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Phys. Rev. Lett. **122**, 040502 — Published 30 January 2019

DOI: 10.1103/PhysRevLett.122.040502

Ancilla-free quantum error correction codes for quantum metrology

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Quantum error correction has recently emerged as a tool to enhance quantum sensing under Markovian noise. It works by correcting errors in a sensor while letting a signal imprint on the logical state. This approach typically requires a specialized error-correcting code, as most existing codes correct away both the dominant errors and the signal. To date, however, few such specialized codes are known, among which most require noiseless, controllable ancillas. We show here that such ancillas are not needed when the signal Hamiltonian and the error operators commute; a common limiting type of decoherence in quantum sensors. We give a semidefinite program for finding optimal ancilla-free sensing codes in general, as well as closed-form codes for two common sensing scenarios: qubits undergoing dephasing, and a lossy bosonic mode. Finally, we analyze the sensitivity enhancement offered by the qubit code under arbitrary spatial noise correlations, beyond the ideal limit of orthogonal signal and noise operators.

Quantum systems can make very effective sensors, but as with most quantum technologies, their performance is limited by decoherence. Typically, a quantum sensor acquires a signal as a relative phase between two states in coherent superposition [1–3]. Its sensitivity therefore depends on how quickly this phase accumulates, and how long the superposition remains coherent. The fundamental strategy to enhance sensitivity is then to increase the rate of signal acquisition (e.g., with entanglement) without equally reducing the coherence time [4]. These competing demands pose a familiar dilemma in quantum engineering: a quantum sensor must couple strongly to its environment without being rapidly decohered by it.

Quantum error correction (QEC) has recently emerged as a promising tool to this end. It is effective with DC signals and Markovian decoherence, important settings beyond the reach of dynamical decoupling, a widely-used tool with the same goal [3, 5, 6]. The typical QEC sensing scheme involves preparing a superposition of logical states, and periodically performing a recovery operation (i.e., error detection and correction). This allows a signal to accumulate as a relative phase at the logical level, while extending the duration of coherent sensing. To enhance sensitivity, however, great care must be taken in designing a QEC code which corrects the noise but not the signal. This new constraint is unique to error-corrected quantum sensing, with no analog in quantum computing or communication. Indeed, most QEC codes developed for those applications do not satisfy the above constraint, and cannot be used for sensing.

Recent works have begun to reveal how—and when—new QEC codes could enhance quantum sensing. Initial schemes [7–12] assumed a signal and a noise source coupled to a sensor in orthogonal directions (e.g., σ_z and σ_x). A two-qubit code utilizing one probing qubit and one noiseless ancilla could restore unitary evolution asymptotically (that is, for recoveries performed with sufficiently high frequency) [7–9]. Refs. [13–15] generalized these results, showing that given access to noiseless ancillas, one can find a QEC code for quantum sensing, provided the sensor's Hamiltonian is outside the so-called "Lindblad span". (Intuitively, the Hamiltonian-not-in-Lindblad-span, or HNLS, condition means that the signal is not generated solely by the same Lindblad error operators one seeks to correct.) Ref. [16] adapted this result to qubits with signal and noise in the same direction, and found numerical evidence that noiseless ancillas were unnecessary in this common experimental scenario.

Noiseless, controllable ancillas are seldom available in experiment. While they have often been assumed in constructing QEC codes for sensing, little is known to date as to whether they are truly necessary, beyond limited counterexamples [10–12, 16]. Similarly, Refs. [15, 16] showed, through perturbative arguments, that QEC can still enhance sensitivity even when the HNLS condition is not exactly met, but the exact sensitivity attainable is unknown. Here we address both of these open questions. First, we give a sufficient condition for error-corrected quantum sensing without noiseless ancillas, and a corresponding method to construct optimal QEC codes. We then present new explicit codes for two archetypal settings: qubits undergoing dephasing, and a lossy bosonic mode. Finally, we introduce a QEC recovery adapted for the former code, and give an exact expression for the achievable sensitivity outside the HNLS limit.

