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Thermodynamic Capacity of Quantum Processes

Philippe Faist,^{1,*} Mario Berta,² and Fernando Brandão^{3,1}

¹*Institute for Quantum Information and Matter, Caltech, Pasadena, CA 91125, U.S.A.*

²*Department of Computing, Imperial College London, London SW7 2AZ, United Kingdom*

³*Google Inc., Venice, CA 90291, U.S.A.*

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Thermodynamics imposes restrictions on what state transformations are possible. In the macroscopic limit of asymptotically many independent copies of a state—as for instance in the case of an ideal gas—the possible transformations become reversible and are fully characterized by the free energy. In this Letter, we present a thermodynamic resource theory for quantum processes that also becomes reversible in the macroscopic limit, a property that is especially rare for a resource theory of quantum channels. We identify a unique single-letter and additive quantity, the thermodynamic capacity, that characterizes the “thermodynamic value” of a quantum channel, in the sense that the work required to simulate many repetitions of a quantum process employing many repetitions of another quantum process becomes equal to the difference of the respective thermodynamic capacities. On a technical level, we provide asymptotically optimal constructions of universal implementations of quantum processes. A challenging aspect of this construction is the apparent necessity to coherently combine thermal engines that would run in different thermodynamic regimes depending on the input state. Our results have applications in quantum Shannon theory by providing a generalized notion of quantum typical subspaces and by giving an operational interpretation to the entropy difference of a channel.

Introduction.—In the quest of extending the laws of thermodynamics beyond the macroscopic regime, the resource theory of thermal operations was introduced to characterize possible transformations which could be carried out at the nano scale [1–5]. By imposing a set of physically motivated rules, an agent can only perform a restricted set of operations on a system, which we refer to generically as *thermodynamic operations*. Here, we will consider as thermodynamic operations either thermal operations [1–3] or Gibbs-preserving maps [6–9]. By characterizing the possible state transformations under these rules, one obtains formulations of the second law of thermodynamics which are valid for small-scale systems out of thermodynamic equilibrium. A natural regime to study such state transformations is a macroscopic regime in which one considers conversions between many independent and identically distributed (i.i.d.) copies of a state, i.e., states of the form $\rho^{\otimes n}$. If we consider transformations on a system S with Hamiltonian H_S , using a heat bath at inverse temperature β and a work storage system W , then the asymptotic work cost per copy of transforming $\rho^{\otimes n}$ into $\sigma^{\otimes n}$ is given by the difference in free energy $F(\sigma) - F(\rho)$ [1]. The free energy is defined as

$$F(\rho) = \text{tr}(H\rho) - \beta^{-1}S(\rho) = \beta^{-1}D(\rho||e^{-\beta H}), \quad (1)$$

expressed either in terms the von Neumann entropy $S(\rho) = -\text{tr}(\rho \ln \rho)$ or the quantum relative entropy $D(\rho||\gamma) = \text{tr}[\rho(\ln \rho - \ln \gamma)]$. Since the cost of asymptotically performing the reverse transformation $\sigma^{\otimes n} \rightarrow \rho^{\otimes n}$ is the negative of the cost of the forward transformation this resource theory becomes reversible (Fig. 1a).

Here, we study the resource theory of thermodynamics for quantum processes themselves. Given a black-box implementation of a process \mathcal{E} , can we simulate a process

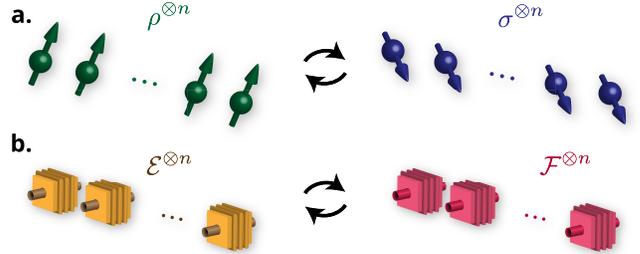


FIG. 1. Reversibility in the many-copy regime. **a.** In the resource theory of thermodynamics, quantum states are reversibly interconvertible, i.e., the work cost of transforming n independent copies of ρ into n independent copies of σ is (approximately) the same as the work that can be extracted in the reverse transformation. Reversibility is a valuable property for a resource theory, as it provides a full characterization of the precise amount of resources that is necessary for any state transformation [10]. **b.** We show that a similar conclusion holds for quantum processes. There is a unique quantity, the thermodynamic capacity, that measures the “thermodynamic value” of the channel in terms of the resources required to create, or extracted while consuming, many copies of a channel. Note that, in this context, reversibility refers to the interconversion of processes themselves, not to recovering the input of a process from its output.

