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Andrew J. A. James, Robert M. Konik, and Neil J. Robinson

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# Nonthermal states arising from confinement in one and two dimensions

Andrew J. A. James,<sup>1,\*</sup> Robert M. Konik,<sup>2,†</sup> and Neil J. Robinson<sup>3,‡</sup>

<sup>1</sup>*London Centre for Nanotechnology, University College London,  
Gordon Street, London WC1H 0AH, United Kingdom*

<sup>2</sup>*Condensed Matter Physics & Materials Science Division,  
Brookhaven National Laboratory, Upton, NY 11973-5000, USA*

<sup>3</sup>*Institute for Theoretical Physics, University of Amsterdam,  
Science Park 904, 1098 XH Amsterdam, The Netherlands*

We show that confinement in the quantum Ising model leads to nonthermal eigenstates, in both continuum and lattice theories, in both one (1D) and two dimensions (2D). In the ordered phase, the presence of a confining longitudinal field leads to a profound restructuring of the excitation spectrum, with the low-energy two-particle continuum being replaced by discrete ‘meson’ modes (linearly confined pairs of domain walls). These modes exist far into the spectrum and are atypical, in the sense that expectation values in the state with energy  $E$  do not agree with the microcanonical (thermal) ensemble prediction. Single meson states persist above the two meson threshold, due to a surprising lack of hybridization with the ( $n \geq 4$ )-domain wall continuum, a result that survives into the thermodynamic limit and that can be understood from analytical calculations. The presence of such states is revealed in anomalous post-quench dynamics, such as the lack of a light cone, the suppression of the growth of entanglement entropy, and the absence of thermalization for some initial states. The nonthermal states are confined to the ordered phase – the disordered (paramagnetic) phase exhibits typical thermalization patterns in both 1D and 2D in the absence of integrability.

**Introduction:** Thermalization, and the associated scrambling of information, is considered to be a generic feature of isolated quantum systems. Understanding how to avoid this, and thus preserve quantum information on long time scales, may prove useful in the development of quantum computing technologies. Two well-studied counter examples to thermalization are known: integrable quantum systems [1–3] and many-body localization [4–6]. In both cases the existence of many local conservation laws allows the system to retain an extensive amount of information and so avoid thermalization [2, 7]. The question of whether integrability is a crucial ingredient in preventing thermalization and scrambling of information has recently attracted much attention with the realization that kinetic constraints, in the absence of integrability, can also help avoid it [8–17]. In this Letter, we will show that thermalization can be avoided in quantum magnets that lack both integrable and kinetic constraints. This occurs in both 1D and 2D, which suggests that nonthermal behavior may be commonplace.

At the heart of understanding thermalization in isolated quantum many-body systems is the eigenstate thermalization hypothesis (ETH) [18–20]. This gives a simple set of criteria under which eigenstate expectation values (EEVs) of local operators agree with the thermal prediction [18–36]. Of importance for this work, ETH proposes that the EEV of a local operator in a state with energy  $E$  becomes a smooth function of  $E$  as the system size increases, with the spread in EEVs at a fixed  $E$  shrinking to zero [20, 21]. The EEV is then, by construction, thermal and coincides with the microcanonical prediction [37].

It is known, however, that in finite systems nonthermal states that violate ETH can also exist [26, 27, 38–41], usually being observed at the very edges of the spectrum

(though not always [38]). The presence of such states can have important consequences for nonequilibrium dynamics [20, 40–44], in particular leading to an absence of thermalization following a quantum quench [39, 41]. Thermalization is used here in the sense that expectations values in the long-time limit agree with the thermal result [22, 23]. Such predictions can now be routinely tested in cold atomic gases, following ground breaking progress in isolating and controlling these systems [45–57]. We also expect, with the ability to probe electronic degrees of freedom on femtosecond time scales [58], to be able to study questions of thermalization in materials, unaffected by the electron-phonon coupling.

