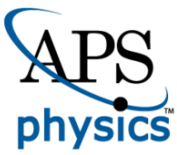

Four-Body Scale in Universal Few-Boson Systems

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Phys. Rev. Lett. **122**, 143001 — Published 10 April 2019

DOI: 10.1103/PhysRevLett.122.143001

The Four-Body Scale in Universal Few-Boson Systems

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(Dated: February 12, 2019)

The role of an intrinsic four-body scale in universal few-boson systems is the subject of active debate. We study these systems within the framework of effective field theory. For systems of up to six bosons we establish that no four-body scale appears at leading order. However, we find that at next-to-leading order a four-body force is needed to obtain renormalized results for binding energies. With the associated parameter fixed to the binding energy of the four-boson system, this force is shown to renormalize the five- and six-body systems as well. We present an original *ansatz* for the short-distance limit of the bosonic A-body wavefunction, from which we conjecture that new A-body scales appear at $\mathrm{N}^{A-3}\mathrm{LO}$. As a specific example, calculations are presented for clusters of helium atoms. Our results apply more generally to other few-body systems governed by a large scattering length, such as light nuclei and halo states, the low-energy properties of which are independent of the detailed internal structure of the constituents.

The universal aspects of few-body systems with large scattering length have attracted a lot of attention in recent years [1, 2], largely owing to well-controlled experiments involving ultracold atomic gases where the scattering length can be tuned arbitrarily via Feshbach resonances [3]. Nuclear physics, where the scattering lengths in both nucleon-nucleon S-wave channels are significantly larger in magnitude than the interaction range set by the pion mass, falls into the same universality class. Another interesting example is given by atomic ^{4}He clusters, where the two-body scattering length also happens to be much larger than the van der Waals length.

These systems share a pronounced separation of scales. When the scattering length a significantly exceeds the force range, the system properties become independent of the force details, which can be represented by contact interactions, in analogy with the multipole expansion of classical electrodynamics. Effective field theory (EFT) implements this idea systematically starting from a leading order (LO) with only Dirac delta functions. Finite-range corrections are accounted for at higher orders through delta functions with derivatives. In the two-body sector, this expansion around the zero-range limit is equivalent [4] to Fermi's pseudopotential [5], to nontrivial boundary conditions [6], and to the effective range expansion [7].

This idea extends to A-body systems: A-body forces, which capture aspects of the underlying potential that only manifests themselves for $A > 2$, are ordered according to their relevance to low-energy physics. For three identical particles with spin statistics not precluding a totally symmetric spatial wavefunction—bosons or nucleons, for instance—the zero-range limit is not well defined with only two-body interactions due to bound-state collapse [8]. In the EFT framework, despite expectations based on dimensional analysis, a three-body contact interaction must enter at LO to ensure renormalization-group (RG) invariance [9, 10]. It introduces a scale that in the unitary limit ($a \to \infty$) determines the position of a geometric tower of three-body bound states [11]. These Efimov trimers have been observed in cold-atom systems [12, 13].

It is of fundamental interest to understand whether this phenomenon repeats in larger systems: when does an additional particle bring in a new scale from a higher-body contact interaction? Once such a scale appears, universality is reduced and the properties of the corresponding system can no longer be predicted entirely on the basis of systems with fewer particles.

The importance of a four-body parameter has in fact been the subject of active debate in the literature [14–20]. In contrast to a zero-range model [17, 20], early EFT studies [14, 15, 18] of the four-body system found that no four-body force (and thus no four-body scale) is required at LO. Recently, Ref. [21] also established the absence of LO higher-body forces for systems of up to six particles. Lack of further LO scales means that Efimov towers exist for more than three bosons [18, 19, 22–25] and that there are correlations between cluster and trimer binding energies, the atomic equivalent of the

nuclear Tjon line [26] for four-boson clusters [14] and its generalization for five- and six-boson clusters [21]. The properties of unitary bosonic matter are universal when written in terms of a three-body energy [27], a property that might be testable in cold-atom experiments [28, 29].