\mathcal{F} using thermodynamic operations, or is there a thermodynamic cost in doing so? We fully answer the question in the i.i.d. regime, and show that the thermodynamic simulation of channels becomes reversible (Fig. 1b). That is, the work cost of executing many realizations of \mathcal{F} using many realizations of \mathcal{E} is the same as the work that can be extracted in the reverse process of implementing \mathcal{E} from uses of \mathcal{F} .

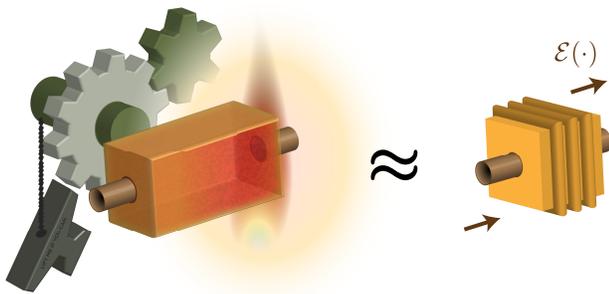


FIG. 2. Universal thermodynamic implementation of a quantum process. Using thermodynamic operations and by furnishing work, the task is to simulate approximately an ideal process \mathcal{E} . The implementation must output a state close to $\mathcal{E}(\sigma)$ for any possible input σ , even relative to a reference system. In this letter, we consider the regime of many independent copies of the channel, the i.i.d. regime, and we show that it is possible to construct a universal implementation of an i.i.d. quantum process at a work cost rate equal to the *thermodynamic capacity* of the channel, defined as the worst-case difference of free energies between the output and the input. The main challenge is that the implementation must work for any input state, including superpositions of input states for which individual input-specific implementations would run in different thermodynamic regimes.

Reversibility is a coveted property for a resource theory [10], as it usually follows from this property that one can establish necessary and sufficient conditions for the interconversion of states, thus fully understanding which state transformations are possible. On the other hand, many natural resource theories do not have this property. For instance, entanglement transformations under local operations and classical communication is a reversible resource theory only for the set of pure states, and not for general mixed states [11, 12]. Also, resource theories of channels are typically not asymptotically reversible [13] except in certain specific cases, for instance simulating a quantum channel with local operations, classical communication and with arbitrary shared entanglement [14–16].

Challenges.—The main problem we would like to solve is to find a universal implementation of any quantum process in the i.i.d. regime using thermodynamic operations (Fig. 2). In the single-shot setting, where the process is only performed once and the implementation must succeed with high probability, we can leverage results from [9] to characterize the corresponding work cost for any input state. The single-shot work cost for a known input state σ is given by the *coherent relative entropy* [9], which in the i.i.d. limit is known to converge to the free energy difference $F(\mathcal{E}(\sigma)) - F(\sigma)$. If we adapt the definition of the coherent relative entropy by imposing that the implementation is accurate for any input state, we obtain a mathematical expression that characterizes the work cost of a single-instance universal implementation of a process. However, it is unclear what this cost becomes

in the i.i.d. limit. More precisely, it is not clear that one can find an implementation of an i.i.d. process that performs accurately for any input state, because the implementation seems to have to be able to work coherently in a superposition of different thermodynamic regimes of energy-to-entropy conversion. To see this, consider the following naive attempt at an implementation of an i.i.d. process $\mathcal{E}^{\otimes n}$. One could determine the input state nondestructively using gentle tomography [17], and then apply the implementation \mathcal{T}_σ given by [9] that is optimal for the given i.i.d. input state $\sigma^{\otimes n}$. Mathematically, this is:

$$\mathcal{T}(\cdot) = \sum_{\sigma} \mathcal{T}_{\sigma}(M_{\sigma}(\cdot)M_{\sigma}), \quad (2)$$

where M_{σ} would be the square root of the POVM effects corresponding to the gentle tomography of [17], which estimates approximately which i.i.d. state $\sigma^{\otimes n}$ the input is while not disturbing the state too much. Crucially, the different optimal processes \mathcal{T}_{σ} might have different work costs, and worse, might be working in different regimes of energy-to-entropy conversion. For instance, the ideal process \mathcal{E} might be a combination of a Landauer erasure for one input but a shift in energy levels for a different input. So what happens if a superposition of different i.i.d. inputs is given? In this case, the state is decohered by the gentle tomography step, destroying the superposition. So we need to somehow find a way of combining the different \mathcal{T}_{σ} coherently, with the difficulty that each of the processes \mathcal{T}_{σ} for different σ might be operating in different thermodynamic regimes.