QEC for sensing. We consider a finite d-dimensional sensor under Markovian noise, whose dynamics is given by a master equation [17–19]

$$\frac{d\rho}{dt} = \mathcal{L}(\rho) = -i[\omega H, \rho] + \sum_i \left(L_i \rho L_i^\dagger - \frac{1}{2}\{L_i^\dagger L_i, \rho\} \right), \quad (1)$$

where ωH is the Hamiltonian from which ω is to be estimated, and $\{L_i\}$ are the Lindblad operators describing

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the noise. The Lindblad span associated with Eq. (1) is $\mathfrak{S} = \text{span}\{I, L_i, L_i^\dagger, L_i^\dagger L_j, \forall i,j\}$, where $\text{span}\{\cdot\}$ denotes the real linear subspace of Hermitian operators spanned by $\{\cdot\}$. One can use noiseless ancillas to construct a QEC code, described by the projector $P = |0_{\text{L}}\rangle\langle 0_{\text{L}}| + |1_{\text{L}}\rangle\langle 1_{\text{L}}|$ onto the code space, which asymptotically restores the unitary dynamics with non-vanishing signal

$$\frac{d\rho}{dt} = -i[\omega H_{\text{eff}}, \rho], \tag{2}$$

where $H_{\text{eff}} = PHP \not\propto P$, if and only if the HNLS condition is satisfied ($H \notin \mathfrak{S}$) [15]. To go beyond this result, we want to find conditions for QEC sensing codes that do not require noiseless ancillas, but still reach the same optimal sensitivity, as quantified by the quantum Fisher information (QFI). According to the quantum Cramér-Rao bound [20–23], the standard deviation $\delta\omega$ of the ω-estimator is bounded by $\delta\omega \geq (N_{\text{exp}} F(t))^{-1/2}$, where N_{exp} is the number of experiments and $F(t)$ is the QFI as a function of the final quantum state. The bound is asymptotically achievable as N_{exp} goes to infinity [23–25]. For a pure state $|\psi\rangle$ evolving under Hamiltonian ωH, $F(t) = 4t^2(\langle\psi|H^2|\psi\rangle - \langle\psi|H|\psi\rangle^2)$. $\delta\omega \propto 1/t$ is the so-called Heisenberg limit in time—the optimal scaling with respect to the probing time t [1–3]. The optimal asymptotic QFI provided by the error-corrected sensing protocol in Ref. [15], maximized over all possible QEC codes, is given by

$$F_{\text{opt}}(t) = 4t^2 \min_{S\in\mathfrak{S}} \|H - S\|^2 \equiv 4t^2 \|H - \mathfrak{S}\|^2, \tag{3}$$

where $\|\cdot\|$ is the operator norm.

Commuting noise. We address here the following open questions: (i) Under what conditions the noiseless sensing dynamics in Eq. (2) can be achieved with an ancilla-free QEC code. (ii) Whether such code can achieve the same optimal QFI in Eq. (3). We give a partial answer to these questions in terms of a sufficient condition on the signal Hamiltonian and the Lindblad jump operators.

Theorem 1 (Commuting noise)**.** *Suppose $H \notin \mathfrak{S}$ and $[H, L_i] = [L_i, L_j] = 0$, $\forall i,j$. Then there exists a QEC sensing code without noiseless ancilla that asymptotically recovers the Heisenberg limit in t. Moreover, it achieves the same optimal asymptotic QFI [Eq. (3)] offered by noiseless ancillas.*

