Quantum Rabi Model with Two-Photon Relaxation

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Quantum Rabi model with two-photon relaxation

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We study a cavity-QED setup consisting of a two-level system coupled to a single cavity mode with two-photon relaxation. The system dynamics is modeled via a Lindblad master equation consisting of the Rabi Hamiltonian and a two-photon dissipator. We show that an even-photon relaxation preserves the Z_2-symmetry of the Rabi model, and provide a framework to study the corresponding non-Hermitian dynamics in the number-parity basis. We discuss the role of different terms in the two-photon dissipator and show how one can extend existing results for the closed Rabi spectrum to the open case. Furthermore, we characterize the role of the Z_2-symmetry in the excitation-relaxation dynamics of the system as a function of light-matter coupling. Importantly, we observe that initial states with even/odd parity manifest qualitatively distinct transient and steady state behaviors, contrary to the Hermitian dynamics that is only sensitive to whether the initial state is parity-invariant or not. Moreover, the parity-sensitive dynamical behavior is not a creature of ultrastrong coupling and is present even at weak coupling values.

Introduction. The Rabi model [1] describes the quantum interaction between a two-level system (TLS) and a bosonic mode. Despite its simple form, the Rabi model represents an important theoretical building block of quantized matter-field interactions and quantum information processing. It is applicable to a broad range of quantum phenomena spanning microscopic to mesoscopic systems, finding realizations in a wide range of quantum platforms, including cavity-QED [2–5], circuit-QED [6–11], nanoelectromechanical [12–15], quantum-dot [16], and trapped-ion [17, 18] systems.

Light-matter interactions within the Rabi model consist of rotating (resonant) and counter-rotating (non-resonant) contributions. Traditionally, Rabi dynamics is analyzed under the rotating-wave approximation (RWA), resulting in the simplified Jaynes-Cummings (JC) model [19], valid when the coupling constant is much weaker than the TLS and mode frequencies. From the perspective of symmetry, RWA fictitiously extends the Z_2-symmetry of the model to a $U(1)$-symmetry, making the total excitation number the second conserved quantity besides the Hamiltonian and therefore facilitates analytical solutions. The JC model has been employed successfully to describe the dynamics of most cavity-QED setups [2–5]. However, with the advent of superconducting quantum devices, it has become feasible to reach ultrastrong [20, 21] and, more recently, deep-strong [22] regimes of interactions. The breakdown of RWA in these regimes motivated various theoretical efforts to revise the Rabi model. First, generalized versions of RWA [23, 24] were introduced that captures correctly stronger couplings. Second, despite the contemporary understanding, Braak [25] argued that the Z_2-symmetry of the Rabi model is sufficient for its integrability, showing that the regular spectrum in each parity subspace can be obtained from the roots of a transcendental function. Moreover, Chen et. al provided a more physical derivation of the Rabi spectrum using Bugoliubov transformations [26], contrary to the Bargmann representation [27] employed by Braak. These early studies paved the way toward ongoing developements of analytical and perturbative methods for determining the spectrum, eigenmodes, and dynamics of the Rabi model under different parameter regimes [28–32].

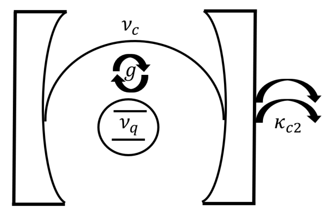

FIG. 1. Schematic of system consisting of a two-level system coupled to a single cavity mode with two-photon relaxation. We discuss possible physical realization of such a relaxation process in the Supplementary Material (SM), revisiting Refs. [33, 34]

To date, however, most works have focused primarily on ideal, closed (Hermitian) properties of the Rabi model, while role of Z_2-symmetry in realistic, open (non-Hermitian) scenarios remains an open question. An even exchange of excitations between a cavity mode and environment conserves the Z_2-symmetry. The latter is particularly important given emerging studies of the dynamics of a single cavity mode under two-photon relaxation [35–42]. Such a relaxation process has been recently implemented in circuit-QED [34] following a four-wave mixing scheme proposed first by Wolinsky and Carmichael [33] (See SM [43]). A major motivation behind recent studies of even-photon relaxation processes is their application to realization of dynamically protected, universal quantum computing paradigms [44–47], in which the quantum information is encoded in logical qubits consisting of Schrödinger cat states with distinct parity that exhibit reliable protection to photon dephasing and single-photon relaxation errors [44].

