi(t) = (\psi_1 - \psi_2)/2$ . This way, large phase excursions can be tracked while remaining in the small angle measurement limit, also greatly suppressing sensitivity to variations or uncertainties in scale factors relating i(t) to  $\phi(t)$ , including uncertainties in atom number [22]. By encoding the spins in hyperfine levels that have an intrinsic splitting, our scheme has the unique capability to greatly increase the unambiguous interval of phase evolution that can be continuously tracked, for example in atomic clocks. While feedback schemes using intermittent non-demolition population measurements have been used to extend this interval in a Ramsey-like sequence [47], our scheme continuously tracks the phase and removes the need for state rotations altogether. It will be interesting to see if this scheme can be adapted to optical clock transitions, perhaps in <sup>87</sup>Sr.

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