Analytically Solvable Renormalization Group for the Many-Body Localization Transition

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Analytically solvable renormalization group for the many-body localization transition

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We introduce a simple, exactly solvable strong-randomness renormalization group (RG) model for the many-body localization (MBL) transition in one dimension. Our approach relies on a family of RG flows parametrized by the asymmetry between thermal and localized phases. We identify the physical MBL transition in the limit of maximal asymmetry, reflecting the instability of MBL against rare thermal inclusions. We find a critical point that is localized with power-law distributed thermal inclusions. The typical size of critical inclusions remains finite at the transition, while the average size is logarithmically diverging. We propose a two-parameter scaling theory for the many-body localization transition that falls into the Kosterlitz-Thouless universality class, with the MBL phase corresponding to a stable line of fixed points with multifractal behavior.

Introduction.— The many-body localization transition (MBLT) separates many-body localized (MBL) and ergodic dynamical phases in isolated quantum systems [1–10]. On the ergodic or thermal side of this transition, the system exchanges energy and information efficiently between its parts, thus quickly loosing its quantum nature. This corresponds to an extensive amount of quantum entanglement in many-body eigenstates. In contrast, the MBL phase is non-ergodic and avoids thermalization by means of an extensive number of local conserved quantities [11–13]. The high energy eigenstates of MBL systems have low area-law entanglement [11, 14] and allow one to encode quantum information even at long times [15, 16].

Although MBLT in one dimension is a subject of intense theoretical [5, 17–26] and experimental studies [27–29], many aspects of this phase transition remain poorly understood or debated. Numerical studies are limited to very small system sizes and are believed to suffer from finite size effects [30]. On the other hand, the pioneering strong disorder renormalization group (RG) approaches by Vosk *et al.* [18] and Potter *et al.* [19] evaded analytical solutions and relied on numerical simulations of simplified RG rules. Recent RG approach by Thiery *et al.* [22] and Thiery *et al.* [23] allowed for "mean-field" approximate solution, however resulting in unphysical exponents.

Recently Zhang *et al.* [21] introduced an exactly solvable RG for the MBLT. However, this RG has an inherent unphysical symmetry between MBL and thermal phases. Typically, the ergodic behavior and tendency to form resonances is very strong in quantum systems. On this basis, one expects that even a sparse network of resonances [19, 26] suffices for delocalization, and thus the critical point between MBL and ergodic phases should be more similar to the localized phase [18, 22]. These expectations are confirmed by numerical studies [26] and also earlier RG studies [18–20, 23].

In this work we present an analytically solvable family of strong-randomness RGs, which can be viewed as a deformation of the RG studied previously in [21]. The deformation is parametrized by $\alpha \leq 1$ that sets the asymmetry between MBL and thermal phase at the transition. We calculate the correlation length exponent ν and fractal dimensions for generic values of α. Upon decreasing value of α we observe that ν diverges, the critical point looks progressively more insulating, and the distribution of thermal blocks tends to a scale-invariant power-law shape. We identify the physical MBLT with the limit of maximal asymmetry $\alpha \to 0$ when critical point is localized with a probability one. By analytically continuing our RG equations to the case $\alpha \to 0$, we find that the MBL phase corresponds to a line of fixed points with the length of thermal inclusions distributed according to the power-law distribution $\rho_T(\ell) \sim 1/\ell^{2+\kappa}$ for large ℓ with $\kappa \geq 0$. The transition to the thermal phase occurs for the critical value $\kappa_c = 0$, when the average size of thermal inclusions diverges (while typical thermal puddles remain finite). We find that the thermal inclusions are renormalized by the surrounding MBL phase upon coarse graining, leading us to a simple two-parameter RG theory in the Kosterlitz-Thouless universality class [31]. This implies that the correlation length is diverging exponentially at the transition, in sharp contrast with previous predictions.

Two-parameter family of RGs.—To develop a theory of the MBLT, we adopt a coarse-grained picture [18–22] and assume that at some intermediate length scale the critical system can be viewed as a set of thermal (ergodic) and insulating (MBL) regions. Starting from this length scale, we build the RG description to account for the competition between ergodic regions that tend to hybridize the nearby insulators, and MBL clusters that absorb thermal regions and prevent resonances.