In this article, we generalize the theory of Z_2-symmetry of the Rabi model to the open quantum case. We first review the spectrum of the closed Rabi Hamiltonian, providing analytical recursion relations for both the eigenfrequency and eigenmodes of the system. Our analysis and calculation are performed in the (cavity) number-(overall) parity representation [48], where the Z_2-symmetry of the model is explicit. For the open scenario, we consider a Lindblad master equation [49, 50] of the Rabi Hamiltonian with two-photon dissipation for the cavity mode. To analyze its spectral properties, we employ an effective Hamiltonian obtained by keeping diagonal decay terms, while neglecting the off-diagonal collapse in the two-photon dissipator. This phenomenological treatment provides a reliable approximation to the complex eigenfrequencies, but not necessarily the eigenmodes and ground state. While an exact definition of a full effective Hamiltonian exists, mapping the Lindblad dynamics into a norm-preserving Schrodinger equation [51], analytical treatment of its associated spectrum seems prohibitive due to its significantly larger Hilbert space compared to the phenomenological model (See SM for comparison). We follow numerical integration of the Lindblad equation for studying the dynamics, while the effective phenomenological Hamiltonian is primarily used for approximate analytical discussion of the spectrum and a better understanding of the observed dynamics.

Model. Our system consists of a TLS coupled to a single cavity mode, engineered such that single-photon is negligible compared to two-photon relaxation, constraining it to exchange only pairs of photons with the environment (Fig. 1). We model the system dynamics via the Linbdlad equation:

$$\dot{\hat{\rho}}(t) = -i[\hat{H}_s, \hat{\rho}(t)] + 2\kappa_{c2}\mathcal{D}[\hat{a}^2]\hat{\rho}(t), \tag{1a}$$

$$\hat{H}_{\mathrm{s}} \equiv \nu_c \hat{a}^\dagger \hat{a} + \frac{\nu_q}{2}\hat{\sigma}^z + g\left(\hat{a}+\hat{a}^\dagger\right)\left(\hat{\sigma}^- + \hat{\sigma}^+\right), \tag{1b}$$

with ν_q, ν_c, and g denoting the qubit frequency, cavity frequency, and light-matter coupling, respectively. Two-photon relaxation is described via the dissipator, $\mathcal{D}[\hat{a}^2](\bullet) = \hat{a}^2(\bullet)(\hat{a}^\dagger)^2 - \frac{1}{2}\left\{(\hat{a}^\dagger)^2\hat{a}^2, (\bullet)\right\}$, with κ_{c2} denoting the two-photon relaxation rate.

We next transform the Lindblad Eq. (1a) such that the Z_2-symmetry of the Rabi Hamiltonian and two-photon relaxation become explicit (See SM). In particular, we define the overall parity operator for the system as:

$$\hat{P} = \hat{P}_q\hat{P}_c = e^{i\pi\hat{\sigma}^+\hat{\sigma}^-}e^{i\pi\hat{a}^\dagger\hat{a}} = -\hat{\sigma}^z e^{i\pi\hat{a}^\dagger\hat{a}}. \tag{2}$$