Proof. A QEC sensing code recovering Eq. (2) should satisfy the following three conditions [15]:

$$PHP \not\propto P, \tag{4}$$

$$PL_iP \propto P, \quad PL_i^\dagger L_j P \propto P, \tag{5}$$

Eq. (5) is exactly the Knill-Laflamme condition to the lowest order in time evolution [26–29] and Eq. (4) is an additional requirement that the signal should not vanish in the code space. We say the code corrects the Lindblad span $\mathfrak{S}$ if Eq. (5) satisfied. Without loss of generality, we consider only a 2-dimensional code $|0(1)_{\text{L}}\rangle = \sum_{k=1}^d \sqrt{\beta_k^{0(1)}}\,|k\rangle$, where $\{|k\rangle\}_{k=1}^d$ is an orthonormal basis under which H and L_i's are diagonal. Define d-dimensional vectors $\mathbf{1}, \boldsymbol{h}, \boldsymbol{\ell}_i$, and $\boldsymbol{\ell}_{ij}$ such that $(\mathbf{1})_k = 1$, $(\boldsymbol{h})_k = \langle k|H|k\rangle$, $(\boldsymbol{\ell_i})_k = \langle k|L_i|k\rangle$ and $(\boldsymbol{\ell_{ij}})_k = \langle k|L_i^\dagger L_j|k\rangle$. Define the real subspace $\mathfrak{S}_{\text{diag}} = \text{span}\{\mathbf{1}, \text{Re}[\boldsymbol{\ell_i}], \text{Im}[\boldsymbol{\ell_i}], \text{Re}[\boldsymbol{\ell_{ij}}], \text{Im}[\boldsymbol{\ell_{ij}}], \forall i,j\} \subseteq \mathbb{R}^d$. The optimal code can be identified from the optimal solution $\tilde{\boldsymbol{\beta}} = \tilde{\boldsymbol{\beta}}^{\mathbf{0}} - \tilde{\boldsymbol{\beta}}^{\mathbf{1}}$ of the following semidefinite program (SDP) [30],

$$\text{maximize} \quad \langle \boldsymbol{\beta}, \boldsymbol{h}\rangle \tag{6}$$

$$\text{subject to} \quad \|\boldsymbol{\beta}\|_1 \leq 2, \text{ and } \langle \boldsymbol{\beta}, \boldsymbol{\ell}\rangle = 0, \ \forall \boldsymbol{\ell} \in \mathfrak{S}_{\text{diag}}. \tag{7}$$

Here $\|\boldsymbol{x}\|_1 = \sum_{i=1}^d |x_i|$ is the one-norm in $\mathbb{R}^d$ and $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = \sum_{i=1}^d x_i y_i$ the inner product. Choosing the optimal input quantum state $|\psi_0\rangle = \frac{1}{\sqrt{2}}(|0_{\text{L}}\rangle + |1_{\text{L}}\rangle)$, the QFI is $F(t) = t^2 \left|\langle \boldsymbol{\beta^0} - \boldsymbol{\beta^1}, \boldsymbol{h}\rangle\right|^2$. Moreover, the optimal value of Eq. (6) is $2\min_{\boldsymbol{\ell}\in\mathfrak{S}_{\text{diag}}} \|\boldsymbol{h} + \boldsymbol{\ell}\|_\infty$ with the argument of the minimum denoted by $\boldsymbol{\ell}^\diamond$. Here $\|\cdot\|_\infty$ denotes the infinity/max norm, defined as the largest absolute value of elements in a vector. The optimal solution $\tilde{\boldsymbol{\beta}}^{\mathbf{0(1)}}$ can be obtained from the constraint that it is in the span of vectors $\boldsymbol{v}$ such that $\langle \boldsymbol{v}, \boldsymbol{h} + \boldsymbol{\ell}^\diamond\rangle$ is the largest (smallest) [30]. In this case, $F(t) = 4t^2 \|\boldsymbol{h} - \mathfrak{S}_{\text{diag}}\|_\infty^2$ is the same as F_{opt} in Eq. (3) for noiseless ancilla. Therefore, we conclude that $\tilde{\boldsymbol{\beta}}^{\mathbf{0(1)}}$ gives the optimal code. □

Theorem 1 reveals that the need for noiseless ancillas arises from the non-commuting nature of the Hamiltonian and Lindblad operators. Indeed, we can find a non-trivial example with $[H, L_i] \neq 0$ for which there exist no ancilla-free QEC codes—even for arbitrarily large d (see Supplemental Material [31]). Another interesting feature of commuting noise is that it allows quantum error correction to be performed with a lower frequency, by analyzing the evolution in the interaction picture [31].