In this Letter, we show that nonthermal states exist away from the edges of the spectrum in paradigmatic models of quantum magnetism, both in 1D and 2D. These states are present both on the lattice and in the continuum limit, and the fraction of these states compared to the Hilbert space dimension is consistent with a weak version of ETH [29, 39, 44]. In the continuum limit, which is not usually the subject of ETH studies, we harness powerful numerical techniques [59] to look at large system sizes, and present systematic analytical calculations that support our results. On the lattice, we use matrix product state methods [60] to show that the observed physics is not a remnant of the scaling limit and so may be possible to probe in experiments on low-dimensional quantum magnets (see, e.g., [61–64]).

**1D lattice and continuum theories:** Let us focus on a particular example of a theory with confinement, the quantum Ising chain with an additional longitudinal field,

$$H_{\text{latt}} = \sum_{j=1}^N J \sigma_j^z \sigma_{j+1}^z + h_x \sigma_j^x + h_z \sigma_j^z. \quad (1)$$

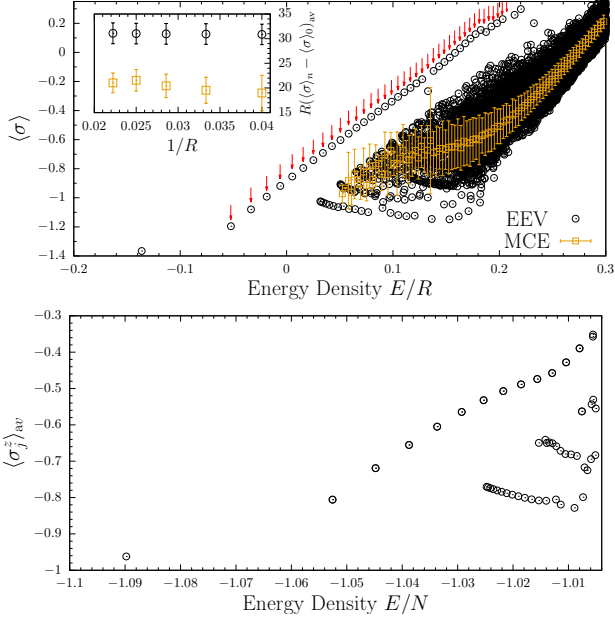


FIG. 1. **(Upper)** EEV spectrum of the spin operator  $\sigma(0)$  as a function of energy  $E$  for the 1D Ising field theory (2) with  $m = 1$ ,  $g = 0.1$ ,  $R = 35$ . Arrows show the energies of the first forty meson states [69]. The MCE result is shown within the continuum, with error bars denoting the standard deviation of results averaged over. **(Inset)** The average magnetization for the  $n = 11 - 15$  meson states compared to the MCE at the same average energy for a number of volumes  $R$ . **(Lower)** EEV spectrum of  $\sum_j \sigma_j^z/N$  as a function of energy  $E$  in the 1D lattice model (1) with  $J = -1$ ,  $h_x = -0.5$ ,  $h_z = 0.05$  for  $N = 40$  sites, computed with DMRG for open boundary conditions. The nonthermal states are meson-like and confined to the vicinity of a boundary.

Here  $\sigma_j^\alpha$  ( $\alpha = x, y, z$ ) are the Pauli matrices acting on the  $j$ th site of the chain,  $J$  is the Ising exchange parameter, and  $h_x$  ( $h_z$ ) is the transverse (longitudinal) field strength. Taking the scaling limit in the vicinity of the critical point ( $h_x = 1$ ,  $h_z = 0$ ), one arrives at the field theory [65, 66]

$$H_{\text{ft}} = \int_0^R dx \left( \bar{\psi} \partial_x \psi - \psi \partial_x \bar{\psi} + im \bar{\psi} \psi + g \sigma \right). \quad (2)$$

Here  $R$  is the system size,  $\bar{\psi}$  ( $\psi$ ) are right (left) moving Majorana fermion fields,  $m$  is the fermion mass ( $m \sim 1 - h_x$ ),  $g$  is the continuum longitudinal field, and  $\sigma(x)$  is the spin operator in the continuum. For generic values of the parameters, both the lattice (1) and the continuum (2) models are nonintegrable [67, 68]. Herein we (mostly) focus on the ordered phase,  $|h_x| < 1$  and  $m > 0$ .

