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Phys. Rev. Lett. **122**, 135101 — Published 5 April 2019

DOI: 10.1103/PhysRevLett.122.135101

Decoupled Cascades of Kinetic and Magnetic Energy in Magnetohydrodynamic Turbulence

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Magnetic energy (ME) and kinetic energy (KE) in ideal incompressible MHD are not global invariants and, therefore, it had been justified to discuss only the cascade of their sum, total energy. We provide a physical argument based on scale-locality, along with compelling evidence that ME and KE budgets statistically decouple beyond a transitional "conversion" range. This arises because magnetic field-line stretching is a large-scale process which vanishes on average at intermediate and small scales within the inertial-inductive range, thereby allowing each of mean ME and KE to cascade conservatively and at an equal rate, yielding a turbulent magnetic Prandtl number of unity over these scales.

Magnetohydrodynamic (MHD) turbulence is of fundamental importance to many fields of science, including astrophysics, solar physics, space weather, and nuclear fusion. The Reynolds numbers of such flows are typically very large, giving rise to plasma fluctuations with power-law spectra over a vast range of scales where both viscosity and resistivity are negligible. We call such a range "inertial-inductive" since ideal dynamics dominate. There are several competing theories for the spectrum of strong MHD turbulence over the inertial-inductive range [1–5], all of which assume scale-locality of the energy cascade, which has been shown to hold [6].

In a scale-local cascade, energy transfer across scale ℓ is predominantly due to scales within a moderate multiple of ℓ [7]. This gives rise to an inertial-inductive scale range over which the flow evolves without direct communication with the largest or smallest scales in the system.

In MHD turbulence, only the sum of magnetic and kinetic energy (KE and ME, respectively), *i.e.* total energy, is a global invariant of the inviscid unforced dynamics. Therefore, it has been justified to discuss only the cascade of total energy, but not of KE or ME separately, which are coupled by magnetic field-line stretching. In principle, the process of magnetic field-line stretching can operate at *all* scales, giving rise to various phenomena such as Alfvén waves.

We shall show here that magnetic field-line stretching is a large-scale process, which operates over a "conversion range" of scales of limited extent and vanishes *on average* at intermediate and small scales in the inertial-inductive range [8]. Over the ensuing part of the inertial-inductive range, mean KE and ME cascade conservatively and at an equal rate to smaller scales despite not being separate invariants.

Our findings are important in subgrid scale modeling of systems such as accretion disks, whose evolution is controlled by magnetic flux through the disk [9–11]. The strength of the magnetic field is determined by a balance between (i) turbulent advection (or turbulent viscosity) which accretes the field radially inward, and (ii) turbulent resistivity which diffuses it outward [12–15]. Other applications are outlined in the conclusion.

We start from the incompressible MHD equations with a constant density ρ:

$$\partial_t \mathbf{u} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla} p + \mathbf{J}\times\mathbf{B} + \nu\nabla^2\mathbf{u} + \mathbf{f}, \quad (1)$$

$$\partial_t \mathbf{B} = \nabla\times(\mathbf{u}\times\mathbf{B}) + \eta\nabla^2\mathbf{B}. \quad (2)$$

Here $\mathbf{u}$ is the velocity, and $\mathbf{B}$ is the magnetic field normalized by $\sqrt{4\pi\rho}$ to have Alfvén (velocity) units. Both fields are solenoidal: $\boldsymbol{\nabla}\cdot\mathbf{u} = \boldsymbol{\nabla}\cdot\mathbf{B} = 0$. The pressure is p, $\mathbf{J} = \boldsymbol{\nabla}\times\mathbf{B}$ is (normalized) current density, $\mathbf{f}$ is external forcing, ν is viscosity, and η is resistivity.