Here, we go beyond LO and address the question whether the naïvely construed next-to-leading-order (NLO) part of the EFT expansion, where range corrections enter in the form of contact interactions with derivatives, is properly renormalized. While subleading orders have been included perturbatively in the three-boson system with success [30, 31], our work is the first to examine more-boson systems in this way. Our central result is that a four-body force is required at NLO, which, once fixed to a single four-body observable, suffices to stabilize clusters of up to at least six bosons. The relatively small resulting changes at NLO bode well for the convergence of the EFT expansion.

EFT description. A system of nonrelativistic spinless bosons of mass m interacting via a short-range force can be described by the Lagrangian density

$$\mathcal{L} = \psi^\dagger \left(\mathrm{i}\partial_0 + \frac{\nabla^2}{2m} \right) \psi - \frac{C_0^{(0)}}{2} (\psi^\dagger \psi)^2 - \frac{D_0^{(0)}}{6} (\psi^\dagger \psi)^3 + \cdots , \tag{1}$$

where ψ is the field operator, $C_0^{(0)}$ and $D_0^{(0)}$ are low-energy constants (LECs), and the ellipsis represents terms with more fields and/or more derivatives, entering at higher orders. The LECs' super- and subscripts denote, respectively, the order in the EFT expansion and the powers of momenta involved.

Translated to the language of ordinary quantum mechanics, the interaction terms in Eq. (1) give rise to delta-function potentials, which need to be regularized. We choose here a separable form, $V_2^{(0)} = C_0^{(0)} |g\rangle\langle g|$, where g represents a Gaussian regulator in momentum space, $\langle \mathbf{q}|g\rangle = \exp(-q^2/\Lambda^2) \equiv g(q^2)$. In coordinate space, this corresponds to a smeared-out delta function which tends to a delta function as the cutoff parameter $\Lambda \to \infty$. Observables must not depend on the arbitrary regularization except for terms that decrease as Λ increases. This is achieved *via* renormalization, when the LEC "runs" with the cutoff, $C_0^{(0)} = C_0^{(0)}(\Lambda)$, in such a way that a chosen observable—for example, the scattering length—remains fixed to its physical value.

The term involving $D_0^{(0)}$ parametrizes the three-body force at LO. We include it in the form $V_3^{(0)} = D_0^{(0)} |\xi\rangle\langle\xi|$, where $\langle \mathbf{q_1 q_2}|\xi\rangle = g(q_1^2 + 3q_2^2/4)$ regulates the three-body system described by the Jacobi momenta q_i. Renormalization is achieved when one three-body observable—for example a trimer energy—is kept fixed. $D_0^{(0)}(\Lambda)$ has a log-periodic form [9, 10] representing an RG limit cycle.

Range corrections enter at NLO in the form of a term that involves four fields and two derivatives, with a new LEC $C_2^{(1)}$ to be determined from a second two-body observable. The corresponding potential can be written in momentum space as

$$\langle \mathbf{k}|V_2^{(1)}|\mathbf{k}'\rangle = C_2^{(1)} g(k^2) \left(k^2 + k'^2\right) g(k'^2) . \tag{2}$$

There are also corrections to the LO LECs which do not introduce new parameters as they merely ensure that the renormalization conditions used at LO remain satisfied at NLO. While the LO interactions must be treated nonperturbatively, NLO consists of a single insertion of the NLO potential. Renormalization cannot be achieved for positive effective range, as is the case here, when an inconsistent subset of higher-order corrections is included by the nonperturbative solution of the Schrödinger equation with the NLO potential [32].

Numerical methods. We employ two independent numerical methods to calculate the A-boson binding energies, both treating NLO corrections perturbatively.