The Z_2-symmetry of the Rabi Hamiltonian (1b) means that $\hat{P}^\dagger\hat{H}_s\hat{P} = \hat{H}_s$. Consequently, the Hilbert space can be partitioned into parity subspaces having even (plus) and odd (minus) total excitation numbers:

$$p = +1: \ \{|0,g\rangle, |1,e\rangle, |2,g\rangle, |3,e\rangle, |4,g\rangle, \ldots\}, \tag{3a}$$

$$p = -1: \ \{|0,e\rangle, |1,g\rangle, |2,e\rangle, |3,g\rangle, |4,e\rangle, \ldots\}. \tag{3b}$$

number-excitation basis	$\vert n,g\rangle$	$\vert n,e\rangle$
number-parity basis	$\vert n,(-1)^n\rangle$	$\vert n,(-1)^{n+1}\rangle$

TABLE I. Correspondence between (cavity) number-(qubit) excitation and (cavity) number- (overall) parity bases.

The adjacent states in each subspace are coupled via both rotating or the counter-rotating terms. If we neglect the latter, each subspace is reduced into a collection of number-conserving Jaynes-Cummings doublets $\{|n-1,e\rangle, |n,g\rangle\}$, given by:

$$p = +1: \ \{|0,g\rangle\}, \{|1,e\rangle, |2,g\rangle\}, \{|3,e\rangle, |4,g\rangle\}, \ldots, \tag{4a}$$

$$p = -1: \ \{|0,e\rangle, |1,g\rangle\}, \{|2,e\rangle, |3,g\rangle\}, \ldots. \tag{4b}$$

Defining a new set of bosonic operators, $\hat{b} \equiv \hat{\sigma}^x\hat{a}$, and replacing $\hat{\sigma}^z$ in terms of the parity operator of Eq. (2), one can rewrite that the Rabi Hamiltonian (1b) as [48]

$$\hat{H}_{\mathrm{s}} = \nu_c\hat{b}^\dagger\hat{b} - \frac{\nu_q}{2}e^{i\pi\hat{b}^\dagger\hat{b}}\hat{P} + g\left(\hat{b}+\hat{b}^\dagger\right). \tag{5}$$

The parity $\hat{P}$ and bosonic $\hat{b}$ operators commute, thus provide a complete basis for the Hilbert space defined as $\hat{b}^\dagger\hat{b}|n,p\rangle = n|n,p\rangle$ and $\hat{P}|n,p\rangle = p|n,p\rangle$ for $n = 0,1,2,\ldots$ and $p = \pm 1$, respectively. Table I summarizes the correspondence between the (old) number-excitation and (new) number-parity bases.

Next, we rewrite the original Lindblad Eq. (1a) in this basis, starting by observing that the two-photon dissipator is also invariant under the parity transformation, i.e. $\hat{P}^\dagger\mathcal{D}[\hat{a}^2]\hat{P} = \mathcal{D}[(-\hat{a})^2] = \mathcal{D}[\hat{a}^2]$, where $\hat{a}^2 = (\hat{\sigma}^x\hat{a})^2 = \hat{b}^2$ also implies that $\mathcal{D}[\hat{a}^2] = \mathcal{D}[\hat{b}^2]$. In the quantum treatment of dissipation, the two contributions to the dissipator are described by decay and collapse terms. The former represents the rate at which a quantum state loses probability while the latter represents the rate at which lower states in the excitation ladder receive probability, in such a way that the net probability is conserved in time, i.e. $\mathrm{Tr}\left(\mathcal{D}[\hat{b}^2]\hat{\rho}\right) = 0$. Separating the two contributions, one can re-express the Lindblad Eq. (1a) to yield,

$$\dot{\hat{\rho}}(t) = -i\left[\hat{H}_{\mathrm{s,ef}}\hat{\rho}(t) - \hat{\rho}(t)\hat{H}^\dagger_{\mathrm{s,ef}}\right] + 2\kappa_{c2}\hat{b}^2\hat{\rho}(t)(\hat{b}^\dagger)^2, \tag{6a}$$

with $\hat{H}_{s,\mathrm{ef}}$ denoting the phenomenological effective Hamiltonian as

$$\hat{H}_{\mathrm{s,ef}} = \nu_c\hat{b}^\dagger\hat{b} - \frac{\nu_q}{2}e^{i\pi\hat{b}^\dagger\hat{b}}\hat{P} + g\left(\hat{b}+\hat{b}^\dagger\right) - i\kappa_{c2}(\hat{b}^\dagger)^2\hat{b}^2. \tag{6b}$$