In the absence of a longitudinal field ( $h_z = 0$ ,  $g = 0$ ) low-energy excitations are spin flips (costing energy  $\sim 2m$ ), which fractionalize into pairs of domain walls that are free to independently propagate. Thus, at low-energies, above energy  $2m$  there is a continuum of two-particle states. The presence of a longitudinal field  $h_z \neq 0$ ,  $g \neq 0$  profoundly changes this. The energy

cost of a domain of flipped spins now grows linearly in the size of the domain. This confining potential between domain walls (much like quarks in quantum chromodynamics (QCD) [70]) leads to a collapse of the low-energy continuum into discrete ‘meson’ excitations, formed from pairs of domain walls [71, 72]. This has been observed in two quasi-1D quantum magnets,  $\text{CoNb}_2\text{O}_6$  [61, 62] and  $\text{SrCo}_2\text{V}_2\text{O}_8$  [63, 64].

The presence of confinement leads to nonthermal states appearing within the spectrum, despite the system being nonintegrable. To show this, we construct eigenstates of the two models, (1) and (2), and measure the average magnetization within each state [73]. On the lattice, we do this via the density matrix renormalization group (DMRG) [60] by targeting up to 100 low-lying eigenstates [74]. In the continuum we use truncated spectrum methods [59] to construct thousands of low-lying eigenstates [75]. Example results are shown in Fig. 1; we see that there are two major features in the EEV spectrum. Firstly, there is a thermal-like continuum of excitations on the right hand side of the plot (confirmed by comparison with the microcanonical ensemble (MCE) in the continuum). With increasing system size, this continuum narrows as predicted from ETH, see [41]. Secondly, there is a line of states that is well separated and above this continuum (see the arrows in both plots), whose EEVs do not coincide with the MCE results. These states remain separated from the thermal continuum up to the largest system sizes that we can reach; extrapolation to infinite volume is consistent with the nonthermal states possessing a different magnetization to the MCE, as shown in the inset. These features are seen in both the continuum and on the lattice; the similarity between the two panels in Fig. 1 is striking.

One advantage of tackling this problem in the continuum is that we have well-controlled analytical approaches, as well as the numerical data, that allow us to understand these nonthermal states. For example, in the upper panel we draw arrows at the energies of the meson (linearly-confined domain walls) excitations, as predicted from a semiclassical analysis [69, 76, 77]. We see that these coincide exactly with the nonthermal states. We also have direct access to the wave functions, and see that these states are well described by the two (domain wall) fermion sector of the theory [59]. The nonthermal states are well approximated by the meson form:

$$|M_n\rangle = \sum_{\nu=\text{NS,R}} \sum_{p_\nu} \Psi_n(p_\nu) a_{p_\nu}^\dagger a_{-p_\nu}^\dagger |\nu\rangle, \quad (3)$$

where  $a_{p_\nu}^\dagger$  creates a fermion of momentum  $p_\nu$  in the  $\nu = \text{NS, R}$  (Neveu-Schwarz (half-integer momenta moding) and Ramond (integer momenta moding) respectively) sector of the Hilbert space [78],  $|\nu\rangle$  is the vacuum within a given sector. The wave function,  $\Psi_n(p)$ , and the mass of the meson,  $M_n$ , can be determined analytically via the Bethe-Salpeter equation, see [79] for details.

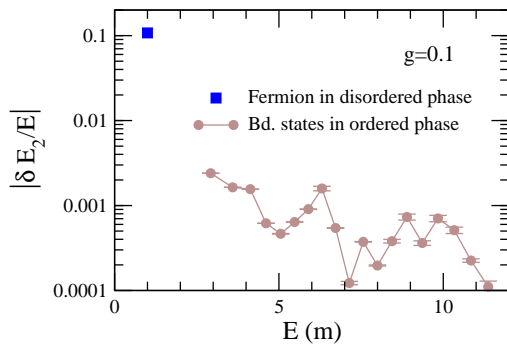


FIG. 2. We plot the relative second order correction to the energy of the first 19 zero momentum mesons ( $\delta E_2/E$ ) coming from mixing with 0 and 4 domain wall states in the 1D Ising field theory (2) with  $m = 1$  and  $g = 0.1$ . For comparison we also plot the energy correction to the zero momentum spin flip excitation in the disordered phase. Note that the correction for the mesons is from  $10^{-2}$  to  $10^{-4}$  that of spin flip excitation.