We now consider two explicit, archetypal examples of quantum sensors dominated by commuting noise. In principle, a QEC code for each example could be found numerically through Theorem 1. Instead, we introduce two near-optimal, closed-form codes which are customized to the application and the errors at hand.

Correlated Dephasing Noise. A common sensing scenario involves a quantum sensor composed of N probing qubits with energy gaps proportional to ω [3]. For such a sensor to be effective, the qubits' energy gaps must depend strongly on ω, which in turn makes them vulnerable to rapid dephasing due to fluctuations in their energies from a noisy environment [32–37]. Assuming for simplicity that each qubit has the same dephasing time T_2, the generic Markovian dynamics for the sensor is

$$\frac{d\rho}{dt} = -i[\omega H, \rho] + \frac{1}{2T_2}\sum_{j,k=1}^N c_{jk}\Big(Z_j\rho Z_k - \frac{1}{2}\{Z_jZ_k, \rho\}\Big). \tag{8}$$

Here, $H = \frac{1}{2}\mathfrak{h} \cdot \boldsymbol{Z}$ where $\boldsymbol{Z} = (Z_1, \ldots, Z_N)$, so qubit j has an energy gap $\omega\mathfrak{h}_j$. The correlation matrix $C = (c_{jk})_{j,k=1}^N$ describing the spatial structure of the noise can be quite general, e.g. depending on their coupling to a nearby fluctuator or a common resonator. In particular, $c_{jk} \in [-1, 1]$ describes the correlation between the fluctuations on qubits j and k, with $c_{jk} = 1, -1$ and 0 signifying full positive, full negative, and the absence of correlations, respectively.

Eq. (8) can be converted to the form of Eq. (1) by diagonalizing C ($C\boldsymbol{v}_j = \lambda_j \boldsymbol{v}_j$) with an orthonormal eigenbasis. Concretely, $L_j = \sqrt{\lambda_j}\boldsymbol{v}_j \cdot \boldsymbol{Z}$ can be viewed as normal modes of the phase noise. The HNLS condition then translates to $\mathfrak{h} \notin \mathrm{col}(C)$, the column space of C, which occurs when one normal mode u overlapping with H (i.e., $\boldsymbol{v}_u \cdot \mathfrak{h} \neq 0$) has a vanishing amplitude, $\lambda_u = 0$. This occurs generically in the limit of strong spatial noise correlations, provided the noise is not uniformly global [16]. Observe that $[H, L_j] = [L_j, L_k] = 0$ here, so Theorem 1 guarantees a QEC code without noiseless ancillas saturating the optimal bound in Eq. (3). One such code, for $N \geq 3$, is given by

$$|0_{\mathrm{L}}\rangle = \bigotimes_{j=1}^{N}\Big(\cos\theta_j|0_j\rangle + i\sin\theta_j|1_j\rangle\Big), \quad |1_{\mathrm{L}}\rangle = X^{\otimes N}|0_{\mathrm{L}}\rangle\,, \tag{9}$$

where $\boldsymbol{\theta} = \frac{1}{2}\arccos\mathfrak{b}^{\diamond}$, defined element-wise, and $\mathfrak{b}^{\diamond}$ is the solution of the following SDP:

$$\text{maximize}\,\langle\mathfrak{b}, \mathfrak{h}\rangle, \quad \text{subject to}\ \|\mathfrak{b}\|_\infty \leq 1,\ \mathfrak{b} \perp \mathrm{col}(C). \tag{10}$$

It is straightforward to show that the code in Eq. (9), with this choice of $\mathfrak{b}^{\diamond}$, satisfies the QEC sensing conditions (4)–(5). It works by correcting all non-vanishing noise modes, but leaving a vanishing mode with the maximum overlap with H uncorrected, through which H affects the logical state. Moreover, it achieves the optimal asymptotic QFI Eq. (3); in this case [31]:

$$F_{\mathrm{opt}}(t) = t^2\,\|\mathfrak{h} - \mathrm{col}(C)\|_1^2\,. \tag{11}$$

