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Non-Topological Majorana Zero Modes in Inhomogeneous Spin Ladders

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We show that the coupling of homogeneous Heisenberg spin-1/2 ladders in different phases leads to the formation of interfacial zero energy Majorana bound states. Unlike Majorana bound states at the interfaces of topological quantum wires, these states are void of topological protection and generally susceptible to local perturbations of the host spin system. However, a key message of our work is that in practice they show a high degree of resilience over wide parameter ranges which may make them interesting candidates for applications.

Introduction: The Majorana fermion has become one of the most important fundamental quasi particles of condensed matter physics. Besides its key role as a building block in correlated quantum matter, much of this interest is motivated by perspectives in quantum information [1–3]. Majorana qubits have unique properties which make them ideal candidates for applications in, e.g., stabilizer code quantum computation [4]. Current experimental attempts to isolate and manipulate Majorana bound states (MBSs) focus on interfaces between distinct phases of symmetry protected topological (SPT) quantum matter. These material platforms have the appealing property that MBSs are protected against local perturbations by principles of topology. In practice, however, topological protection may play a lesser role than one might hope, and various obtrusive aspects of realistic quantum materials appear to challenge the isolation and manipulation of MBSs. Specifically, in topological quantum wires based on the hybrid semiconductor-superconductor platform [5] or on coupled ferromagnetic atoms [6], all relevant scales are confined to narrow windows in energy. In this regard, proposals to realize MBSs in topological insulator nanowires [7] may offer superior solutions. However, these realizations require a high level of tuning of external parameters, notably of magnetic fields, and may be met with their own difficulties.

In this Letter, we suggest an alternative hardware platform for the isolation of zero-energy MBSs. Our proposal does not engage topology. Specifically, local perturbations of the microscopic Hamiltonian may induce nonlocal correlations between the emergent Majorana quantum particles. However, we argue below that in practice this problem is less drastic than one might fear, and that the current architecture may grant a high level of effective protection. The numerical evidence provided below certainly points in this direction.

The material platform we suggest is based on spin ladder materials. Their phases can be classified by combining standard Landau-Ginzburg symmetry breaking with the presence of SPT order [8, 9]. We show here that combining ladders in different phases provides a systematic means to generating interface MBSs. The formal bridge between the physics of spin ladders and that of Majorana fermions is provided by a two-step mapping, first representing the spin degrees of freedom by bosons, followed by refermionization of the latter into an effective Majorana theory [10]. We will discuss how numerous spin ladder properties that are difficult to access in the spin language are made simple and transparent in Majorana representation. In particular, SU(2) invariant spin ladders with two legs are described by a theory of four massive Majorana fermions, comprising a triplet and a singlet of different masses, together with a global parity constraint. The ground state (g.s.) degeneracies of the spin systems are then encoded entirely in zero-energy MBSs localized on the boundaries of the system.

Two surprising findings arise from this Majorana representation. The first is that additional g.s. degeneracies can appear in *inhomogeneous* ladders, where the spin-spin interactions vary spatially along the ladder. In the fermionic language, these degeneracies manifest themselves in new MBSs appearing at the phase boundaries via the Jackiw-Rebbi mechanism [11], according to which a sign change in the fermion mass creates a zero mode. This may happen even if all of the bulk phases composing the ladder do not support MBSs on their own. The

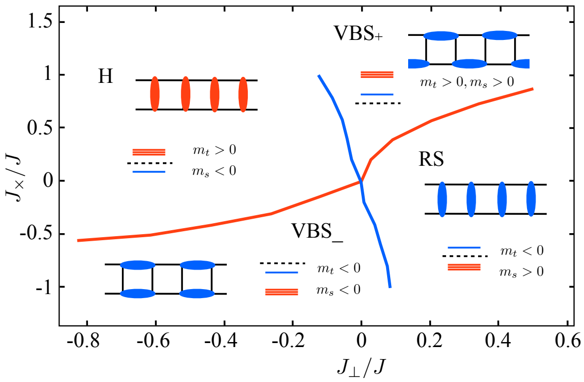

FIG. 1. Phase diagram for the Hamiltonian (1) obtained from SU(2) DMRG simulations of a 100×2 site ladder with bond dimension $\chi = 1500$ states. The red (blue) phase boundary shows the critical line with Majorana fermion mass $m_t = 0$ ($m_s = 0$). Inset figures show schematic representations of singlet (blue) and triplet (red) bond order within each phase, and the corresponding signs of m_t and m_s.

second finding is that zero-energy MBSs exist only if the spatial variation of spin couplings about the boundary is sufficiently gentle (a few lattice sites, in practice). The spatial smoothness across the interface is required to stabilize the mapping onto a continuum description and to prevent the coupling of distant MBSs via higher-energy states [12]. This condition manifests the lack of topological protection. (For other zero energy modes in topologically trivial phases, see Refs. [13–17].) However, we present numerical evidence that these MBSs are nonetheless close to zero energy over parametrically wide regions.