**Meson stability above thresholds:** The persistence of well-separated single meson excitations above the two-meson threshold is, at first glance, surprising. Analogously to QCD (see, e.g., [70]), one might expect these single mesons to be unstable, with open decay channels to multimeson states. As shown in Fig. 1, this intuition is incorrect. To shed some light on this, we consider the (two domain wall) meson excitations, described by (3), and compute the second order energy correction that comes from hybridization with four domain wall states and vacuum [80]. Below we will see that this correction is exceedingly small compared to the bound state energy,  $E = M_n$ , in contrast to the second order correction coming from the spin flip excitations within the disordered (paramagnetic) phase, where confinement is absent [81]. We find that the meson corrections are orders of magnitude smaller than those of the paramagnetic spin flip.

We give explicit details of the second order energy computation in the supplemental (see also [41]), only schematically sketching the calculation here. The problem is split into three parts,  $H = H_{\text{meson}} + H_{\text{free}} + H_{\text{int}}$ . The first part,  $H_{\text{meson}}$ , describes the single meson part of the problem, whose eigenstates are given by (3). In  $H_{\text{free}}$  we describe the non-interacting part of all the other ( $n > 2$ ) fermion sectors of the theory. Finally,  $H_{\text{int}}$  contains all interaction vertices, except the two-fermion-to-two-fermion case, which was taken into account in  $H_{\text{meson}}$ . We specifically consider vertices involve two-to-four fermions and two-to-zero fermions. A similar calculation is performed for the second order correction of the single particle excitations in the disordered phase.

We present results of our computations in Fig. 2, showing the relative second order corrections to the zero momentum energy for the first nineteen meson states (brown circles). For comparison we present the corresponding

computation for the zero momentum fermion in the disordered phase (blue square). We see that the energy corrections for all of the mesons range from  $10^{-5}$  to  $10^{-3}$  of their unperturbed energy. Moreover the energy corrections for those mesons which lie above the four domain wall continuum, i.e. that are not nominally kinematically stable, are no larger than those below the threshold. We also see that the meson energy corrections are at least two orders of magnitude smaller than the correction of the fermion (spin flip) for the disordered phase. Thus the meson excitations, states of the form (3), appear quasi-stable to mixing with four domain wall states. This supports the results of the previous section; by slightly dressing the states (3), we form *completely stable* nonthermal states in the finite volume. Note that this is counter to the usual intuition from QCD, where one would expect the single meson to be unstable to kinematic decay above the two meson threshold. Even at higher orders in perturbation theory, the two domain wall sector of the theory appears to continue to mix only very weakly with the sectors containing  $n \geq 4$  domain walls, despite there being scattering processes induced by the longitudinal magnetic field that remain finite into the thermodynamic limit. While we have not extended our second order correction to account for mixing with six domain wall states explicitly, we expect such mixing to be considerably smaller because of phase space considerations [82].

**Extension to 2D:** Surprisingly, the above analysis in 1D extends in a straightforward manner to higher dimensions. Consider the following 2D Hamiltonian:

$$H_{2D} = \sum_j \left( H_j + J_{\perp} \int_0^R dx \sigma_j(x) \sigma_{j+1}(x) \right), \quad (4)$$

formed from individual Ising continuum chains

$$H_j = \int_0^R dx \left( \bar{\psi}_j \partial_x \bar{\psi}_j - \psi_j \partial_x \psi_j + im \bar{\psi}_j \psi_j \right), \quad (5)$$

coupled by a local spin-spin interaction of strength  $J_{\perp}$ . For this system the coupling  $J_{\perp}$  between neighboring ordered ( $m > 0$ ) chains provides a confining potential. Meson-like approximate eigenstates of (4), of the form