In a statistically steady state, the space-averaged KE and ME budgets are, respectively,

$$\langle S_{ij}B_iB_j\rangle = \epsilon^{\text{inj}} - \nu\langle|\nabla\mathbf{u}|^2\rangle, \quad (3)$$

$$\langle S_{ij}B_iB_j\rangle = \eta\langle|\nabla\mathbf{B}|^2\rangle, \quad (4)$$

where $\langle ... \rangle$ is a spatial average, $S_{ij} = (\partial_j u_i + \partial_i u_j)/2$ is the strain rate tensor, and $\epsilon^{\text{inj}} = \langle\mathbf{f}\cdot\mathbf{u}\rangle$ is kinetic energy injection rate. It is clear from eqs. (3)-(4) that mean KE-to-ME conversion due to magnetic field-line stretching is positive and bounded: $0 \leq \langle \mathbf{B}\cdot\mathbf{S}\cdot\mathbf{B}\rangle \leq \epsilon^{\rm inj}$. The bound holds in the presence of an arbitrarily strong uniform magnetic field $\mathbf{B}_0$, indicating significant cancellations. This can be understood by considering that in a turbulent flow, the strain $\mathbf{S}$, being a derivative, is dominated by the small-scales, whereas $\mathbf{B}$ is dominated by the large-scales, near the magnetic spectrum's peak, leading to decorrelation effects.

To analyze how magnetic field-line stretching operates at different length-scales, we utilize a coarse-graining approach for diagnosing multi-scale dynamics [7, 16]. A coarse-grained field which contains modes at length-scales $> \ell$ is defined by $\overline{f}_\ell(\mathbf{x}) = \int d\mathbf{r}\, G_\ell(\mathbf{r}-\mathbf{x}) f(\mathbf{r})$, where $G_\ell(\mathbf{r}) \equiv \ell^{-3} G(\mathbf{r}/\ell)$ is a normalized kernel with its main weight in a ball of diameter ℓ. Coarse-grained MHD equations can then be written to describe $\overline{\mathbf{u}}_\ell$ and $\overline{\mathbf{B}}_\ell$, along with corresponding budgets for the quadratic invariants at scales $\geq \ell$, for *arbitrary* ℓ in contrast to the mean field approach [17, 18] (see [16] and references therein). Hereafter, we drop subscript ℓ when possible.

KE and ME density balance at scales $> \ell$ are,

$$\begin{aligned} &\partial_t\left(\frac{|\overline{\mathbf{u}}|^2}{2}\right) + \nabla\cdot[\cdots] \\ &\quad = -\overline{\Pi}^u_\ell - \overline{S}_{ij}\overline{B}_i\overline{B}_j - \nu|\nabla\overline{\mathbf{u}}|^2 + \overline{\mathbf{f}}\cdot\overline{\mathbf{u}}, \qquad (5) \\ &\partial_t(|\overline{\mathbf{B}}|^2) + \nabla\cdot[\cdots] \\ &\quad = -\overline{\Pi}^b_\ell + \overline{S}_{ij}\overline{B}_i\overline{B}_j - \eta|\nabla\overline{\mathbf{B}}|^2, \qquad (6) \end{aligned}$$

where $\nabla\cdot[\cdots]$ represents spatial transport terms. Dissipation terms, $\nu|\nabla\overline{\mathbf{u}}|^2$ and $\eta|\nabla\overline{\mathbf{B}}|^2$, are mathematically guaranteed to be negligible [16, 19] at scales $\ell \gg (\ell_\nu, \ell_\eta)$, with ℓ_ν and ℓ_η the viscous and resistive length scales, respectively.

The first term on the RHS of eq.(5), $\overline{\Pi}^u_\ell$, appears as a sink in the KE budget of large scales $> \ell$ and as a source in the KE budget of small scales $< \ell$ [16]. It quantifies the KE transfer *across* scale ℓ, and is defined as $\overline{\Pi}^u_\ell \equiv -\overline{S}_{ij}\overline{\tau}_{ij}$, where $\overline{\tau}_{ij} = \tau_\ell(u_i,u_j) + \tau_\ell(B_i,B_j)$ is the sum of both the Reynolds and the Maxwell stress generated by scales $< \ell$ acting against the large-scale strain, $\overline{S}_{ij}$. Subscale stress is defined as $\tau_\ell(f,g) = \overline{(fg)}_\ell - \overline{f}_\ell\overline{g}_\ell$ for any two fields f and g. Similarly, $\overline{\Pi}^b_\ell \equiv -\overline{\mathbf{J}}_\ell\cdot\overline{\boldsymbol{\varepsilon}}_\ell$ in eq. (6) quantifies the ME transfer *across* scale ℓ, where $\overline{\boldsymbol{\varepsilon}}_\ell \equiv \overline{\mathbf{u}\times\mathbf{B}} - \overline{\mathbf{u}}\times\overline{\mathbf{B}}$ is (minus) the electric field generated by scales $< \ell$ acting on the large-scale current, $\overline{\mathbf{J}} = \nabla\times\overline{\mathbf{B}}$, resulting in a "turbulent Ohmic dissipation" to the small scales.