In the first approach, which is more efficient for a precise numerical determination of $D_0^{(0)}(\Lambda)$, we calculate $A = 3, 4$ binding energies by solving, respectively, the Faddeev and Faddeev-Yakubovsky (FY) equations. We employ the same numerical framework as in Ref. [33], which is an implementation of the formalism discussed in Refs. [14, 15, 34]. The central idea is to decompose the full wavefunctions into Faddeev(-Yakubovsky) components which are related by Bose symmetry. For $A = 3$, we set the wavefunction $|\Psi_3^{(0)}\rangle = (1 + P)|\psi\rangle + |\psi_3\rangle$ in terms of the Faddeev components $|\psi\rangle$ and $|\psi_3\rangle$, where $1 + P$ with $P = P_{12}P_{23} + P_{13}P_{23}$ is an operator that enforces Bose symmetry through a combination of appropriate permutations P_{ij} of the individual particles. One obtains the system of equations

$$\begin{aligned} |\psi\rangle &= G_0\, t\, P|\psi\rangle + G_0\, t\, |\psi_3\rangle , \\ |\psi_3\rangle &= G_0\, t_3\, (1 + P)|\psi\rangle , \end{aligned} \tag{3}$$

where G_0 denotes the free three-body Green's function and the operators t and t_3 are solutions of Lippmann-Schwinger equations with, respectively, $V_2^{(0)}$ and $V_3^{(0)}$ as driving terms. It is an advantage of the separable regulator we use that these operators can be derived analytically in closed form. The solution for Eq. (3) is obtained in momentum space by projection onto partial-wave states $|q_1 q_2, \ell_1 \ell_2\rangle$, where $\ell_{1,2}$ are orbital angular-momentum quantum numbers corresponding to the Jacobi momenta $q_{1,2}$; they are coupled to total angular momentum zero for the states we consider in this work. Upon discretization on a momentum grid, Eq. (3) yields a homogeneous matrix equation that depends on energy via G_0 and t. Bound states are found at those energies where the matrix has a unit eigenvalue. The wavefunction components are obtained by solving the corresponding homogeneous equations. Similarly, for $A = 4$, $|\Psi_4^{(0)}\rangle = (1+P_{34}+PP_{34})(1+P)|\psi_A\rangle + (1+P)(1+\tilde{P})|\psi_B\rangle$ involves the additional permutation operator $\tilde{P} \equiv P_{13}P_{24}$

as well as components $|\psi_A\rangle$ and $|\psi_B\rangle$ that correspond to partitions into, respectively, $3+1$ and $2+2$ clusters.

In the second approach, which is more efficient for systems with more particles, we expand the coordinate-space wavefunction in a correlated Gaussian basis [35],

$$\Psi_A^{(0)}(\boldsymbol{\eta}) = \sum_i c_i \hat{\mathcal{S}} \exp\left(-\frac{1}{2}\boldsymbol{\eta}^T A_i \boldsymbol{\eta}\right) , \qquad (4)$$

where $\boldsymbol{\eta}$ collects the $A-1$ Jacobi vectors $\boldsymbol{\eta}_j$, A_i is an $(A-1)\times(A-1)$ real, symmetric, and positive-definite matrix, and $\hat{\mathcal{S}}$ is a symmetrization operator. The coefficients $\{c_i\}$ and the energy are determined by solving a generalized eigenvalue problem. An important feature of the Gaussian basis is that it can deal with both short ($\sim \Lambda^{-1}$) and long ($\sim a$) length scales. To optimize our basis we use the stochastic variational method (SVM) [35], where the elements of the matrix A_i are chosen randomly taking at each step the element that gives the lowest energy. By the variational principle, the method is guaranteed to give upper bounds for the binding energies. The implementation of this method here follows Ref. [21].

Our choice of a separable Gaussian regulator significantly simplifies the Faddeev equations. With SVM we could verify that our results are reproduced with a non-separable regulator made of local Gaussians in configuration space.

Results. While our conclusions are generally valid for other universal systems, such as ultracold atomic gases or atomic nuclei, for concreteness we calculate here the energies of small clusters of ^{4}He atoms. The ^{4}He atomic system is characterized by a scattering length $a \approx 90\,\text{Å}$, which is much larger than the van der Waals length $\approx 5.4\,\text{Å}$ that describes the long-range part of the interatomic potential. The dimer was measured experimentally to have a binding energy of about 1.5 mK [36–38]. Two Efimov trimers were measured [39, 40], which are the remains of the otherwise infinite geometric tower of Efimov states that emerges as $a \to \infty$. Larger clusters are predicted [41–44] by modern He-He pair potentials [45, 46], but have not yet been observed.