Note that since signal and noise are both along σ_z on each qubit, the usual repetition code [38] is not suitable for sensing as it also corrects the signal Hamiltonian H. Remarkably, while the domain of the SDP in Eqs. (6)–(7) has dimension $O(2^N)$, that of Eq. (10) only has dimension $O(N)$: our ansatz in Eq. (9) renders the QEC code optimization efficient. An approximate solution to Eq. (10) is $\tilde{\mathfrak{b}}^{\diamond} = \gamma\,\mathrm{proj}_{\ker(C)}\mathfrak{h}$, where γ is an adjustable parameter in the range $[-\gamma_{\max}, \gamma_{\max}] \setminus \{0\}$, $\gamma_{\max} = \|\mathrm{proj}_{\ker(C)}\mathfrak{h}\|_\infty^{-1}$. The code using $\boldsymbol{\theta} = \frac{1}{2}\arccos\tilde{\mathfrak{b}}^{\diamond}$ always satisfies the QEC sensing conditions exactly [Eqs. (4)–(5)], although it needs not saturate the optimal QFI in Eq. (11). In the important case of a single vanishing noise mode [i.e., $\mathrm{nullity}(C) = 1$], however, $\tilde{\mathfrak{b}}^{\diamond}$ achieves the optimal QFI at $\gamma_{\max}$.

Lossy bosonic channel. Boson loss is often the dominant decoherence mechanism in a bosonic mode [39], described by the master equation

$$\frac{d\rho}{dt} = -i\Big[\sum_{i=1}^{s}\zeta_i(a^\dagger a)^i, \rho\Big] + \kappa\Big(a\rho a^\dagger - \frac{1}{2}\{a^\dagger a, \rho\}\Big), \tag{12}$$

where a is the annihilation operator and κ the boson loss rate. We only consider Hamiltonians that are a function of the boson number $a^\dagger a$, applying a cutoff at the s-th power, where $s > 1$ is a positive integer. We also truncate the boson number at M, to keep the system dimension finite. According to the HNLS condition, while ζ_1 cannot be sensed at the Heisenberg limit, $\omega := \zeta_s$ asymptotically can, with the optimal code for $s = 2$ provided in Ref. [15].

To sense ω, it is important to filter out all lower-order signals $\sum_{i=1}^{s-1}\zeta_i(a^\dagger a)^i$ using the QEC code. Therefore, we should use the following modified Lindblad span [31]:

$$\mathfrak{S} = \{I, a, a^\dagger, (a^\dagger a)^i,\ 1 \leq i \leq s-1\}. \tag{13}$$

Note that the boson loss noise is not commuting because $[a, (a^\dagger a)^i] \neq 0$. Still, this type of off-diagonal noise can be tackled by simply ensuring the distance of the supports (non-vanishing terms) of $|0_{\mathrm{L}}\rangle$ and $|1_{\mathrm{L}}\rangle$ is at least 3.

To obtain the optimal code, we could solve the SDP in Eqs. (6)–(7). However, when M is sufficiently large, we obtain a near-optimal solution analytically by observing that for large M, minimizing $\|(a^\dagger a)^s - \sum_{i=0}^{s-1}\chi_i(a^\dagger a)^i\|$ over all $\{\chi_i\}_{i=0}^{s-1}$ is equivalent to approximating a s-th degree polynomial using an $(s-1)$-degree polynomial. The optimal polynomial is the Chebyshev polynomial [40] and the near-optimal code, that we call the *s-th order Chebyshev code*, is supported by its max/min points:

$$|0(1)_{\mathrm{L}}\rangle = \sum_{k\ \mathrm{even(odd)}}^{[0,s]} \tilde{c}_k\,\big|\lfloor M\sin^2(k\pi/2s)\rfloor\big\rangle\,, \tag{14}$$

where $\lfloor x\rfloor$ denotes the largest integer $\leq x$, and $|\tilde{c}_k|^2$ can be obtained from solving a linear system of equations of size $O(s^2)$. $|\tilde{c}_k|^2$ is approximately equal to $\frac{2}{s} - \frac{1}{s}\delta_{ks} - \frac{1}{s}\delta_{k0}$ for sufficiently large M. (Detailed calculations are in Ref. [31].)