Term $\overline{\mathbf{B}}_\ell\cdot\overline{\mathbf{S}}_\ell\cdot\overline{\mathbf{B}}_\ell$ appears as a sink in eq.(5) and a source in eq.(6), representing KE expended by the large-scale flow to bend and stretch large-scale $\overline{\mathbf{B}}$-lines. Unlike the cascade terms $\overline{\Pi}^u_\ell$ and $\overline{\Pi}^b_\ell$, which involve large-scale fields acting against subscale terms ($\overline{\boldsymbol{\tau}}_\ell$ and $\overline{\boldsymbol{\varepsilon}}_\ell$), $\overline{\mathbf{B}}_\ell\cdot\overline{\mathbf{S}}_\ell\cdot\overline{\mathbf{B}}_\ell$ is purely due to large-scale fields and does not participate in energy transfer *across* scale ℓ. A more refined scale-by-scale analysis in [6] showed how energy lost or gained from one field ($\mathbf{u}$ or $\mathbf{B}$) by line stretching reappeared in or disappeared from the other field at the *same* scale.

In a steady state, space-averaging eqs. (5),(6) at any scale ℓ in the inertial-inductive range, $L \gg \ell \gg (\ell_\nu, \ell_\eta)$, yields

$$\langle\overline{\Pi}^u_\ell\rangle = \epsilon^{\rm inj} - \mathcal{C}^{ub}(\ell), \qquad (7)$$

$$\langle\overline{\Pi}^b_\ell\rangle = \mathcal{C}^{ub}(\ell), \qquad (8)$$

where we have dropped the dissipation terms and assumed that forcing is due to modes at scales $\sim L \gg \ell$, such that $\overline{\mathbf{f}}_\ell = \mathbf{f}$. Mean conversion, $\mathcal{C}^{ub}(\ell) \equiv \langle\overline{S}_{ij}\overline{B}_i\overline{B}_j\rangle$, in eqs. (7),(8) quantifies the cumulative KE-to-ME conversion at *all* scales $> \ell$.

Using scale-locality of the cascade terms, $\overline{\Pi}^u_\ell$ and $\overline{\Pi}^b_\ell$, which was proved in [6], we will now argue that mean magnetic field-line stretching is primarily a large-scale process which vanishes at intermediate and small scales within the inertial-inductive range. Note that the scale-locality discussed in [6, 7] is "diffuse" [20] and states that contributions from disparate scales decay only as a power-law of the scale ratio.

Define ℓ_d as the largest scale at which non-ideal microphysics becomes significant, $\ell_d = \max(\ell_\nu, \ell_\eta)$. Define the cumulative KE-to-ME conversion at scales $> \ell_d$ by $\mathcal{C}^{ub}_d \equiv \mathcal{C}^{ub}(\ell_d)$, which is not necessarily equal to the unfiltered $\langle\mathbf{B}\cdot\mathbf{S}\cdot\mathbf{B}\rangle$ due to possible contributions from scales $< \ell_d$ [see discussion shortly after eq. (10) below].

Define ℓ_s as the largest scale at which $\mathcal{C}^{ub}(\ell_s) = \mathcal{C}^{ub}_d$. We'll argue that (i) $\ell_s \neq \ell_d$ and (ii) $\mathcal{C}^{ub}(\ell) = \mathcal{C}^{ub}_d$ for all scales $\ell_s > \ell \gg \ell_d$.