Three data points are needed to fix the coefficients of our EFT up to NLO, which we choose as the two-body scattering length and effective range, as well as the binding energy of the excited trimer. In order to compare with heavier-cluster predictions, we take the values calculated from a potential, in particular the modern PCKLJS potential [43, 46]. Once enough data on helium clusters become available, we can let go of potential-model input. For now, we use this two-body potential as a possible representation of short-distance physics; the inclusion of more complicated interactions [47] would not affect our conclusions. The dimer binding energy here is $B_2 = 1.615$ mK, and indeed our EFT converges well toward this value, with $B_2^{\text{LO}} = 0.918\,B_2$ and $B_2^{\text{NLO}} = 0.991\,B_2$. We use the Faddeev equations to fix $D_0^{(0)}(\Lambda)$ and then find good agreement between the two methods for the ground state trimer binding energy, $B_3^{\text{LO}} = 98.1\,B_2$ and $B_3^{\text{NLO}} = 73.1\,B_2$, to be compared with the direct potential-model result $B_3 = 81.6\,B_2$ [43].

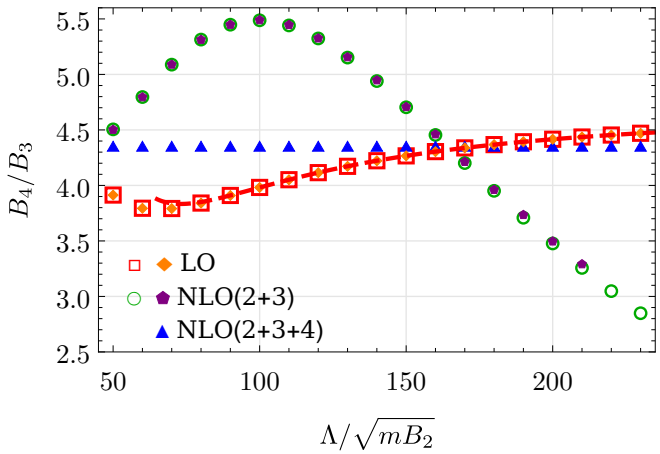

FIG. 1. (Color online) The tetramer binding energy in units of the trimer ground-state energy is plotted as function of the cutoff in units of $\sqrt{mB_2}$. LO and NLO results without a four-body force from the FY (orange diamonds and purple pentagons) and SVM (red squares and green circles) methods are in very good agreement. They are compared to the result calculated directly [43] from the PCKLJS potential, which coincides (by construction) with the NLO result with a four-body force (blue triangles). The red dashed curve is a fit in powers of Λ^{-1}.

EFT calculations for four-atom systems so far are only available at LO. Here we confirm the pioneering result [14] that the LO tetramer ground-state energy converges as the cutoff Λ is increased. We proceed for the first time to NLO, where we observe that, in contrast, the tetramer energy does *not* converge once range corrections are included—it instead diverges roughly linearly within the investigated cutoff range. Our LO and NLO results for the tetramer ground-state energy as a function of Λ are shown in Fig. 1. The two methods agree very well.

The observed divergence is a clear indication that a four-body force is required at NLO, much earlier than one would expect from a naïve counting of many-body forces. This promotion is analogous to that of the three-body force to LO. The simplest four-body force is a contact interaction without derivatives: $V_4^{(1)} = F_0^{(1)}|\zeta\rangle\langle\zeta|$, where $\langle \mathbf{q}_1\mathbf{q}_2\mathbf{q}_3|\zeta\rangle = g\big(q_1^2 + 3q_2^2/4 + 2q_3^2/3\big)$ in the same regularization as before. The LEC $F_0^{(1)}(\Lambda)$ is determined by demanding that the tetramer energy is fixed at the value calculated directly from the potential model [43].