Spin ladders: Ladder geometries provide an important viewport on the physics of strongly correlated electron systems [18] and are a research focus of condensed matter physics in their own regard. They are close enough to being one-dimensional (1D) that powerful theoretical techniques can be deployed in their understanding running the gamut from field theory [10, 19–21] and Bethe ansatz [22] to density matrix renormalization group (DMRG) [23–28]. However, they are also far enough removed from 1D that they capture the physics of two-dimensional systems. We here focus on ladders where the fluctuations of spin-1/2 degrees of freedom are dominant (e.g., $SrCu_2O_3$ [29]) over ladders where charge degrees are mobile (e.g., $Sr_{14-x}Ca_xCu_{24}O_{41}$ [30]). For concreteness, we consider the two-leg ladder Hamiltonian

$$\begin{aligned} H = J \sum_{\ell=1,2} \sum_{r=1}^{N-1} \boldsymbol{S}_{\ell,r} \cdot \boldsymbol{S}_{\ell,r+1} + J_\perp \sum_{r=1}^{N} \boldsymbol{S}_{1,r} \cdot \boldsymbol{S}_{2,r} \\ + J_\times \sum_{r=1}^{N-1} \Big(\boldsymbol{S}_{1,r} \cdot \boldsymbol{S}_{1,r+1} \Big) \Big(\boldsymbol{S}_{2,r} \cdot \boldsymbol{S}_{2,r+1} \Big), \end{aligned} \tag{1}$$

where $S^a_{\ell,r}$ is the $a = x, y, z$ spin-1/2 operator located on leg ℓ and rung r of the ladder. The exchange parameters $J := 1$, $J_\perp$, $J_\times$ characterize leg, rung, and plaquette interactions, respectively. For uncoupled Heisenberg chains, the total spin of each leg would be conserved, and we could work in a representation where $S^z_\ell = \sum_r S^z_{\ell,r}$ are good quantum numbers. Assuming an even number N of sites per chain, both $S^z_\ell \in \mathbb{Z}$ are integer valued. The coupling $J_\perp$ exchanges spin in integer units, $S^z_1 \to S^z_1 \pm 1, S^z_2 \to S^z_2 \mp 1$, violating the conservation of the individual S^z_ℓ, but still constraining the even and odd combinations, $S^z_\pm = S^z_1 \pm S^z_2$, to have identical *parity*,

$$S^z_+ \equiv S^z_- (\mathrm{mod} 2). \tag{2}$$

We thus expect an effective fermionized theory of the system to display a U(1) symmetry reflecting the conservation of S^z_+ plus a $\mathbb{Z}_2$ parity condition implementing (2). The latter introduces correlation between the S^z_+ and the S^z_- sector and will play a key role throughout.

Phase diagram: Depending on the couplings $J_\perp, J_\times$, the Hamiltonian (1) supports different phases. For strong positive rung interaction $J_\perp$ and weak plaquette interaction $J_\times$, the formation of rung singlets (RS) is favored, cf. the lower right part of Fig. 1. For strong negative couplings $J_\perp$, rung triplets are formed instead and effectively implement an $S = 1$ Haldane-Heisenberg chain (Haldane phase, H). For strong $J_\times$, one may anticipate 'valence bond solids' (VBS) distinguished by different types of periodically repeated intra-chain dimerization, VBS_+ and VBS_- (see Fig. 1). While the existence of different dimerization patterns is relatively easy to anticipate, it takes more effort to determine the symmetries characterizing them, the respective order parameters, the g.s. degeneracies, and the phase boundaries. For example, the Haldane phase is an SPT phase without a local order parameter. It exhibits a four-fold g.s. degeneracy due to two spin-1/2 degrees of freedom dangling at the boundaries. In particular, the identification of the symmetries of the VBS phases is a non-trivial matter [31, 32]. The boundaries between the phases as well as the ensuing g.s. degeneracies can be established via DMRG simulations [12]: in Fig. 1 we present the phase diagram and in Tab. I the g.s. degeneracies.