First, assume $\ell_s = \ell_d$. This implies that as functions of ℓ, $\mathcal{C}^{ub}(\ell) = \langle\overline{\Pi}^b_\ell\rangle = \epsilon^{\rm inj} - \langle\overline{\Pi}^u_\ell\rangle$ depends on dissipative parameters ν or η. However, $\langle\overline{\Pi}^u_\ell\rangle$ and $\langle\overline{\Pi}^b_\ell\rangle$ are scale-local in the inertial-inductive range [6] and are insensitive to the microphysics when $\ell \gg \ell_d$. Therefore, $\ell_s \neq \ell_d$. Second, if $\mathcal{C}^{ub}(\ell) \neq \mathcal{C}^{ub}_d$ over

$\ell_s > \ell \gg \ell_d$, then $\mathcal{C}^{ub}(\ell)$, which we assume is continuous, will have an extremum at a scale ℓ_* within that range [since $\mathcal{C}^{ub}(\ell_s) = \mathcal{C}^{ub}(\ell_d) = \mathcal{C}^{ub}_d$]. Therefore, $\langle\overline{\Pi}^u_\ell\rangle$ and $\langle\overline{\Pi}^b_\ell\rangle$ will also have extrema, indicating the existence a special scale ℓ_* in the inertial-inductive range, in conflict with scale-invariance of the ideal MHD dynamics.

Therefore, $\mathcal{C}^{ub}(\ell) \to \mathcal{C}^{ub}_d$ within a conversion range $L > \ell > \ell_s$ and, over the ensuing range $\ell_s > \ell \gg \ell_d$, it saturates at $\mathcal{C}^{ub}(\ell) = \mathcal{C}^{ub}_d$. Since $\mathcal{C}^{ub}(\ell)$ measures the cumulative KE-to-ME conversion at all scales $> \ell$, saturation implies a zero contribution from $\ell_s > \ell \gg \ell_d$. We conclude that mean KE-to-ME conversion, $\langle S_{ij}B_iB_j\rangle$, is a large-scale process within the inertial-inductive range, acting over a conversion range $L > \ell > \ell_s$ of limited extent, *i.e.* the scale-range does not increase asymptotically with the Reynolds number. Mean KE and ME budgets decouple in the absence of conversion over the "decoupled range" of scales, $\ell_s > \ell \gg \ell_d$:

$$\langle\overline{\Pi}^u_\ell\rangle = \epsilon^{\mathrm{inj}} - \mathcal{C}^{ub}_d, \tag{9}$$

$$\langle\overline{\Pi}^b_\ell\rangle = \mathcal{C}^{ub}_d. \tag{10}$$

With the RHS of eqs.(9),(10) being independent of scale ℓ, KE and ME each cascades conservatively after the mechanism coupling them halts. Scale-locality suggests that the normalized KE and ME cascade rates, $\langle\overline{\Pi}^u\rangle/\epsilon^{\mathrm{inj}}$ and $\langle\overline{\Pi}^b\rangle/\epsilon^{\mathrm{inj}}$, should have a universal value of order unity over $\ell_s > \ell \gg \ell_d$, regardless of the forcing, $Pm = \nu/\eta$, or $\mathbf{B}_0$. Note that scale ℓ_s at which the budgets decouple is within the inertial-inductive range, despite the well-known non-equipartition of KE and ME spectra in that range [21–24] (See Fig. 8 in the supplemental material (SM) [25]).

While the above argument suggests that $\mathcal{C}^{ub}(\ell)$ should become constant at scales smaller than the conversion range, it only applies within the inertial-inductive range, $L \gg \ell \gg \ell_d$. It is possible for $\mathcal{C}^{ub}(\ell)$ to vary again when transitioning to scales $\lesssim \ell_d$. An example is the viscous-inductive (Batchelor) range, $\ell_\nu \gg \ell \gg \ell_\eta$, over which a scale-by-scale analysis in [6] showed that magnetic field-line stretching can act as a forcing term in the ME budget, consistent with our understanding of high Pm flows [26, 27]. The above argument for saturation of $\mathcal{C}^{ub}(\ell)$ breaks down at scales $\lesssim \ell_d$, such as in the viscous-inductive range where scale-locality does not hold due to a smooth velocity field [6].