In quantum sensing, the s-th order Chebyshev code corrects the Lindblad span (Eq. (13)) and provides a near optimal asymptotic QFI for ω

$$F(t) \approx F_{\mathrm{opt}}(t) \approx 16t^2\left(\frac{M}{4}\right)^{2s}, \tag{15}$$

for sufficiently large M. Note that the $(s-1, \frac{M}{s}-1)$ binomial code [41] also corrects Eq. (13), but it gives a QFI that is exponentially smaller than the optimal value by $O\big(s(2/e)^{2s}\big)$ for sufficiently large M.

Enhancing sensitivity beyond HNLS. Previous works have focused on regimes where the HNLS condition is exactly satisfied. However, QEC can still enhance quantum

sensing well beyond this ideal scenario, even if the sensor's encoded dynamics is not unitary (even asymptotically for $\Delta t \to 0$). Indeed, decoherence at the logical level can often be made weaker than at the physical level—while still maintaining signal—giving a net enhancement in sensitivity.

To show how, we generalize the dephasing qubits example to this more realistic setting. When HNLS is satisfied, the code in Eq. (9) corrects noise modes with non-zero amplitude $\lambda_j > 0$, but leaves a mode with $\lambda_u = 0$ uncorrected. In experiments, the noise correlation matrix C is generically full-rank, meaning that the HNLS condition is not satisfied. Yet, non-trivial noise correlations will generally cause C to have a non-uniform spectrum, yielding some subdominant eigenvalues and corresponding L_j's. It is thus possible to design a code that still accumulates signal at the cost of leaving uncorrected just one subdominant noise mode ($\lambda_u \approx 0$) through an appropriate choice of $\boldsymbol{\theta}$ in Eq. (9). To reach a closed-form expression for the resulting sensitivity, we use $\tilde{\mathfrak{b}}^{\diamond}$ as a starting point rather than an SDP formulation, setting

$$\boldsymbol{\theta} = \frac{1}{2}\arccos(\gamma \boldsymbol{v}_u), \tag{16}$$

defined element-wise, where $|\gamma| \in (0, \gamma_{\max}]$ is again adjustable, now with $\gamma_{\max} = \|\boldsymbol{v}_u\|_\infty^{-1}$.

The natural figure of merit for a sensor with uncorrected noise is not the Fisher information: decoherence eventually causes $F(t)$ to decrease, rather than grow unbounded as in Eq. (3). Instead, it is sensitivity, defined as the smallest resolvable signal per unit time [3]. For a single qubit with an energy gap $A\omega$ and dephasing time T_2/B, the best achievable sensitivity is [16]

$$\eta = \min_{t>0} \frac{1}{\sqrt{F(t)/t}} = \frac{\sqrt{B}}{A}\sqrt{\frac{2e}{T_2}}. \tag{17}$$

Taking $\mathfrak{h}_j = 1$ in Eq. (8), each physical qubit ($A = B = 1$) gives $\eta_1 = \sqrt{2e/T_2}$. N such qubits operated in parallel give $\eta_{\text{par}} = \eta_1/\sqrt{N}$, while for entangled states one could reach $A = N$, often at the cost of an increased B. For example, a Greenberger-Horne-Zeilinger (GHZ) sensing scheme with the same N qubits gives

$$\eta_{\text{GHZ}} = \frac{\|D_C^{1/2} V^\top \mathfrak{h}\|_2}{N}\sqrt{\frac{2e}{T_2}}, \tag{18}$$

where $V = (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_N)$ and $D_C = \text{diag}(\lambda_1, \ldots, \lambda_N)$ [42]. Note that for uncorrelated noise we have $\|D_C^{1/2} V^\top \mathfrak{h}\|_2 = \sqrt{N}$, thus negating any gains from entanglement.