Aiming for a simple description [21], we assume that each region can be characterized by a single parameter, ℓ, which we refer to as "length". At each RG step one removes the shortest thermal (insulating) segment by merging it with adjacent insulating (thermal) regions,

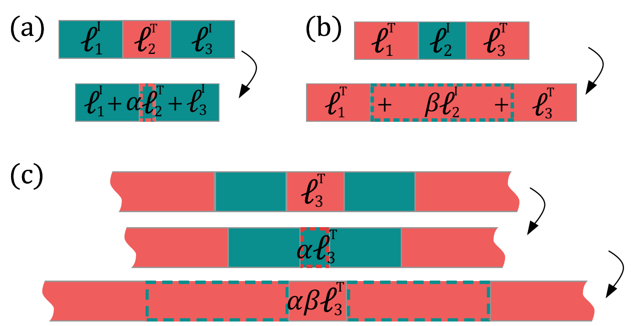

Figure 1. Illustration of RG rules for the decimation of thermal (a) and insulating segments (b). (c) The length of the central thermal block ℓ_2^T is recovered with unit prefactor after two decimation steps if $\alpha\beta = 1$.

see Fig. 1. The length of a new region reads

$$\ell^I_{\rm new} = \ell^I_{n-1} + \alpha \ell^T_n + \ell^I_{n+1},\ \ell^T_{\rm new} = \ell^T_{n-1} + \beta \ell^I_n + \ell^T_{n+1}, \quad (1)$$

where the length of the decimated segment is multiplied by a parameter α if it is thermal, and by β if it is insulating.

Rules (1) describe a two-parameter family of RGs, which reduces to (over)simplified RG in Ref. [21] for symmetric point $\alpha = \beta = 1$. We seek a deformation away from this point that makes the critical point more MBL-like. Intuitively, such asymmetry reflects the very strong tendency of quantum chaotic systems to develop entanglement and form resonances. Hence, even a small fraction of thermal blocks should suffice to drive the transition. Such deformation can be achieved by taking $\alpha \ll 1 \ll \beta$ so that the insulating segments do not increase much in size when a thermal block is decimated. On the other hand, when two thermal blocks absorb an insulating segment, the resulting thermal region has a significantly larger length, see Fig. 1(b), hence being less likely to be decimated again.

The rules in Eq. (1) with $\alpha \ll 1 \ll \beta$ are physically motivated if we interpret the length ℓ as setting the hybridization time τ through the corresponding segment. For insulating blocks it is natural to assume the time to be exponentially large in ℓ^I, $\tau^I \propto \exp(\ell^I/\xi_0)$, where ξ_0 is the (bare) localization length. In contrast, for thermal segments such time is expected to scale as $\tau^T \propto \ell^T$. When a thermal segment is decimated, its contribution to the hybridization rate of a new segment is negligible, motivating a small value of α. Similarly, the large value of β mimics the dominant contribution of the I segment to the hybridization time of a TIT block, see supplementary material [32] for more details. Moreover, this limit of variables will be justified in the following by the condition $\alpha\beta = 1$ and by the absence of fractality of the insulating regions at criticality when $\alpha = 0$.

Generalized length-preserving line $\alpha\beta = 1$.—We can obtain an additional relation between the parameters α, β by imposing that the contribution of each segment does not depend on its previous history in the RG. For instance, Fig. 1(c) shows a microscopic thermal segment of length ℓ_3^T that first was absorbed into an insulating segment, but later becomes again part of a thermal region. Requiring that this segment contributes by the amount ℓ_3^T to the effective length of the final thermal region, we obtain the condition $\alpha\beta = 1$. When $\alpha\beta = 1$, one can define a generalized total length, $\ell_{\rm tot} = \sum_n (\alpha \ell_n^T + \ell_n^I)$, that is preserved along the RG flow. If $\alpha \to 0$, it results in the conservation of the total length of insulating regions, which guarantees the correct scaling between the tunneling time and the total length when flowing into the localized phase. In what follows we restrict to the line $\alpha\beta = 1$, using value of $\alpha < 1$ as a control parameter. Critical behavior for generic values of α, β will be reported elsewhere [33].