Our conclusions are supported by a suite of pseudospectral Direct Numerical Simulations (DNS) up to 2,048^3 in resolution with phase-shift dealiasing, using hyperdiffusion and other parameters summarized in Table I.

TABLE I. Each suite of Runs was carried out at different Reynolds numbers at 256^3, 512^3, and 1,024^3 resolutions. Run V was also conducted at 2,048^3 resolution. $Pm = \nu/\eta$ is magnetic Prandtl number. $B^{\max}_k = \sqrt{\max_k[E^b(k)]}$ is at the magnetic spectrum's $[E^b(k)]$ peak. ABC (helical) and TG (non-helical) forcing were applied at wavenumber k_f. More details are in the SM [25].

Run	Forcing	k_f	Pm	$\lvert\mathbf{B}_0\rvert/B^{\max}_k$
I	ABC	2	1	0
II	ABC	2	1	10
III	TG	1	1	0
IV	ABC	1	2	0
V	ABC	2	1	2

Figure 1 shows results from the five flows we consider, at the highest resolution (see SM [25] for lower resolution runs and evidence of convergence). In all runs, total energy, being a global invariant, is transferred conservatively across scales $L \gg \ell \gg \ell_d$, as indicated by a scale-independent total energy flux, $\langle\overline{\Pi}_\ell\rangle = \langle\overline{\Pi}^u_\ell + \overline{\Pi}^b_\ell\rangle$. Both $\overline{\Pi}^u_\ell$ and $\overline{\Pi}^b_\ell$ decay to zero at scales $\lesssim \ell_d$, when the nonlinearities shut down in the dissipation range. Mean KE-to-ME conversion, $\mathcal{C}^{ub}(\ell)$, increases from 0 at the largest scales to $\approx \mathcal{C}^{ub}_d \approx \epsilon^{\mathrm{inj}}/2$ at an intermediate scale ℓ_s within the inertial-inductive range. Over the ensuing range, $\ell_s > \ell \gg \ell_d$, $\mathcal{C}^{ub}(\ell)$ is scale-independent, indicating a negligible contribution to magnetic field-line stretching at these scales. There is a slight increase in $\mathcal{C}^{ub}(\ell)$ in the dissipation range, at scales $\lesssim \ell_d$ where our argument is not expected to hold due to a lack of scale-locality. In all cases, $\langle\overline{\Pi}^b_\ell\rangle \approx \mathcal{C}^{ub}(\ell)$ and $\overline{\Pi}^u_\ell \approx \epsilon^{\mathrm{inj}} - \mathcal{C}^{ub}(\ell)$ over the inertial-inductive range, consistent with eqs. (7),(8). Beyond the conversion range, scale-transfer becomes independent of ℓ, $\langle\overline{\Pi}^u_\ell\rangle \approx \epsilon^{\mathrm{inj}} - \mathcal{C}^{ub}_d$ and $\langle\overline{\Pi}^b_\ell\rangle \approx \mathcal{C}^{ub}_d$ over $\ell_s > \ell \gg \ell_d$, consistent with eqs. (9),(10), and indicative of a conservative cascade of KE and ME energy, respectively. In all runs, we observe that the KE and ME cascade rates become equal in magnitude, $\langle\overline{\Pi}^u_\ell\rangle \approx \langle\overline{\Pi}^b_\ell\rangle$, over $\ell_s > \ell \gg \ell_d$, with magnetic field-line stretching channeling $\approx 1/2$ of the injected energy to the magnetic field, regardless of the forcing, Pm, or $\mathbf{B}_0$, consistent with scale-locality.