To find the sensitivity offered by the QEC code described above, we compute the sensor's effective Liouvillian, $\mathcal{L}_{\text{eff}} = \mathcal{R} \circ \mathcal{L} \circ \mathcal{P}$, under frequent recoveries $\mathcal{R}$, where $\mathcal{P}(\rho) = P\rho P$ [16]. The usual QEC recovery (i.e., the transpose channel) results in population leakage out of the codespace due to the uncorrected error L_u, even when $\Delta t \to 0$, which complicates the analysis [28, 43]. To prevent such leakage at leading order in $\Delta t/T_2$, we modify the usual recovery so that the state is returned to the codespace after an error L_u, though perhaps with a logical error. This modification results in a Markovian, trace-preserving effective dynamics over the two-dimensional codespace, given by $\mathcal{L}_{\text{eff}}$. Specifically, the sensor's effective dynamics becomes that of a dephasing qubit with $A = \gamma|\boldsymbol{v}_u \cdot \mathfrak{h}|$ and $B = \gamma^2\lambda_u$, giving $\eta_{\text{QEC}}^{(u)} = \eta_1\sqrt{\lambda_u}/|\boldsymbol{v}_u \cdot \mathfrak{h}|$. The optimal choice of u is the one that minimizes this quantity, giving:

$$\eta_{\text{QEC}} = \frac{1}{\|D_C^{-1/2} V^\top \mathfrak{h}\|_\infty}\sqrt{\frac{2e}{T_2}}, \tag{19}$$

valid for arbitrary noise correlation profile C [44]. The straightforward but lengthy calculation is given in [31].

Eq. (19) identifies the C's for which this QEC scheme provides enhanced sensitivity over parallel and GHZ sensing. Notice that while HNLS is satisfied only in a measure-zero set of C's, QEC can enhance sensitivity over a much larger set, regardless of whether it can approach the Heisenberg limit in t.

Eq. (19) admits a broad range of η_{QEC} vs. N scalings due to the critical dependence of η_{QEC} on $C = C(N)$. The same is true of the Fisher information in the HNLS limit as we show in [31].

Discussion. We have shown that noiseless ancillas, while frequently invoked, are not required for a large family of error-corrected quantum sensing scenarios where the Hamiltonian and the noise operators all commute. Our proof is constructive, and gives a numerical method for designing QEC codes for sensing through semidefinite programming, analogous to the techniques from Refs. [45, 46] for quantum computing. Commuting noise, however, is not necessary for ancilla-free codes [10–12] and it is an interesting open problem to refine Theorem 1 into a necessary and sufficient condition.

We also introduced near-optimal, closed-form QEC codes and associated recoveries for two common sensing scenarios. For dephasing qubits, we found an expression for the sensitivity enhancement offered by our QEC scheme under arbitrary Markovian noise, even when the Heisenberg limit in t could not be reached. Our results raise the questions of whether there exists a simple geometric condition defining the set of C's for which QEC can enhance sensitivity, and whether or not Eq. (19) is a fundamental bound for QEC schemes. More broadly, our results show that ancilla-free, task-oriented QEC code design through convex optimization is a promising tool to enhance near-term quantum devices.

Acknowledgments. We thank Victor Albert, Kyungjoo Noh, Florentin Reiter and John Preskill for inspiring discussions. We acknowledge support from the ARL-CDQI (W911NF-15-2-0067, W911NF-18-2-0237), ARO (W911NF-18-1-0020, W911NF-18-1-0212), ARO MURI (W911NF-16-1-0349, W911NF-15-1-0548), AFOSR MURI (FA9550-14-1-0052, FA9550-15-1-0015), DOE (DE-SC0019406), NSF (EFMA-1640959, EFRI-

ACQUIRE 1641064, EECS1702716) and the Packard Foundation (2013-39273).

Author contributions. S.Z. conceived Theorem 1, the SDP for the dephasing code, and the bosonic code. D.L. devised the dephasing code ansatz, its closed-form approximate solution, and its extension beyond HNLS. P.C. and L.J. supervised the project. All authors discussed the results and contributed to the final manuscript.

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