Phase	m_t/m_s	d_+/d_-	g.s. deg.
H	$+/-$	4/2	4
RS	$-/+$	1/2	1
VBS_+	$+/+$	4/4	8
VBS_-	$-/-$	1/1	1

TABLE I. Phases of the spin model, the signs of their fermion masses, m_t/m_s, the g.s. degeneracies, $d_\pm$, of their even/odd sectors ($S^z_\pm$) before the parity restriction (2) is applied, and finally their overall actual g.s. degeneracies from SU(2) DMRG.

The presence of distinct dimerization patterns also provides a first clue as to the formation of zero energy degrees of freedom if chains of competing order are coupled

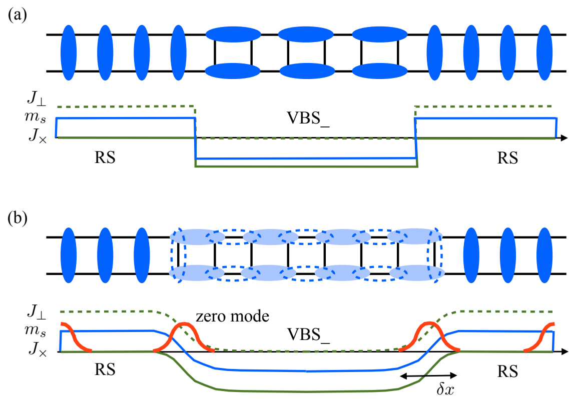

FIG. 2. Bond formation patterns when parameters $(J_\times, J_\perp)$ are varied (in green) in order to form a RS–VBS$_-$–RS ladder. a) For sudden parameter changes at interfaces. b) For smooth parameter changes, Jackiw-Rebbi zero modes emerge when the singlet mass, m_s, changes sign (lowest sketch).

by interfaces of sufficiently smooth variation. As an example, consider the RS–VBS$_-$–RS setup in Fig. 2. The VBS$_-$ chain breaks a translational $\mathbb{Z}_2$ symmetry via the choice of the links harboring singlet configurations (indicated as blue ovals). If the interface is sharp, one such configuration is rigidly pinned between two RS phases, and the ground state is unique. However, for a smooth interface, dimerization patterns of either parity can be put at no difference in energy (cf. the bottom part of the figure). This leads to a $\mathbb{Z}_2$ g.s. degeneracy between phases whose ground states are individually non-degenerate.

Majorana representation: All the structures and phenomena alluded to above afford a simple and surprisingly quantitative description in a language of Majorana fermions. The passage to this representation involves the abelian bosonization [33] of the spin ladder as an intermediate step. In a second step, the bosonic degrees of freedom are mapped to an equivalent system of Majorana fermions [10]. Within the bosonized framework, smooth and rapid changes of the spin magnetization in the interaction terms are represented as gradient ('current-current') and transcendental ('massive') perturbations of the boson fields, respectively [12]. Within the fermion language, these in turn correspond to interaction terms and bilinear fermion operators, where, crucially, the former turn out to be irrelevant in a renormalization group sense. This means that, perhaps counterintuitively, the spin ladder is represented by a system of two *non-interacting* fermion fields, representing the sum and the difference $S_\pm$ of the magnetization, respectively. The fermion bilinears describe scattering between left and right moving fermions, plus effectively superconducting correlations in the S^z_- sector reflecting the absence of U(1) symmetry. Much as for the case of topological superconducting wires [2], it then pays off to switch to a Majorana fermion representation. As a result, one arrives at the low-energy continuum Hamiltonian

$$H = \int \mathrm{d}x \Big[-\frac{iv}{2}\Big(\xi^0_R \partial_x \xi^0_R - \xi^0_L \partial_x \xi^0_L\Big) - i m_s \xi^0_R \xi^0_L - \frac{iv}{2}\Big(\boldsymbol{\xi}_R \partial_x \boldsymbol{\xi}_R - \boldsymbol{\xi}_L \partial_x \boldsymbol{\xi}_L\Big) - i m_t \boldsymbol{\xi}_R \cdot \boldsymbol{\xi}_L \Big], \tag{3}$$

where $\xi^{0,1,2,3}$ are Majorana fields arranged into a singlet, ξ^0, and a triplet, $\boldsymbol{\xi} = (\xi^1, \xi^2, \xi^3)$, subject to masses [10]

$$m_t \propto 9J_\times/\pi^2 - J_\perp, \qquad m_s \propto 9J_\times/\pi^2 + 3J_\perp. \tag{4}$$