Among the five cases in Fig. 1, the conversion

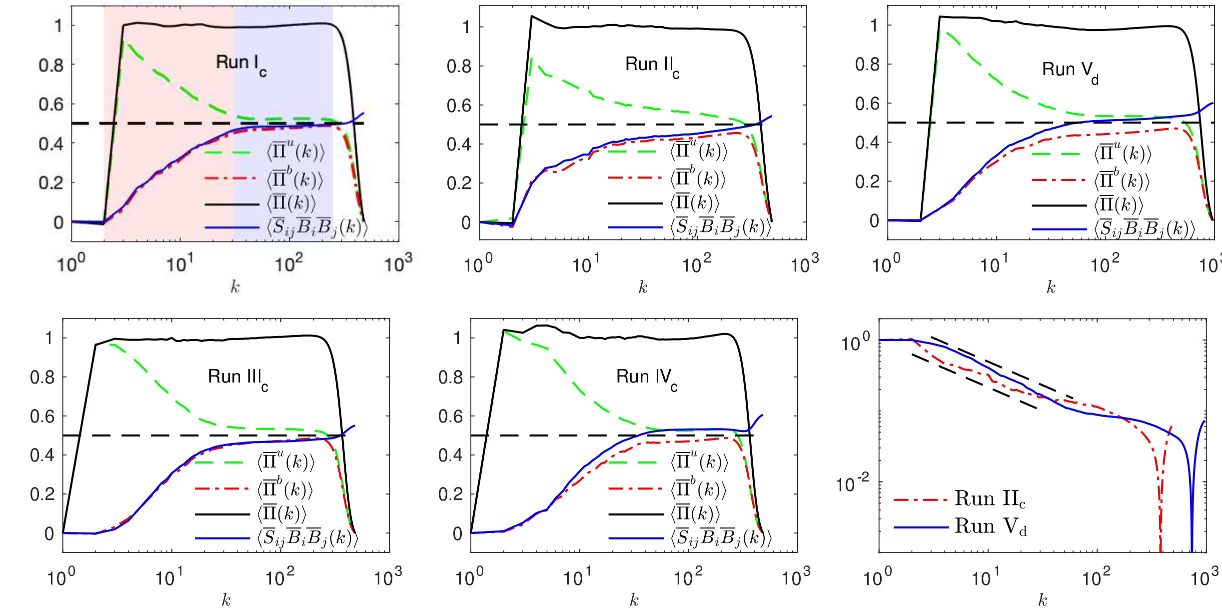

FIG. 1. The first five panels show $\langle\overline{\Pi}\rangle = \langle\overline{\Pi}^u + \overline{\Pi}^b\rangle$, $\langle\overline{\Pi}^u\rangle$, $\langle\overline{\Pi}^b\rangle$, and $\langle \overline{S}_{ij}\overline{B}_i\overline{B}_j\rangle$ as a function of $k \equiv 2\pi/\ell$ from our highest resolution Runs (1,024^3 for Runs I to IV and 2,048^3 for Run V. See SM [25] for lower resolutions). In top-left panel, conversion (decoupled) range is shaded red (blue). All plots are time-averaged and normalized by $\epsilon^{\rm inj}$. The horizontal straight dashed line is at 0.5. Bottom-right panel shows a log-log plot of relative residual conversion, $\mathcal{R}^{ub}(k)/\mathcal{C}_d^{ub}$, and a reference black-dashed line with a $-2/3$ slope, suggesting that KE-to-ME conversion saturates in a manner consistent with scale-locality [6].

range is widest in the presence of $|\mathbf{B}_0|/B_k^{\rm max} = 10$ (Run II$_{\rm c}$). However, according to our argument, its extent cannot increase indefinitely with an increasing dynamic range of scales (or Reynolds number, Re). After all, $\langle\mathbf{B}{\cdot}\mathbf{S}{\cdot}\mathbf{B}\rangle$ is bounded even in the $|\mathbf{B}_0| \to \infty$ limit. Indeed, a plot of the relative residual conversion $\mathcal{R}^{ub}(\ell)/\mathcal{C}_d^{ub} \equiv \langle\overline{\mathbf{B}}_{\ell_d}{\cdot}\overline{\mathbf{S}}_{\ell_d}{\cdot}\overline{\mathbf{B}}_{\ell_d} - \overline{\mathbf{B}}_{\ell}{\cdot}\overline{\mathbf{S}}_{\ell}{\cdot}\overline{\mathbf{B}}_{\ell}\rangle/\langle\overline{\mathbf{B}}_{\ell_d}{\cdot}\overline{\mathbf{S}}_{\ell_d}{\cdot}\overline{\mathbf{B}}_{\ell_d}\rangle$ in Fig. 1 (and Fig. 5 in SM [25]) decays at least as fast as a power-law as $\ell \to \ell_d$, consistent with what is expected from scale-locality (we take ℓ_d as the scale at which $\langle\overline{\Pi}_\ell\rangle = \epsilon^{\rm inj}/2$). Moreover, plots of $\mathcal{C}^{ub}(\ell)$ at increasing Re (Fig. 4 in SM [25]) show a clear convergence to $\mathcal{C}_d^{ub} \approx \epsilon^{\rm inj}/2$.