At the heart of this problem is the fact that we require the implementation to be accurate even for non-i.i.d. inputs. For instance, if we are developing a specialized computing chip that implements a given quantum gate in a quantum computer, the different inputs to the gate might very well be correlated, as they would depend on earlier stages of the computation. Here, we may appeal to the postselection technique [18], which essentially states that if an implementation \mathcal{T}_n of an i.i.d. process $\mathcal{E}^{\otimes n}$ is accurate for any i.i.d. input state with an accuracy that decreases exponentially in the number of copies n , then the implementation \mathcal{T}_n is also accurate for any input state and not only for i.i.d. inputs. Unfortunately, this actually even precludes the use of nondestructive or gentle tomography as suggested above, because the error induced by this step is polynomial in n instead of exponential. So we are back to square one.

A different natural approach is to follow the proof logic of Refs. [15, 19]: One can define an entropic quantity associated with the single-shot problem instance, as described above, and exploit properties of the relevant entropy measures. In fact, this proof strategy works for systems described by a trivial Hamiltonian $H = 0$. There, the single-shot work cost for i.i.d. processes reduces to a single conditional max-entropy quantity using the post-selection

technique [18]. Then, exploiting a quasi-convexity-like property of the conditional max-entropy [20] allows to prove our result for the special case of trivial Hamiltonians (technical details will be published elsewhere [21]). Attempting to generalize this proof approach to non-trivial Hamiltonians fails because the coherent relative entropy does not display the required quasi-convexity property.

Main result.—We solve the above problems by explicitly constructing a universal implementation of any i.i.d. quantum process, using a new notion of quantum typicality that intuitively allows to coherently combine different entropy-to-energy conversion regimes. By construction, the implementation does not depend on the input state, and when considered as a channel, it is close in diamond norm to the ideal channel $\mathcal{E}^{\otimes n}$ [22]. The rate at which work has to be supplied is characterized by the *thermodynamic capacity* of the channel, given as

$$T(\mathcal{E}) = \max_{\sigma} \left\{ F(\mathcal{E}(\sigma)) - F(\sigma) \right\}. \quad (3)$$

That is, the work cost of such an implementation coincides with the worst-case cost of implementing the process over all possible i.i.d. input states. Surprisingly in light of the above challenges, there is no intrinsic overhead in the work cost rate associated with the implementation being ignorant of the input state—the work cost rate is no worse than the rate corresponding to the worst-case input state.

We may combine our main result with the following result by Navascués *et al.* [23]: From a black-box access to many copies of a process \mathcal{E} , it is possible to extract work at a rate that is asymptotically equal to $T(\mathcal{E})$. Hence the work cost rate associated with converting $\mathcal{E}^{\otimes n}$ to $\mathcal{F}^{\otimes n}$ is simply $T(\mathcal{F}) - T(\mathcal{E})$, corresponding to extracting as much work as possible from the copies of \mathcal{E} and then simulating $\mathcal{F}^{\otimes n}$ using our procedure. This work cost is reversible, i.e., all the work invested in the transformation can be recovered in the reverse transformation $\mathcal{F}^{\otimes n} \rightarrow \mathcal{E}^{\otimes n}$. Hence, this work cost is optimal, and the resource theory becomes reversible, with the thermodynamic capacity being the unique measure of the “thermodynamic value” of the quantum channel (see also Fig. 1).

The thermodynamic capacity (3) generalizes the notion of capacity for quantum channels to the context of thermodynamics, by measuring how much free energy can be conveyed through the use of the channel. The thermodynamic capacity is expressed as a single-letter formula and can be computed efficiently, even analytically for some simple examples, as it can be formulated as a convex optimization problem [24]. The thermodynamic capacity is also additive [23], and does not need to be regularized as for other channel capacity measures. It is tightly related to other channel entropy measures, especially the amortized entanglement of a channel [25] and the entropy of a channel [26].

We announce our results in this Letter, providing the

physical background and intuition surrounding our construction of a universal implementation of any i.i.d. process. Fully detailed technical proofs will be published elsewhere [21].