Flow equations.—In order to describe the critical point, we derive RG flow equations for distributions of lengths of MBL and thermal segments [34–36]. At each step of the RG, the smallest block of length $\Gamma \equiv \min \ell_n$ is decimated according to the rules (1). Let $\rho_\Gamma^{I,T}(\ell)$ be the distributions of insulating and thermal block lengths respectively, with cutoff Γ. It is convenient to define the rescaled dimensionless length $\eta = (\ell - \Gamma)/\Gamma$ and associated probability distributions $\rho_\Gamma^{I,T}(\ell) = (1/\Gamma) Q_\Gamma^{I,T}(\eta)$. The RG equations which describe the flow of these rescaled probability distributions with the cutoff Γ read as [36, 37]:

$$\frac{\partial Q^I_\Gamma(\eta)}{\partial \ln \Gamma} = \partial_\eta \left[(1+\eta) Q^I_\Gamma(\eta)\right] + Q^I_\Gamma(\eta)[Q^I_\Gamma(0) - Q^T_\Gamma(0)] + Q^T_\Gamma(0)\theta(\eta - \alpha - 1) \int_0^{\eta-\alpha-1} d\eta' Q^I_\Gamma(\eta') Q^I_\Gamma(\eta - \eta' - \alpha - 1), \quad (2a)$$

$$\frac{\partial Q^T_\Gamma(\eta)}{\partial \ln \Gamma} = \partial_\eta \left[(1+\eta) Q^T_\Gamma(\eta)\right] + Q^T_\Gamma(\eta)[Q^T_\Gamma(0) - Q^I_\Gamma(0)] + Q^I_\Gamma(0)\theta(\eta - \beta - 1) \int_0^{\eta-\beta-1} d\eta' Q^T_\Gamma(\eta') Q^T_\Gamma(\eta - \eta' - \beta - 1). \quad (2b)$$

Here the first term on the right hand side originates from the overall rescaling, the second term corresponds to decimation of the smallest block at cutoff with $\eta = 0$, and the last convolution term accounts for the creation of new I or T block of length η.

Fixed point solutions for finite α.—In order to find

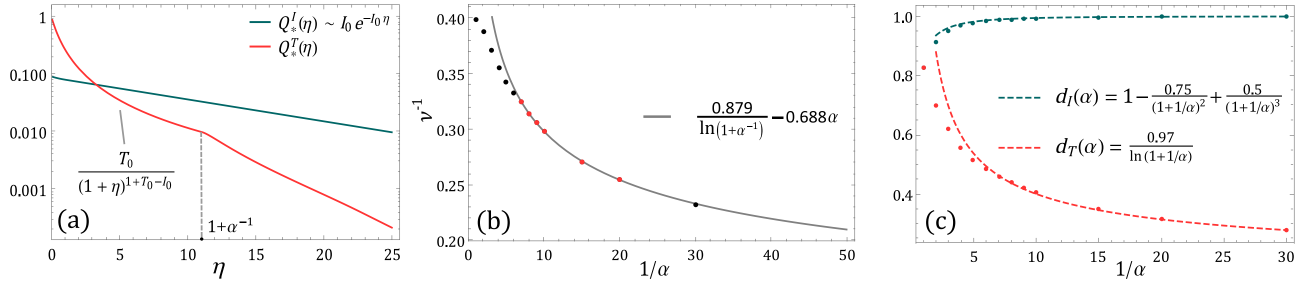

Figure 2. (a) For small $\alpha = \beta^{-1} = 1/10$ the fixed point distributions for thermal $Q_T^*(\eta)$ blocks (red line) behave as a power-law for $\eta \leq \alpha^{-1}+1$, and decays exponentially for larger values of η. Insulating blocks are approximately distributed exponentially for all η (blue line). (b) Slow decay of inverse critical exponent ν^{-1} with α^{-1} is well approximated by the analytical asymptotic. The dots marked with red have been used for the extrapolation of $\nu^{-1}(\alpha)$ for smaller values of α. Value of $\nu^{-1}(1/30)$ collapses well onto the numerical fit. (c) Fractal dimension d_I of insulating regions rapidly approaches one when $\alpha \to 0$, whereas fractal dimension of thermal inclusions slowly decays to zero.