The negligible mean KE-to-ME conversion at small scales within the decoupled range might seem counterintuitive at first. After all, a hallmark of MHD turbulence are Alfvén waves which are fastest at small scales. The decoupling of ME and KE budgets poses no contradiction since it is only in the *mean*, which allows for decorrelation effects at small scales similar to those arising in compressible turbulence [28, 29]. Utilizing the simultaneous information in both scale and space afforded by our coarse-graining approach, we analyze $\overline{\mathbf{B}}_{\ell}{\cdot}\overline{\mathbf{S}}_{\ell}{\cdot}\overline{\mathbf{B}}_{\ell}(\mathbf{x})$ acting on scales $> \ell$ and the residual conversion within the inertial-inductive range, $\overline{\mathbf{B}}_{\ell_d}{\cdot}\overline{\mathbf{S}}_{\ell_d}{\cdot}\overline{\mathbf{B}}_{\ell_d}(\mathbf{x}) - \overline{\mathbf{B}}_{\ell}{\cdot}\overline{\mathbf{S}}_{\ell}{\cdot}\overline{\mathbf{B}}_{\ell}(\mathbf{x})$, as a function of space $\mathbf{x}$ in Fig. 2. For an intermediate scale $\ell = 2\pi/30$ from Run II$_{\rm c}$ (and Run I$_{\rm c}$ in Fig. 7 of SM [25]), Fig. 2 shows how magnetic field-line stretching, which is concentrated in magnetic filaments, is an order of magnitude more intense at scales smaller than $\ell = 2\pi/30$ compared to larger scales. Yet, the small-scale contribution fluctuates vigorously in sign, yielding a mere 17% (10% in Run I$_{\rm c}$ in Fig. 7 of SM [25]) to the space average. To illuminate the role of waves, we repeat in the SM [25] the analysis above on two examples of non-colliding Alfvén waves, a monochromatic wave and a wavepacket, which are exact solutions of the MHD equations and which lack energy transfer between scales.

In conclusion, small-scales of the magnetic field in the decoupled scale range are maintained, on average, by turbulent Ohmic dissipation (the ME cascade), $\langle\overline{\Pi}^b\rangle = \langle\overline{\mathbf{J}}{\cdot}\overline{\boldsymbol{\varepsilon}}\rangle$. Mean magnetic field-line stretching acts as a large-scale driver of the ME cascade, justifying the inclusion of a low-mode forcing in the induction eq. (2) when resolving the transitional conversion range is unimportant, such as in high-Re asymptotic scaling studies of MHD turbulence [30–32]. Our results will help in deriving