Neglecting the coupling induced by collapse, the last term in Eq. (6a), the dissipative dynamics is approximated by $\hat{H}_{\mathrm{s,ef}}$. This framework is a middle ground in which the unitary part of the system dynamics is treated quantum

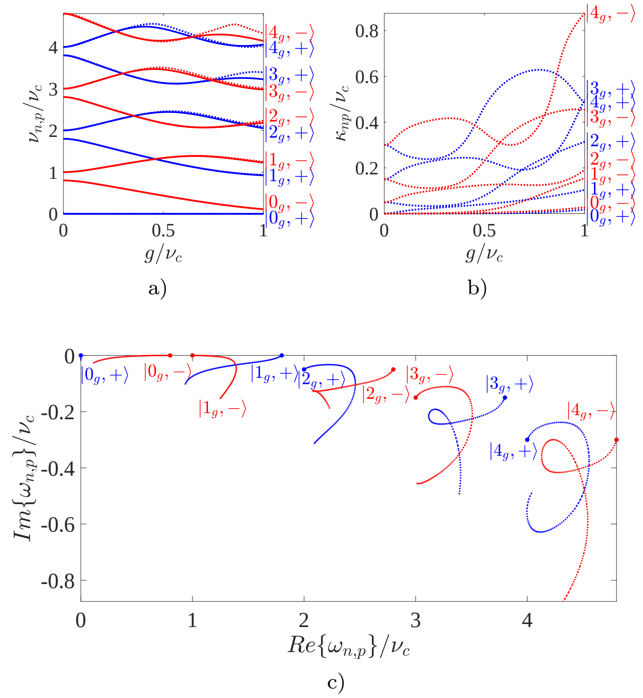

FIG. 2. Phenomenological open Rabi eigenfrequencies $\omega_{n,p} \equiv \nu_{n,p} - i\kappa_{n,p}$ for $\nu_q = 0.8\nu_c$ and $\kappa_{c2} = \nu_c/40$. a) Real part (frequency), b) imaginary part (decay rate), and c) complex spectrum as a function of light-matter coupling g. Solid lines in a) show the result for the closed ($\kappa_{c2} = 0$) case while dotted lines are for $\kappa_{c2} = \nu_c/40$. The labels $|n_g, \pm\rangle$ in a) and b) are ordered based on values at $g = 0$. The frequencies in a) are plotted relative to the ground state $|0_g, +\rangle$.

mechanically, while the dissipation is treated phenomenologically. Essentially, such an approach provides a good approximation for the complex spectrum of the problem, while ignoring proper characterization of the modal and ground state information (See Sec. IV of the SM for further discussion).

Spectrum. Here, we first revisit the spectrum of the closed Rabi model and benchmark our solution against those of Braak [25]. For the open case, we study the impact of two-photon relaxation via $\hat{H}_{\mathrm{s,ef}}$ of Eq. (6b). In particular, we show that the typical solution obtained for the closed case can be generalized to yield the complex eigenfrequencies of the open system.