fixed point distribution we set $\partial_\Gamma Q_\Gamma^{I,T}(\eta) = 0$ in Eqs. (2). We find that the value of the fixed point probability distributions $Q_*^{I,T}(\eta)$ at the cutoff can be determined as $I_0 \equiv Q_*^I(0) = \alpha/(1+\alpha)$ and $T_0 \equiv Q_*^T(0) = 1/(1+\alpha)$ [32]. Physically I_0 and T_0 correspond to the probability to decimate insulating and thermal segments. When $\alpha = 1$ we recover the symmetric result of Ref. [21] where fixed point was insulating/thermal with probability 1/2. However, in the limit $\alpha \ll 1$, $T_0 \to 1$ the fixed point is dominated by insulating regions. Note that the number of blocks scales as $N_{\rm tot} \propto 1/\Gamma$ with the cutoff since $I_0 + T_0 = 1$, so that the total length of the system $\ell_{\rm tot} \propto \Gamma N_{\rm tot}$ is asymptotically conserved in the RG for $\alpha\beta = 1$.

Using the boundary conditions at $\eta = 0$, each equation in system (2) can be solved iteratively. For the initial region $\eta \in [0, \beta+1]$, the integral term in Eq. (2b) vanishes, resulting in a power-law form of $Q_*^T(\eta)$. When $\eta \in [\beta+1, 2(\beta+1)]$ one can use the known solution for smaller values of η and solve the resulting non-uniform differential equation. Repeating such iterations for both $Q_*^{I,T}(\eta)$, we obtain the fixed point distributions shown in Fig. 2(a) for $\alpha = 1/\beta = 1/10$. The initial power-law region of $Q_*^T(\eta) \sim (1+\eta)^{-(1+T_0-I_0)}$ for $\eta \leq 1+\alpha^{-1}$ is followed by an exponential decay. Since α is small, the power-law region in $Q_*^I(\eta)$ is very short and the distribution can be approximated as $Q_*^I(\eta) = I_0 \exp(-I_0\eta)$.

Critical exponent and fractal dimensions.—From the fixed point distributions, we obtain the correlation length critical exponent ν and the fractal dimensions that characterize the fixed point. To extract ν, we consider weak perturbations around the fixed point, parametrized as $Q_\Gamma^{I,T}(\eta) = Q_*^{I,T}(\eta) + \Gamma^{1/\nu} f^{I,T}(\eta)$. The critical exponent ν controls the behavior of the perturbation upon increasing the cutoff, with $\nu > 0$ for a relevant perturbation. We have $\int_0^\infty d\eta\, f^{I,T}(\eta) = 0$ since $Q_*^{I,T}(\eta)$ is normalized to one. Linearizing the RG flows (2), we obtain an eigenvalue system of functional equations, $(1/\nu) f^{I,T}(\eta) = \hat{O}_{I,T} f^{I,T}(\eta)$ where the explicit form of the integro-differential operator $\hat{O}_{I,T}$ is given in [32]. Solving this eigenvalue problem, we obtain a single relevant eigenvalue $1/\nu$, which is real and positive and thus sets the critical exponent. The critical exponent $\nu(\alpha)$ takes its minimal value for $\alpha = 1$, $\nu(1) \approx 2.50$ [21] and increases for smaller values of α. We note that the increase of ν when the fixed point becomes more MBL-like qualitatively agrees with other RG approaches [18, 19] which predict more MBL-like fixed points and suggest $\nu \approx 3.5$. The inverse exponent $1/\nu$ decays to zero when $\alpha \to 0$. We predict that $\nu^{-1}(\alpha) \approx 1/\ln(1+\alpha^{-1}) + \mathcal{O}(\alpha)$ as $\alpha \to 0$ [32]. This is consistent with our results obtained via numerical diagonalization of $\hat{O}_{I,T}$, see Fig. 2(b).