Implementation based on quantum typicality.—We exploit two main ingredients for our universal implementation. First, using Schur-Weyl duality one may estimate approximately the spectrum of an i.i.d. state, and hence its entropy, using a global measurement on the n systems [16, 17, 27, 28]. Let us denote by $\{P_s^n\}_s$ the positive operator-valued measure (POVM) that produces an estimate s of the entropy per copy of an i.i.d. state over n systems. Second, for noninteracting systems, a global energy measurement will provide a sharp statistics for any i.i.d. state due to large deviation bounds (cf. for example [29]); thus the average energy per copy h of an i.i.d. state can be estimated to a good approximation by this measurement, whose POVM effects we denote by $\{Q_h^n\}$.

Our construction is then based on the following idea. Given an i.i.d. process $\mathcal{E}_{X \rightarrow X'}$, we may consider a Stinespring isometry $V_{X \rightarrow X'E}$ into an environment E , satisfying $\mathcal{E}(\cdot) = \text{tr}_E(V(\cdot)V^\dagger)$. Using the POVMs mentioned above, we may write $V = (\sum_{s',h'} P_{s'} Q_{h'}) V (\sum_{s,h} P_s Q_h)$, since the elements of a POVM sum to the identity operator. An important observation however is that not all combinations of outcomes s, h, s', h' are possible. Namely, we know that for any i.i.d. input state $\sigma^{\otimes n}$, we must have $-s' + \beta h' + s - \beta h \approx -S(\mathcal{E}(\sigma)) + \beta \langle H_{X'} \rangle_{\mathcal{E}(\sigma)} + S(\sigma) - \beta \langle H_X \rangle_{\sigma} = \beta F(\mathcal{E}(\sigma)) - \beta F(\sigma) \leq \beta T(\mathcal{E})$. So we may enforce this condition explicitly in the decomposition above, and by pushing all the POVM effects through the isometry, we get a candidate implementation of the form $\mathcal{T}_{X^n \rightarrow X'^n} = \text{tr}_{E^n}(W(\cdot)W^\dagger)$ with $W_{X \rightarrow X'E} = M_{X'E} V_{X \rightarrow X'E}$, and where M is defined as

$$M_{X'E} = \sum_{-s'+\beta h'+s-\beta h \leq \beta T(\mathcal{E})} P_{s'} Q_{h'} (V P_s Q_h V^\dagger). \quad (4)$$

The operator M can be interpreted as a fully quantum, universal smoothing operator for bipartite states that counts entropy relative to another operator, and which is a natural generalization of universal and relative typical subspace projectors [16, 17, 30–37]. The approximations above are related to how well the POVMs $\{P_s^n\}$ and $\{Q_h^n\}$ are able to resolve the entropy and energy per copy, and the corresponding error vanishes in the limit $n \rightarrow \infty$. That the implementation process $\mathcal{T}_{X^n \rightarrow X'^n}$ is close in diamond norm to the ideal i.i.d. process $\mathcal{E}^{\otimes n}$ follows from the postselection technique [18], which allows us to focus on i.i.d. input states, and from the fact that the omitted terms in (4) only account for an exponentially small weight for any input i.i.d. state. Finally, we invoke a mathematical characterization of the work cost of processes in the framework of Gibbs-preserving maps [9]: We show that $\mathcal{T}(e^{-\beta H_{X^n}}) \lesssim e^{nT(\mathcal{E})} e^{-\beta H_{X'^n}}$, which in turn implies that the candidate implementation \mathcal{T} requires $T(\mathcal{E})$ work per

copy. The complete proof of these statements is provided in Ref. [21].

Thermal Operations.—The framework of Gibbs-sub-preserving maps is particularly generous, and it is a priori not clear that all such maps are implementable at no work cost. Alternatively, we may consider the framework of thermal operations, where only energy-conserving unitary interactions with a heat bath are allowed. It turns out that in this framework it is also possible to construct a universal implementation of i.i.d. processes at a work cost of $T(\mathcal{E})$ per copy, yet our proof is restricted to processes that are time-covariant, i.e., that commute with the time evolution. The assumption of time covariance allows us to sidestep issues of coherence between energy levels [38–42]. This result directly implies that the asymptotic reversibility of the resource theory of quantum processes also holds in the context of thermal operations for time-covariant processes. Our proof for this second main result is presented in detail in Ref. [21], and follows a considerably different strategy than above; we make use of recent ideas from quantum information theory, including the convex-split lemma and position-based decoding [43–48].