With the NLO four-body force thus determined, one may wonder whether higher-body forces appear at the same order. We find that this is not the case when we study the pentamer ground-state energy up to NLO with SVM, as shown in Fig. 2. At LO the results converge with

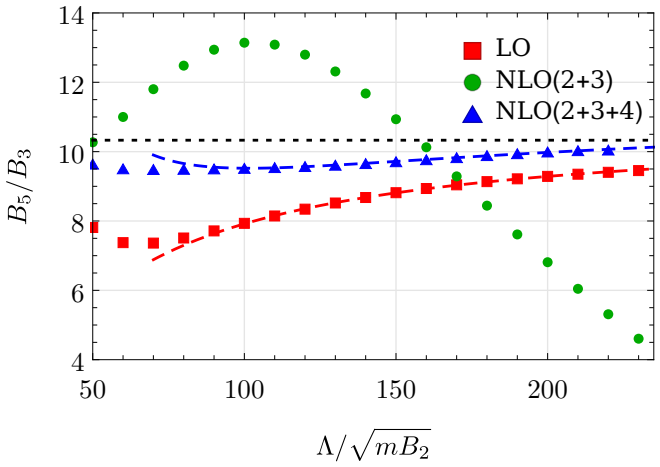

FIG. 2. (Color online) The pentamer ground-state energy is plotted as function of the cutoff in the same units as Fig. 1. Shown are results at LO (red squares), NLO without a four-body force (green circles), and NLO with the four-body force that renormalizes the four-body system (blue triangles). The colored dashed curves are fits in powers of Λ^{-1}. The result calculated [41] from the LM2M2 potential is the dotted line.

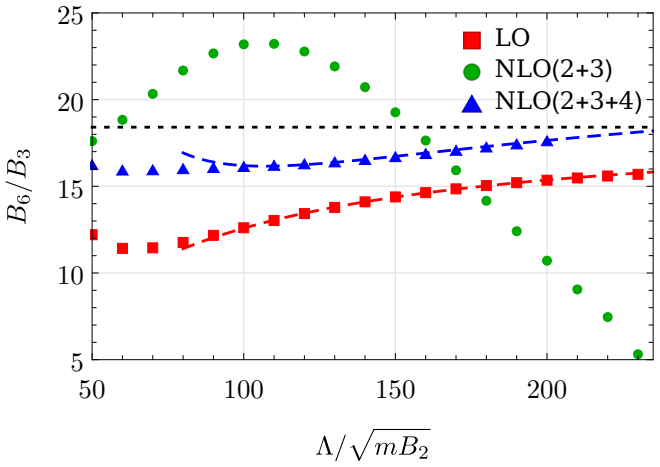

FIG. 3. (Color online) The hexamer ground-state energy is plotted as function of the cutoff in the same units as Fig. 1. Symbols are as in Fig. 2.

	LO	NLO	PCKLJS	LM2M2
B_4/B_3	4.8(1)	4.35(∗)	4.35	4.44(1)
B_5/B_3	10.8(5)	11.3(3)	—	10.33(1)
B_6/B_3	18(2)	22(3)	—	18.41(2)

TABLE I. The A-body ^{4}He binding energies, in units of the trimer binding energy, for $A = 4, 5, 6$. (∗) indicates a fit value. Our results are compared to those obtained with the PCKLJS [43] and LM2M2 [41] potentials.

Λ, in agreement with the conclusion of Ref. [21] that no five-body term is needed at this order. Without a four-body force a divergence is observed at NLO, analogously to the one observed for the tetramer energy, but once the NLO four-body force is included, we find the five-body system properly renormalized as well. This adds confidence in our order assignment for the dominant four-body force. Similar conclusions hold for the six-atom system, as can be seen from the hexamer ground-state energy in Fig. 3. While the SVM calculation becomes more difficult as the number of particles increases and the results are therefore less conclusive for $A = 6$, we see overall the same pattern as before. There is no need for a six-body force up to NLO, either.

Numerical calculations are limited to finite cutoffs. In a renormalized theory, the residual cutoff dependence can be absorbed in higher-order operators, which scale as inverse powers of the breakdown scale. Asymptotic ($\Lambda \to \infty$) values obtained from fitting the numerical results with polynomials in Λ^{-1} are given in Tab. I, where the reported error is that from the extrapolation alone. A reasonable estimate of the EFT truncation error at NLO is the square of the relative change from LO to NLO. No results based on the PCKLJS potential are available to compare with; however, since the tetramer energies based on the PCKLJS and LM2M2 differ by only 2%, we list the latter [41] as representative results.