We begin with the eigenvalue problem for the closed Rabi model, $\hat{H}_{\mathrm{s,p}} |n_g, p\rangle = \omega_{np} |n_g, p\rangle$, where n_g labels the eigenvalue/eigenmodes at a nonzero g and p is the corresponding parity subspace. Expanding the unknown eigenmodes in terms of the number-parity basis, $|n_g, p\rangle = \sum_{m=0}^{\infty} c_{np,m} |m, p\rangle$, one finds that the eigenfrequencies ω_{np} are obtained by the roots $G_p(\omega_{np}) = 0$, where $G_p \equiv \lim_{m\to\infty} G_{p,m}$ and $G_{p,m}$ satisfies the following recursion relation (See SM):

$$G_{p,m} = \alpha_{np,m} G_{p,m-1} - \beta_{p,m-1}\gamma_{p,m} G_{p,m-2}, \tag{7}$$

subject to initial conditions, $G_{p,0} = \alpha_{p,0}$ and $G_{p,1} = \alpha_{p,0}\alpha_{p,1} - \beta_{p,0}\gamma_{p,1}$. The coefficients in the recursion Eq. (7) read

$$\begin{aligned} \alpha_{np,m} &\equiv \omega_{np} - m\nu_c + \frac{p}{2}(-1)^m \nu_q, \\ \beta_{p,m} &\equiv -\sqrt{m+1}g, \;\; \gamma_{p,m} \equiv -\sqrt{m}g. \end{aligned} \tag{8}$$

Similarly, the corresponding eigenmodes are determined by yet another recursion relation for the probability amplitudes $c_{np,m}$, given by:

$$\alpha_{np,m} c_{np,m} + \beta_{p,m} c_{np,m+1} + \gamma_{p,m} c_{np,m-1} = 0, \tag{9}$$

with initial conditions, $\alpha_{p,0} c_{p,0} + \beta_{p,0} c_{p,1} = 0$. An illustrative example of the variation of the spectrum with respect to g is shown in Fig. (2a), with parameters chosen to compare our results with those in Fig. 2 of Ref. [25].

Within the phenomenological treatment of relaxation, the system dynamics are determined by $\hat{H}_{\mathrm{s,ef}}$ of Eq. (6b). Here, we find that the recursion relations determining the eigenfrequencies (7) and eigenmodes (9) have the same form as those of the closed system, except that the coefficients $\alpha_{np,m}$ are modified as $\alpha_{np,m} \to \alpha_{np,m} + im(m-1)\kappa_{c2}$ (See SM). To understand the changes induced by phenomenological decay (compared to the closed), we first consider the regime of zero coupling $g = 0$, where the decay terms are diagonal in the number basis. In this scenario, the mth bare cavity mode acquires a decay rate of $\kappa_{c2} m(m-1)$, resulting in nonzero values for all cavity number states except the ground and first-excited state, for each parity. As the coupling g is turned on, the hybridization between the qubit and the cavity mode allows these terms not only to induce additional decay, but also modify the real frequency of each state. Figure 2 (dotted lines) shows such hybridization as a function of g, as calculated by the phenomenological model. We note that an analogous phenomenological model based on the JC model can be solved analytically and result in decay rates that plateau at ultrastrong coupling values of g and hence micharacterizes the interplay of light-matter coupling and two-photon relaxation (See SM for comparison).

Excitation-relaxation dynamics. Here, we study the dissipative dynamics of the system and discuss the role of Z_2-symmetry. For concreteness, we consider the situation in which the cavity is initially prepared with even/odd number of photons, and describe the ensuing dynamics of the cavity photon and qubit population as a function of both time and g. In particular, we consider two scenarios of starting with two ($\hat{\rho}(0) = |2, g\rangle\langle 2, g|$) or three ($\hat{\rho}(0) = |3, g\rangle\langle 3, g|$) initial cavity photons and qubit in ground state, as representatives of the plus/minus parity subspaces. Due to pair-exchange of photons with the