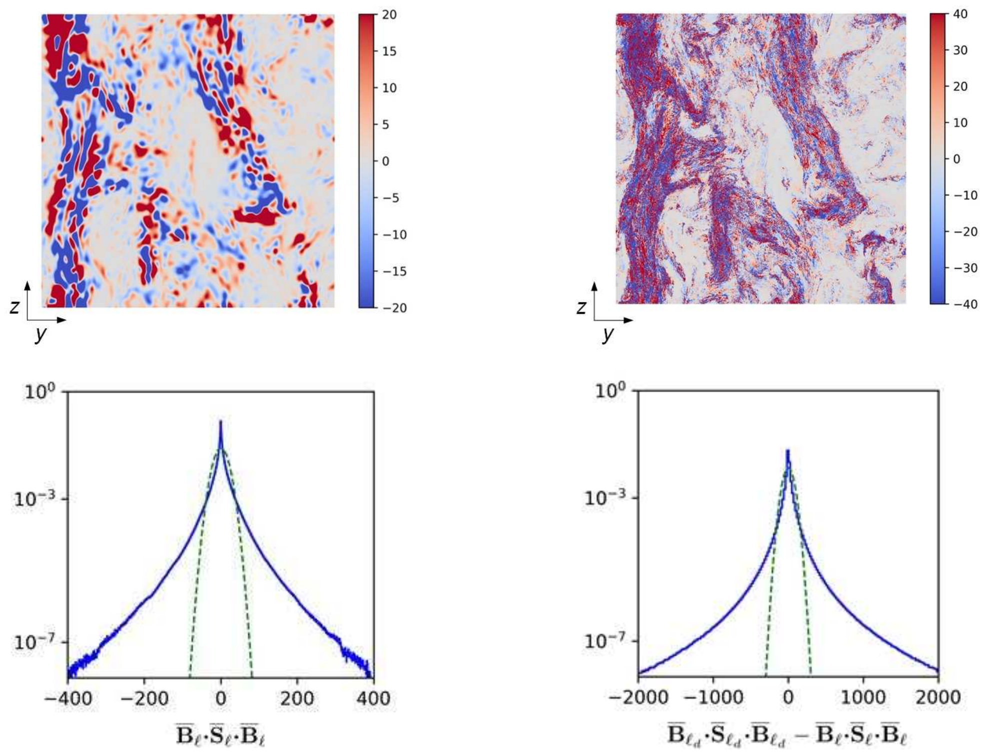

FIG. 2. For scale $\ell = 2\pi/30$ ($k = 30$) from Run II$_c$ in Figure 1 at one instant in time: top two panels show a 2D slice from the 3D domain of pointwise conversion at large scales, $\overline{\mathbf{B}}_\ell{\cdot}\overline{\mathbf{S}}_\ell{\cdot}\overline{\mathbf{B}}_\ell(\mathbf{x})$ (top left), and small scales, $\overline{\mathbf{B}}_{\ell_d}{\cdot}\overline{\mathbf{S}}_{\ell_d}{\cdot}\overline{\mathbf{B}}_{\ell_d}(\mathbf{x}) - \overline{\mathbf{B}}_\ell{\cdot}\overline{\mathbf{S}}_\ell{\cdot}\overline{\mathbf{B}}_\ell(\mathbf{x})$ (top right). $\mathbf{B}_0$ is in the z-direction. Bottom two panels show probability density function of conversion as a function of $\mathbf{x}$ at large scales (bottom left) and small scales (bottom right). The large-scale distribution has mean of 0.43 and variance of 223.54. The small-scale distribution has mean of 0.09 and variance of 3060.84. Quantities are normalized by energy injection rate ϵ^{inj}. Unnormalized Gaussians (green dashed lines) are added to both plots.

relations equivalent to the Politano-Pouquet relations [33] but for the separate cascades of KE and ME, with potential implications on the scaling in MHD turbulence. This work can also help sub-grid scale model development and testing in Large Eddy Simulations of MHD turbulence [34, 35]. For example, they provide a direct measure of the turbulent magnetic Prandtl number, which is unity within decoupled range due to equipartition of the cascades, $\langle\overline{\Pi}^u_\ell\rangle = \langle\overline{\Pi}^b_\ell\rangle$, which has important implications to astrophysical flows such as accretion disks [12, 14, 15]. Our findings are also relevant for turbulent magnetic reconnection [22, 36, 37] since they imply that the net bending and twisting of magnetic field lines at length scales in the decoupled range is driven by the effective electric field, $-\overline{\boldsymbol{\varepsilon}}_\ell$, rather than by the flow's strain, giving independent support to previous studies [22, 38]. Our framework for quantifying field-line stretching at various scales may also prove insightful in magnetic dynamo studies[39–42].

HA thanks G. Eyink for discussions. This work was supported by the DOE FES grant DE-SC0014318. HA was also supported by NASA grant 80NSSC18K0772, DOE grant DE-SC0019329, and DOE NNSA award DE-NA0003856. Computing time was provided by NERSC under Contract No. DE-AC02-05CH11231.

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