To quantify the spatial structure of insulating and thermal regions at criticality, we consider their fractal dimensions. For example, the insulating fractal dimension quantifies the scaling of the total length of microscopic insulating segments $\propto \ell^{d_I}$ that are contained in a piece of insulator segment of size ℓ after coarse graining, and that were insulating at all RG steps. As for the critical exponent ν, we obtain the fractal dimensions by solving a linearized eigenvalue problem [21, 32]. The insulating fractal dimension rapidly tends to one as $d_I(\alpha) = 1 - \mathcal{O}(\alpha^2)$ for small α, see Fig. 2(c). On the physical grounds fractal thermal inclusions in an MBL region can lead to big rare thermal regions after coarse graining [19, 21, 38]. In contrast, fractal insulating regions are most likely unphysical as a fractal set of insulating blocks in an otherwise thermal system cannot lead to localization of the full system. We therefore identify $\alpha \to 0$ as a limit susceptible to describe the actual MBLT since in that limit $d_I = 1$. In this limit the thermal fractal dimension $d_T(\alpha)$ slowly approaches zero. As we discuss below, this is consistent with the physical picture provided by our RG.

Our analytical results reveal that the RG fixed point becomes increasingly MBL-like as α is decreased. The critical point in the limit $\alpha \to 0$ is localized with probabil-

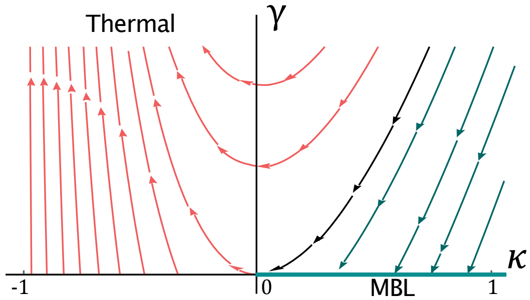

Figure 3. Two-parameter RG flows in the limit $\alpha \to 0$ has the half-line of stable fixed points $\gamma = 0$, $\kappa > 0$ describing a multifractal MBL phase. For $\gamma = 0, \kappa < 0$ the line of fixed points is unstable and gives rise to a flow to strong coupling which corresponds to thermalization. The black line separates the set of initial conditions that flow to the MBL and thermal phases.

ity $T_0 = 1$, with fractal thermal inclusions. In that limit, the insulating fixed point distribution becomes uniform $Q^I_*(\eta) = \lim_{I_0 \to 0} I_0 \exp(-I_0\eta)$, intuitively corresponding to localized blocks of all lengths. Thermal inclusions are power-law distributed when $\alpha \to 0$, $Q^*_T(\eta) = (1+\eta)^{-2}$, consistent with other RGs [18, 20, 23] – although the exponent differs slightly. Note, that the average length of thermal blocks, $\langle \eta^T \rangle$, diverges logarithmically at the transition, while the typical value $\langle \eta^T \rangle_{\rm typ}$ remains finite. This is consistent with a rare events-driven transition.

Two-parameter scaling for the MBLT.—In the limit $\alpha \to 0$, the eigenfunctions corresponding to the eigenvalue $\nu^{-1} \to 0$ can be determined analytically as $f^I(\eta) = f_{I_0}(1 - I_0\eta)e^{-I_0\eta}$ and $f^T(\eta) = f_{T_0}(1 - \ln(1+\eta))/(1+\eta)^2$ [32]. Since this perturbation becomes marginal for $\alpha \to 0$, we need to go beyond linear order to analyze the critical behavior. Motivated by the form of the eigenfunctions, we propose the following two-parameter ansatz

$$Q^I_\Gamma(\eta) = \gamma e^{-\gamma\eta}, \quad Q^T_\Gamma(\eta) = \frac{1+\kappa}{(1+\eta)^{2+\kappa}}, \tag{3}$$

where γ and κ depend on Γ and parametrize deformations of the critical point solution. Both functions are properly normalized provided $\kappa > -1$ and $\gamma > 0$. Moreover, the linear terms in the expansion of Eq. (3) in γ, κ are proportional to the critical eigenmodes $f^{I,T}(\eta)$ in the limit $\alpha \to 0$. Plugging this ansatz into Eqs. (2), we find that there is an exact line of RG fixed points for $\gamma = 0$ parametrized by κ. For small γ the approximate flow equations read:

$$\Gamma \frac{d\gamma}{d\Gamma} = -\gamma\kappa, \qquad \Gamma \frac{d\kappa}{d\Gamma} = -\gamma(1+\kappa). \tag{4}$$

In contrast to linearized case, the variables η and Γ do not fully separate, and we neglected a term logarithmic in η to get a closed equation for $d\kappa/d\Gamma$ [32]. However, the equation for $d\gamma/d\Gamma$ is accurate for small γ, and both equations correctly predict a line of fixed points for $\gamma = 0$.