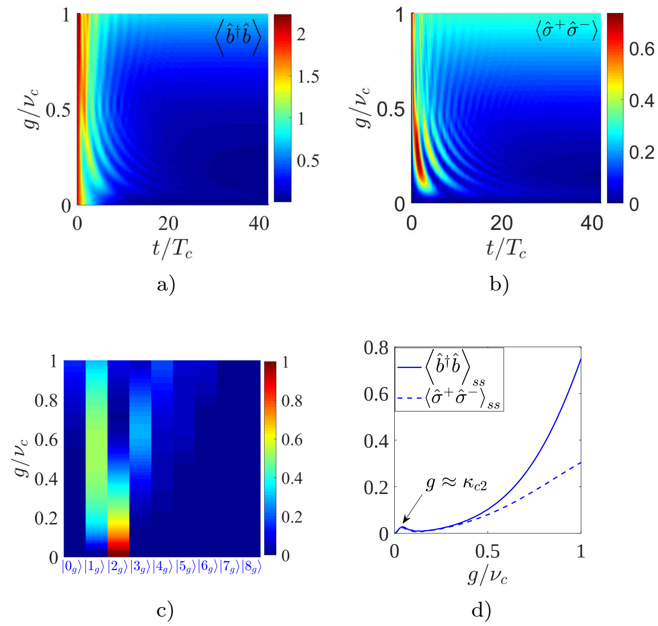

FIG. 3. Excitation-relaxation dynamics of the system of Fig. (2) when the system is prepared with two cavity photons and the qubit is in the ground state, i.e. $\hat{\rho}(0) = |2, g\rangle \langle 2, g| = |2, +\rangle \langle 2, +|$, as a function of light-matter coupling g. a) Cavity photon number, b) qubit excitation number, and c) mapping of the bare state $|2, g\rangle$ to the eigenmodes in the even $(+)$ parity subspace. For convenience, we omit the parity index in the x-axis. d) Steady state populations. Model parameters are the same as in Fig. 2. The time axis in a) and b) is normalized to half of the cavity round-trip time $T_c \equiv \pi/\nu_c$. The two-photon relaxation time reads $T_{\kappa 2} \equiv 1/\kappa_{c2} = 40T_c/\pi$. The cavity mode Hilbert space cut-off is chosen as $N_c = 9$.

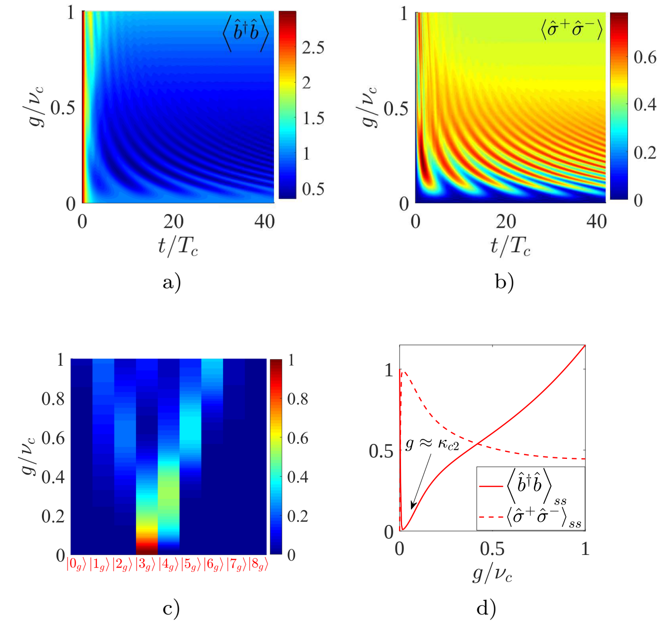

FIG. 4. Excitation-relaxation dynamics when the system is prepared with three cavity photons and the qubit is in the ground state, i.e. $\hat{\rho}(0) = |3, g\rangle \langle 3, g| = |3, -\rangle \langle 3, -|$, as a function of light-matter coupling g. The figure follows the same format as Fig. (3), except that the bare state $|3, g\rangle$ is instead mapped to eigenmodes in the odd (-) parity subspace. Other parameters are the same as in Fig. 3.

environment, we intuitively expect states with even/odd initial cavity photons to exhibit different transient and steady state behavior.