Extensions.—Our proof techniques allow us to prove some related results. First, we exhibit a one-shot conditional erasure protocol that is valid for systems described by a non-trivial Hamiltonian and states that are time-covariant, thus generalizing the protocol of Ref. [49]. The work cost is given in terms of a beyond-i.i.d. generalization of the relative entropy called the *hypothesis testing relative entropy* [50–55], that quantifies how well two states can be distinguished by a hypothesis test and which is closely-related to other one-shot information measures [56–59]. Our result implies that it is possible to implement any time-covariant process for a fixed time-covariant input state in the single-shot regime, using thermal operations and a battery, at a cost given by the coherent relative entropy.

Also, we show that if the input is a fixed i.i.d. state, it is possible to implement any arbitrary, not necessarily time-covariant i.i.d. channel using thermal operations, a battery, and a sub-linear amount of coherence, at the same asymptotic work cost per copy as it would take to implement it with Gibbs-preserving maps. We thus conclude that although Gibbs-preserving maps are more powerful in general than thermal operations [7], they become asymptotically equivalent in the macroscopic limit in terms of implementing i.i.d. processes on given i.i.d. input states.

Discussion.—Quantum resource theories of channels has developed into a hot topic of interest in recent years, as they display various features that are not mirrored in corresponding resource theories of state transformations [13, 26, 60, 61]. In this context, we show that the thermodynamic resource theory of channels is asymptotically reversible in the i.i.d. regime, as is the case for quantum states. Asymptotically, there exists a unique

monotone, the thermodynamic capacity, which characterizes the “thermodynamic value” of a channel. In this sense our result is the thermodynamic analogue of the reverse Shannon theorem for quantum channels [15, 16].

Our statements and proofs are also fully noncommutative in the sense that they cannot be simplified to a problem about classical probability distributions in a fixed basis—a feature that is still rather uncommon in quantum thermodynamics. Our universal implementation of an i.i.d. process is accurate even for superpositions of input states for which individual optimal implementations would require thermal engines running in different regimes of energy-to-entropy conversion. Moreover, standard proof techniques developed for quantum channel simulations do not readily apply to our problem at hand; attempting to mimick the proof in Refs. [15, 19] fails because the coherent relative entropy is not quasi-convex [21].

Whether or not it is possible to universally implement any i.i.d. channel that is not time-covariant using thermal operations is still an open question. We expect that such a protocol might in general need a very large amount of coherence, much like the requirement of large embezzling states for the reverse Shannon theorem [15, 16]. Indeed, if the input is a superposition of two different i.i.d. states of different energy, the environment must be able to coherently compensate for any energy difference caused by the process without disturbing the process. However, we have shown that for fixed i.i.d. input states, any i.i.d. channel can be implemented optimally using thermal operations, so this suggests that a tighter connection between thermal operations and Gibbs-preserving maps remains to be uncovered.

For a trivial Hamiltonian, the thermodynamic capacity reduces (up to a sign) to the *minimal entropy gain* of a channel \mathcal{E} , defined as $\min_{\sigma}[S(\mathcal{E}(\sigma)) - S(\sigma)]$. This quantity was introduced as a measure of information for channels [62–68]. Our results thus exhibit a physical and operational interpretation for this quantity.

Given the relevance of entropy measures for a wide range of physical and information-theoretic situations, we expect our results to find applications beyond thermodynamic interconversion of processes. For instance, we note that a quantity closely related to the coherent relative entropy has found applications in studying dissipative dynamics of many-body systems [69]. Also, in contrast to standard smooth entropy measures for quantum states [58], our channel smoothing in terms of the diamond norm leaves one of the marginals invariant when applied to quantum states (cf. the very recent related works [60, 61]). This might offer some insights on the quantum joint typicality conjecture in quantum communication theory [34, 70, 71], on which recent progress has been made [37]. One may also study how our results are modified if we replace the diamond norm condition on the implementation by other channel distinguishability measures, such as introduced in Ref. [72]; this would be

particularly relevant for settings with memory effects, for instance implementations of gates in a quantum computer.

Finally, that there exists optimal universal thermodynamic implementations of channels indicates that low-dissipation components for future quantum devices can in principle be developed, that function accurately for all inputs, and still dissipate no more than required by the worst case input.

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* phfaist@caltech.edu

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