Our numerical calculations are limited by a cutoff value above which an unphysical, deep trimer state appears [9, 10]. However, considering an appropriate *ansatz* for the wavefunction, the need for a four-body force at NLO can also be derived analytically. By using the RG framework of Ref. [48], we can connect the counting of the A-body LECs with the power-law behavior of the A-body wavefunction at short distances. Our *ansatz* for the latter is $\Psi_A^{(0)} \to \phi_A/(\prod_j |\eta_j|)$ with ϕ_A a function that does not exhibit power-law behavior for $\eta_j \to 0$. This *ansatz* is derived from the Schrödinger equation with short-range forces in the unitary limit: the $1/|\eta_j|$ factors are a trivial consequence of the absence of long-range interactions, while ϕ_A is a non-trivial consequence of the (complicated) boundary conditions induced by the short-range forces. The exact form of ϕ_A is not known in general, but it is irrelevant for power counting. This counting follows from considering matrix elements of A-body contact potentials between the wavefunctions, the short-distance behavior of which determines at which order they are required. For $A = 2, 3$ our conjecture is readily verified analytically. Our numerically obtained $A = 3, 4$ wavefunctions approximate this behavior for distances shorter than the trimer/tetramer sizes and larger than Λ^{-1}, *i.e.*, for the expected domain of validity of the *ansatz*, which we can check by assuming a log-periodic form for ϕ_A. Further trust in this *ansatz* stems from its correct prediction of LO two- and three-body forces and the absence of a four-body force at LO. The validity of the *ansatz* for $A \geq 5$ is a conjecture, which implies even less significance for

all higher-body forces, in agreement with our numerical results. As such, it establishes the power counting of all A-body forces for A-boson systems: new scales appear at $\mathrm{N}^{A-3}\mathrm{LO}$.

Conclusions. We find a large dependence of the ground-state energies for $A = 4, 5, 6$ bosons on the regulator when NLO two-body range corrections are added perturbatively to LO. A four-body force is necessary and sufficient for renormalization at this order. For $A = 4$ this result applies also to fermions with four internal states, such as the nucleon. Previous calculations for the ^{4}He nucleus [49–51] could not observe this effect because range corrections were treated nonperturbatively, thereby breaking RG invariance already at the two-body level. It will be interesting to investigate in future work to what extend the enhancement of many-body forces discussed here is modified in nuclear systems with $A > 4$ due to the Pauli principle.

The low order of the dominant four-body force offers an explanation for the controversy on the importance of a four-body scale [14–20], on one hand, and the need for an additional parameter in the description of ^{4}He droplets [52], on the other. The absence of higher-body forces up to NLO ensures that correlations between higher-body and four-body energies survive at this order. The relatively small size of the full NLO corrections suggests that the EFT expansion is working well for the ^{4}He clusters considered here, despite their large binding compared with the trimer. We plan to extend our calculations to light nuclei in the near future.

We thank Lorenzo Contessi and Francesco Pederiva for useful discussions. This work was supported in part by the US NSF under awards PHY15-15738 (JK) and PHY–1614460 (SK); the US DOE, NUCLEI SciDAC Collaboration award DE-SC0008533 (SK) and Office of Science, Office of Nuclear Physics, under award No. DE-FG02-04ER41338 (UvK); the ERC Grant No. 307986 STRONGINT (SK); the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Projektnummer 279384907 – SFB 1245 (SK); the National Natural Science Foundation of China under Grants No. 11522539 and No. 11735003 (MPV); the Fundamental Research Funds for the Central Universities (MPV); the Thousand Talents Plan for Young Professionals (MPV); the Israel Science Foundation under Grant No. 1308/16 (NB); and the European Union Research and Innovation program Horizon 2020 under grant agreement No. 654002 (UvK). Some of the numerical computations were performed on the Lichtenberg high-performance computer of the TU Darmstadt as well as at the Jülich Supercomputing Center.

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