We conjecture that the two-parameter flow equations (4) correctly capture the critical behavior of the MBLT. The RG flows are plotted in Fig. 3, and are equivalent to the celebrated Kosterlitz-Thouless (KT) equations for small γ, κ [31, 39]. The MBL phase corresponds to a stable line of fixed points with $\gamma = 0$ and $\kappa > 0$. This phase has insulating segments of all lengths, with ergodic inclusions distributed algebraically as $\sim \eta^{-(2+\kappa_\infty)}$, where $\kappa_\infty \geq 0$ parametrizes position on the line of fixed points. While the average length of ergodic regions is finite, the distribution $Q^T_\Gamma(\eta)$ in (3) implies that sufficiently high moments of $\langle (\ell^T)^n \rangle \propto \Gamma^n \langle \eta^n \rangle_{Q^T}$ with $n \geq 1+\kappa_\infty$ diverge, suggesting a multifractal behavior in the MBL phase near the transition [40–42]. The critical point is reached when the (renormalized) exponent κ_∞ becomes equal to the critical value $\kappa_c = 0$, which corresponds to the divergence of the average length of thermal inclusions. In our description, the critical point of the MBLT is a smooth continuation of the MBL phase, just like the critical point in the usual KT transition is a superfluid. In the thermal phase, γ flows to strong coupling corresponding to short insulating regions (in that regime, our RG equations break down), while κ goes to -1 corresponding to infinitely broadly-distributed thermal regions.

The RG trajectories are parametrized by $\gamma_\Gamma - \kappa_\Gamma + \ln(\kappa_\Gamma + 1) = C$ where $C > 0$ corresponds to the flow into strong coupling thermal phase, while for $C < 0$ trajectories flow to the MBL phase. Near the MBLT, we have $C = C_0(W_c - W) + \ldots$ where W corresponds to the bare disorder strength, and $W_c = W$ at the transition. Following usual scaling arguments, the correlation length that sets the crossover to the phases diverges as $\xi \propto \exp(c/\sqrt{|W - W_c|})$, where c is some non-universal positive constant. This is in sharp contrast with previous approaches that measured a large but finite critical exponent ν, and that also observed a finite probability to thermalize at criticality drifting with system size [18–20]. Note that the presence of logarithmic finite size corrections characteristic of the KT transitions would make this scaling very hard to observe on finite size systems. The exponent κ_∞ on the MBL side of the transition is non-universal, but vanishes as $\kappa_\infty = A\sqrt{W - W_c}$.

Summary and discussion.—We presented a one-parameter family of RGs that in the limit $\alpha \to 0$ provides a sensible description of the MBLT, yet allows for an analytic solution. Our simple two-parameter KT scaling predicts an exponentially diverging correlation length at the transition. The distribution of thermal regions has a power-law form both in the MBL phase, and at the transition. In addition, we recover the absence of fractal insulating regions and a sparse structure of thermal regions that have vanishing fractal dimension.

These results can be interpreted within a Griffiths picture [38, 43, 44]: in the MBL phase, thermal inclusions of size ℓ require only $\mathcal{O}(\log \ell)$ independent rare microscopic events with small probability p and therefore occur with algebraic probability $p_T(\ell) \sim p^{\mathcal{O}(\log \ell)} \sim 1/\ell^{2+\kappa}$. In contrast, rare insulating inclusions of size ℓ on the thermal side require $\mathcal{O}(\ell)$ rare events and are therefore exponentially distributed $p_I(\ell) \sim \mathrm{e}^{-\ell/\xi}$, leading to subdiffusive transport properties [18, 19, 38, 45–47]. This picture implies that thermal Griffiths inclusions are even sparser than has been previously assumed, since they formally have fractal dimension $d_T = 0$. The transition to the thermal side then occurs when $\kappa = \kappa_c = 0$. At this point the average size of the thermal inclusions diverges as $\log L$ with system size L, and is barely enough to percolate and thermalize the whole system upon coarse graining. It would be very interesting to see if our KT scaling scenario could be tested in other RG schemes [18, 19, 23], numerical studies [25, 26, 45, 46] or experiments [10, 29].

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