The doublets (ξ^1, ξ^2) and (ξ^0, ξ^3) represent the S^z_+ and S^z_- sectors, respectively. In the Majorana language, the U(1) $\simeq$ O(2) symmetry of the S^z_+ sector is realized as a continuous rotation symmetry between the mass-degenerate fields (ξ^1, ξ^2), and the $\mathbb{Z}_2$ symmetry of the S^z_- sector via sign inversion of $\xi^{0,3}$. Importantly, these Majorana fields are not independent but correlated via the spin parity relation (2). In the present language, the global $S^z_\pm$ quantum numbers assume the form $S^z_+ = i\sum_a \xi^2_a \xi^1_a/2$ and $S^z_- = i\sum_b \xi^3_b \xi^0_b/2$, where $\sum_{a,b}$ is a formal sum over all eigenmodes of the system. (In translational invariant cases, these are momentum modes. However, for systems with boundaries or interfaces, the situation gets more interesting.) The constraint (2) thus translates to

$$\exp(\pi \sum_a \xi^1_a \xi^2_a/2) = \exp(\pi \sum_b \xi^3_b \xi^0_b/2), \tag{5}$$

introducing entanglement between the four Majorana sectors [12].

Interfacial Majorana states: In the Majorana representation, the g.s. degeneracy of a phase is diagnosed via the appearance of MBSs localized at the system's boundaries. Here the vacuum can be represented as a fictitious Majorana system with infinitely large negative mass [34]. A vacuum interface of a system with bulk positive mass then amounts to the zero-crossing of a spatially dependent mass function $m(x)$, where the Jackiw-Rebbi mechanism implies the presence of a zero-energy MBS at each end. Since two MBSs define a fermion Hilbert space of dimension two, prior to imposing the parity constraint (5), the g.s. degeneracy of a system of definite $(J_\times, J_\perp)$ is given by $d = d_+ d_-$, $d_+ = 2^{2\Theta(m_t)}$, $d_- = 2^{\Theta(m_t)+\Theta(m_s)}$, where Θ is the Heaviside function and we use Eq. (4). For $d > 1$, (5) then implies a factor of two reduction in the actually realized g.s. degeneracy, $d \to d/2$. This integer agrees exactly with the DMRG results listed in Tab. I. The same g.s. degeneracies also follow from the bosonized formulation (II of Ref. [12]) from a truncated conformal space approach [35, 36] for sine-Gordon like models [37–51].

What happens at interfaces between ladders of different symmetry can now be understood in equally straightforward ways. Let us then return to the RS–VBS$_-$–RS hybrid, see Fig. 2. Provided the interface varies smoothly

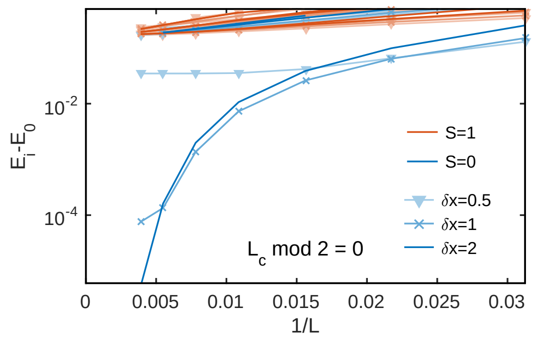

FIG. 3. Finite size scaling of low-energy SU(2) DMRG eigenstates in RS–VBS$_-$–RS ladders of total length L. Blue/red indicates singlet/triplet states ($S = 0/1$) (the g.s. is not shown). We use $J = 1$, with the other couplings varied as $J_\perp(x) = \frac{4}{3}(1 - w(x))$ and $J_\times(x) = -\frac{4}{3}w(x)$, with $w(x) = f(x - x_+) - f(x - x_-)$, $f(x) = [1 + \exp(\frac{x}{\delta x})]^{-1}$, $L_c \equiv x_+ - x_-$, and $x_\pm = (L \pm L_c)/2$. The width δx controls the interface smoothness. The g.s. degeneracy develops quickly with increasing δx and is only marginally affected by the length L_c of the VBS$_-$ region (see IV of [12]).