$$|E_n\rangle = \sum_{\nu_k = \text{NS,R}} \sum_{j=1}^N \sum_{p_{\nu_j}} \Psi_n^{\{\nu\}}(p_{\nu_j}) A_j^{\dagger}(p_{\nu_j}) A_j^{\dagger}(-p_{\nu_j}) |\{\nu\}\rangle, \quad (6)$$

can be found via an analogous Bethe-Salpeter equation [83]. Here  $N$  is the number of chains,  $A_j^{\dagger}(p_{\nu})$  creates a fermion in the  $j$ th chain with momentum  $p_{\nu}$  in the  $\nu = \text{NS,R}$  sector, and  $|\{\nu\}\rangle = \otimes_{j=1}^N |\nu_j\rangle$  are the vacuum states of the system, formed from the individual  $\nu_j$ -vacua in each chain. The physical character of the wave function  $\Psi_n^{\{\nu\}}(p_{\nu_j})$  is similar to the 1D case, Eq. (3).

With meson states (6) (i.e., approximate two fermion eigenstates) defined, one can proceed in a similar manner to the previous section, and compute their self-energies. This calculation is essentially identical to the previous case, leading us to conclude that meson excitations in 2D are extremely long-lived excitations. We can no longer construct the EEV spectrum in 2D (cf. Fig. 1 in 1D), but a mean field decoupling of the 2D system into 1D chains suggests that these meson-like excitations should behave similarly to those analogous excitations in 1D, i.e. they are nonthermal states. In the next section we provide further evidence of this.

**Nonequilibrium dynamics in 2D:** Having argued that nonthermal states exist in the 2D theory with confinement, Eq. (4), we now support this with evidence that the nonequilibrium dynamics is anomalous [84]. This is one of the signatures of the presence of nonthermal states in the spectrum. Nonequilibrium dynamics is induced by a quench of the interaction  $J_{\perp} = 0 \rightarrow J_{\perp} \neq 0$ . Both the initial state and subsequent time-evolution are computed in the chain array matrix product state (ChainAMPS) framework [59]. This methodology blends truncated spectrum methods with MPS algorithms, and has been used to study the entanglement entropy and spectrum of the 2D Ising model [85], and to compute the time-evolution following a quantum quench [86].

In Fig. 3 we present results for the time-dependence of the connected spin correlation function between chains,  $|\langle \sigma_{i+y}(x, t) \sigma_i(x, t) \rangle - \langle \sigma_{i+y}(x, t) \rangle \langle \sigma_i(x, t) \rangle|$ , and the entanglement entropy  $S_E$  for quenches from the  $J_{\perp} = 0$  ground state to  $J_{\perp} \neq 0$ , for both ordered ( $m > 0$ ) and disordered ( $m < 0$ ) chains. Here  $S_E$  is defined as the Von Neumann entanglement when the system is partitioned into two semi infinite arrays of chains. For ordered chains the correlation function does not show the usual light cone behavior following the quench, with response instead being strongly suppressed and correlations remaining local. In the presence of confinement this is consistent with the quasiparticle picture of Calabrese and Cardy [87, 88]: the quench generates pairs of quasiparticles with opposing momenta (forming mesons), which propagate away from one another. At fixed energy density (as set by the quench), the particles can only separate a finite distance before the confinement potential saturates the available energy, and hence the light cone is suppressed. In contrast, the disordered case, where confinement is absent, displays a clear light cone spread of correlations. This suppression of the propagation of quasiparticles also impacts the growth of  $S_E$  (with entanglement being carried by these quasiparticles), as is shown in Fig. 3.

Before concluding, we note that similar effects have been observed in the 1D nonequilibrium dynamics of (1) and (2). In the lattice problem (1), Kormos *et al.* [89] observed both a suppression of the light cone and the growth of the entanglement entropy following a global