First, consider the simplest case of $g = 0$. This choice of parameter decouples the qubit and hence corresponds to the problem of a single cavity mode with two-photon relaxation, which has been studied in detail using multiple methods [38–42]. In this case, initial states having even/odd numbers of cavity photons end up with zero/one cavity photons in the steady state [52].

Next, we move on to characterize the interplay of two-photon relaxation and the qubit for $g \neq 0$. Here, closed form analytical solutions of the evolution operator at arbitrary g seem intractable, and instead we employ numerical integration of the Lindblad Eq. (6a). The time-evolution of the cavity/qubit excitations as a function of g is studied in Figs. 3 and 4 for the cases of two and three initial cavity photons, correspondingly. In both cases, it is generally observed that as g is increased, more complex beatings between various normal modes emerge. Such beatings can be approximately understood from the mapping of the initial cavity state to the corresponding eigenmodes of the open Rabi model. This shows which modes are more active at a given value of g in each parity subspace (Figs. 3c and 4c). For example, for the case of $\hat{\rho}(0) = |2, g\rangle \langle 2, g|$, the initial probability is shared between states $|1_g, +\rangle$ and $|2_g, +\rangle$ up to intermediate values of g ($0 < g \lesssim 0.5\nu_c$), beyond which $|1_g, +\rangle$ and $|3_g, +\rangle$ dominate. The corresponding frequency and decay rate of the modes can be obtained from Figs. 2a-2b.

Despite this generic similarity, it is observed that due to the non-trivial interplay of light-matter coupling and two-photon relaxation, the two cases under consideration have different transient and steady state characteristics. For the case of two initial cavity photons, we observe that the system reaches steady state on a time scale that is more or less given by the two-photon relaxation rate κ_{c2} (Figs. 3a-3b). On the other hand, in the case of three initial cavity photons, the transient dynamics has more features. Generally, at small g, the dynamics can be described as follows (Figs. 4a-4b): First, a fast depletion of the initial three cavity photons into one photon, with timescale roughly determined by κ_{c2}. This can be seen by the sharp transition of the cavity excitation number from 3 to approximately 1 (red to blue in Fig. 4a). Second, a slower depletion of the remaining cavity photon after a large number of Rabi exchanges between the qubit and the cavity, with timescale roughly determined by the decay rate of state $|1_g, -\rangle$. Essentially, since two-photon relaxation only allows pairs of exchange with the environment, the quantum state $|1_g, -\rangle$ acts like a dark state at $g = 0$ (i.e. $|1, g\rangle$). As g is increased, the decay rate of this state is barely modified up until $g/\nu_c \approx 0.5$ (See Fig. 2b), consistent with the observed long-lived excitations in the qubit/cavity dynamics (Figs. 4a-4b).

Steady state excitations have also been studied as a function of g in Figs. (3d-4d). In the case of two initial photons, we observe that the steady state populations of the cavity and qubit increase non-monotonically with increasing g, exhibiting a local maximum close to $g \approx \kappa_{c2}$. The case of three initial photons is more complicated. For small $g < \kappa_{c2}$, one observes fast relaxation of two photons, while the remaining photon energy is transferred to the qubit at steady state. At intermediate values of g, the excitation is shared between the cavity and the qubit while at very large g, the qubit excitation saturates and the cavity photon population increases linearly (Fig. 4d). The overall increase observed in the steady state populations arises from the fact that the coupling in Eq. (6b) appears effectively as an incoherent drive on the cavity. Lastly, we note that the steady state quantities obtained from the Linbdlad formalism will become less accurate at large values of g, as one needs to account for the renormalization of the dissipator arising from the underlying system-bath formalism [53], resulting in a Bloch-Redfield master equation [54]. Using Rayleigh-Schrödinger perturbation theory, however, one can show that disspator renormalizations are higher order in g compared to the ones for the Hamiltonian. This leaves a window, at intermediate values of coupling, where the use of bare dissipators is still justified.

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