enough, the system is described by the Majorana theory with $m_t < 0$ but m_s changing from positive values to negative and back. We thus have MBSs at both interfaces with spatial extension determined by the width of the interface region. Naively, one might think that the same principle secures the existence of MBSs in the complementary case of VBS$_-$–RS–VBS$_-$ hybrids as well. However, there is a catch: The above argument does not make reference to the parity constraint (5). In the RS–VBS$_-$–RS case, since $m_s > 0$ in the outer RS segments, MBSs will not only exist at the internal interfaces but also at the outer vacuum boundaries, cf. Fig. 2(b). This implies that changes in the occupation of the internal MBS system can be compensated by the outer MBS system, which may act as a 'parity sink' to restore the condition (5). In concrete terms, the + sector of the RS–VBS$_-$–RS ladder is even parity and has a unique g.s. as $d_+ = 1$ for the RS and VBS$_-$ segments. On the other hand, the $-$ sector is nominally 4-fold degenerate (as $d_- = 2$ for each RS segment), but only two of the four states have even parity, thus leaving only two allowed states once we combine the $\pm$ sectors.

In Fig. 3 we present DMRG results showing that the RS–VBS$_-$–RS ladder indeed has a doubly degenerate ground state for smooth interfaces. If $J_\perp$ and $J_\times$ defining these phases vary too sharply, the ground state remains unique. We explain why this is so field theoretically in III of [12]. However, once the scale of variation extends over just a few lattice sites, one rapidly approaches a two-fold degenerate g.s. We also note that the energy gap protecting the g.s. degeneracy is rather large for the example in Fig. 3. It is remarkable that MBSs are generated in the RS–VBS$_-$–RS example, where none of the individual parts, VBS$_-$ or RS, support such states. Those MBSs also provide a means to distinguish two different SPT-trivial phases, cf. Refs. [31, 32]. The situation is rather different for the VBS$_-$–RS–VBS$_-$ system. Since one of the two fermion states formed from the central MBS pair is parity blocked, MBSs are effectively removed from the zero energy Hilbert space [52]. See IV.B.2 of [12] for verification of this via DMRG. In this way, the parity constraint trumps the Jackiw-Rebbi principle.

Interfaces between phases of enriched symmetry define higher-dimensional MBS systems. As an example, consider the RS–H–RS hybrid. Although the g.s. degeneracy of the central H segment (the outer RS phases) is only four-fold (unique), the interfaces harbor a potential 32D zero-energy space, with four MBSs at either side of the H segment since four masses change sign upon crossing from one phase into the other. Parity, as in the RS–VBS$_-$–RS ladder, reduces this by one-half (see IV.B.3 of [12]).

Reality check: The above constructions demonstrate that spin ladder materials provide a remarkably rich platform for the isolation of zero energy MBSs, with sizeable energy gaps to higher-lying states. In view of the general interest in MBSs it is imperative to ask how our non-topological MBSs fare in comparison to topologically protected MBSs. At first sight, the absence of topological protection appears to be a crucial setback. However, at present the probably most obtrusive effect hampering Majorana device functionality is the buildup of long-range MBS hybridizations. In topological devices the hybridization exponentially approaches zero with increasing distance but can nonetheless be large in practice. For example, in hybrid semiconductor wires, topological protection crucially relies on the rather tiny superconducting proximity gaps [2, 53–56]. In the present setup, the lack of topological protection manifests itself in long-range correlations between MBSs when short range correlations of the underlying spin chains are changed (in particular, the interface roughness). However, the degrees of freedom behind such changes are highly inert in realistic systems since they require energy scales comparable to the exchange couplings. Even though these energy scales do not grow with system size, they can be sufficiently large to provide efficient MBS protection at low temperatures.

Outlook: A promising aspect of our approach is that it brings a plethora of material platforms into play. While we have focused on spin ladders, similar considerations apply to many quasi-1D materials, in particular those that admit a bosonization treatment, e.g., N-leg Heisenberg ladders with SU(2) spin symmetry [18, 27, 57] or a more general SU(M) symmetry [19, 28, 58], coupled chains of itinerant electrons [21, 59–61], or coupled Luttinger liquid systems [62, 63]. In addition, our setup directly comes with an intrinsic source of strong entanglement. Indeed, the Majorana parity constraint (5) plays a similar role to the strong Coulomb charging energy [64–

67] in mesoscopic MBS systems, where a related parity constraint implies qubit functionality [68, 69]. The question of how this entanglement mechanism may be turned into an operational resource, and how the MBSs discussed here can be probed and/or manipulated, is an interesting subject for future study.

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