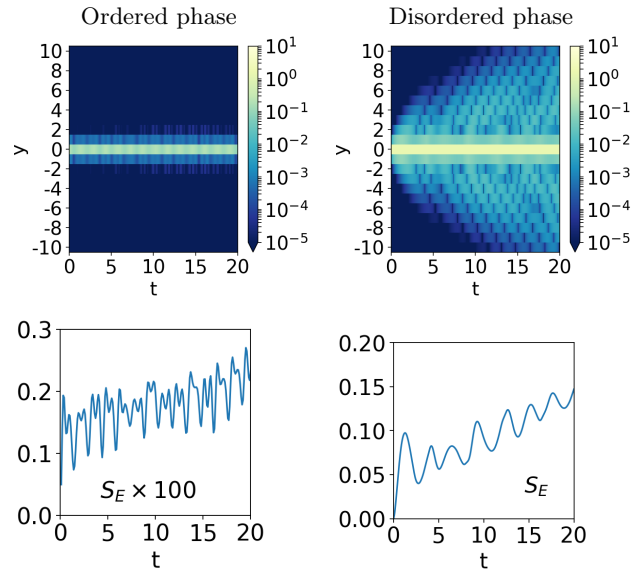


FIG. 3. **(Upper)** The time-evolution of the connected correlation function,  $|\langle \sigma_{i+y}(x, t) \sigma_i(x, t) \rangle - \langle \sigma_{i+y}(x, t) \rangle \langle \sigma_i(x, t) \rangle|$ , following quenches in the 2D quantum Ising model (4) for  $R = 8$ ,  $J_{\perp} = 0$  to  $-0.15$  and (left) ordered chains  $m = 1$ , (right) disordered chains  $m = -1$ . Both cases start from a  $J_{\perp} = 0$  ground state. Dynamics are computed via ChainAMPS [59]. **(Lower)** The time-evolution of the entanglement entropy  $S_E$  following the same quenches. Note the  $y$ -axis of the lower left panel has been increased by a factor of 100. A detailed discussion of the simulations is provided in the Supplemental.

quantum quench. Nonequilibrium dynamics following quenches in the field theory (2) have also shown clear signatures of the meson excitations [41, 90, 91].

**Conclusions:** In this Letter, we have seen that nonthermal states appear in the Ising model, in 1D and 2D, when confinement is present. The nonthermal states have EEVs that do not match the MCE prediction, highlighting their nonthermal nature, despite an absence of integrability. We saw this very explicitly in Fig. 1, in both the continuum and on the lattice, by computing the EEV spectrum of the longitudinal magnetization.

We identified the nonthermal states as being meson-like, in that the state is well approximated by linearly confined pairs of domain walls, as expressed in Eqs. (3) and (6). The mesons hybridize only very weakly with the thermal continuum of multimeson states, see Fig. 2. From controlled numerical and analytical calculation in 1D, we turned our attention to 2D and argued that such meson states exist there, with essentially the same calculations applying in 1D and 2D. The presence of such nonthermal states can lead to anomalous nonequilibrium dynamics, illustrated in Fig. 3, such as suppression of the lightcone and entanglement growth, as well as an absence of thermalization [92] (for a recent example of this in a quantum quench of a 1D lattice model, see Ref. [93]).

While we have focused on Ising models in 1D and 2D, it is natural to expect that the physics of nonthermal states

carries over to other theories with confinement. Recently, holographic theories with confinement have shown an absence of thermalization [94], a hallmark of the presence of nonthermal states. A natural test of this conjecture could be provided by the Schwinger model in an electric field, which has been the subject of a number of recent works [95–100] (the disordered Schwinger model has also recently been shown to display confinement driven non-ergodic behavior [101, 102]).

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\* [andrew.james@ucl.ac.uk](mailto:andrew.james@ucl.ac.uk)

† [rmk@bnl.gov](mailto:rmk@bnl.gov)

‡ [n.j.robinson@uva.nl](mailto:n.j.robinson@uva.nl)

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- [75] For Fig. 1 we use the truncated spectrum approach, with energy cut-off  $E_\Lambda = 10.5$ , corresponding to constructing the Hamiltonian with the lowest 23,500 basis states. The MCE was constructed by averaging over states within an energy window of size  $\Delta E = 0.1$ . We provide some further details of the truncated spectrum approach in Section S2 of the Supplemental Material [103], and we also refer the reader to the recent review article [59]. We note that the truncation effects affect data on the right hand side of our plot, Fig. 1, above energy densities of  $E/R \sim 0.2$ . Further information about truncation effects, convergence, etc., is presented in Ref. [41].
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