This is the accepted manuscript made available via CHORUS. The article has been published as:

Coulomb Blockade of a Nearly Open Majorana Island
Dmitry Pikulin, Karsten Flensberg, Leonid I. Glazman, Manuel Houzet, and Roman M. Lutchyn
Phys. Rev. Lett. 122, 016801 — Published 3 January 2019
DOI: 10.1103/PhysRevLett.122.016801
**Coulomb blockade of a nearly-open Majorana island**

Dmitry Pikulin,¹ Karsten Flensberg,² Leonid I. Glazman,³ Manuel Houzet,⁴ and Roman M. Lutchyn¹

¹Microsoft Quantum, Microsoft Station Q, University of California, Santa Barbara, California 93106-6105 USA
²Center for Quantum Devices, Niels Bohr Institute, University of Copenhagen, DK-2100 Copenhagen, Denmark
³Department of Physics, Yale University, New Haven, CT 06520, USA
⁴Univ. Grenoble Alpes, CEA, INAC-Phelipps, F-38000 Grenoble, France

(Dated: November 30, 2018)

We consider the ground-state energy and the spectrum of the low-energy excitations of a Majorana island formed of topological superconductors connected by a single-mode junction of arbitrary transmission. Coulomb blockade results in $e$-periodic modulation of the energies with the gate-induced charge. We find the amplitude of modulation as a function of reflection coefficient $\mathcal{R}$. The amplitude scales as $\sqrt{\mathcal{R}}$ in the limit $\mathcal{R} \to 0$. At larger $\mathcal{R}$, the dependence of the amplitude on the Josephson and charging energies is similar to that of a conventional-superconductor Cooper-pair box. The crossover value of $\mathcal{R}$ is small and depends on the ratio of the charging energy to superconducting gap.

The Coulomb blockade phenomenon is associated with the localization of charge in a small conductor with appreciable charging energy. The Coulomb blockade results in the observable quantities being periodic functions of the charge induced by an applied gate voltage. For a normal system, this periodicity in the induced charge is $e$ while for an island of conventional ($s$-wave) superconductor, a so-called Cooper-pair box, the periodicity is $2e$.

With a junction between the island and a lead, charging effects are smeared by delocalization of the electrons. Remarkably, the Coulomb blockade is fully suppressed by the presence of even a single reflectionless channel in the junction [1]. The way oscillations vanish depends on the relevant low-energy excitations. For normal-state conductors, the spectrum is continuous and gapless; the effect of weak reflection can be read off from known results for a quantum impurity in a Luttinger liquid [2, 3]. When the island and the lead are $s$-wave superconductors, the ground state is non-degenerate and separated from the continua by gaps. In this case, the description of the Coulomb blockade is given by an imaginary-time version of the Landau-Zener diabatic crossing of two in-gap levels, with the off-diagonal matrix element being proportional to the backscattering amplitude [4].

In this Letter, we elucidate the nature of the suppression of Coulomb blockade in a nearly-open system made of topological superconductors, illustrated in Fig. 1. The topological superconductors are characterized by a finite gap in the energy spectrum, coexisting with a nontrivial degeneracy of the ground state, which causes the periodicity in the induced charge to be $e$ and not $2e$. This difference in the states and spectra from both conventional superconductors and normal metals results in a different underlying physics of the disappearance of Coulomb blockade oscillations at perfect transmission. We show that it is related to the physics of diabatic transitions between a discrete state and a continuum of itinerant states, and we formulate a quantitative theory valid for the crossover from a regime where the amplitude of Coulomb blockade oscillations is proportional to the reflection amplitude, to a regime where the physics is similar to a conventional Cooper-pair box [5].

The system shown in Fig. 1 has become experimentally relevant since the appearance of viable theoretical models of one-dimensional topological superconductors [6–9]. Several recent experiments reported data consistent with topological superconductivity in Coulomb blockade devices [10–12], thus opening a perspective for the experimental study of the quantum charge fluctuations considered here. Moreover, topological superconducting islands have been the basis for several proposals for Majorana-based qubits [13–16], some of which [13, 14] use control of the charging energy to lift the ground-state degeneracy. The theory of such control is another application of our work.

Conventional transmon qubit is a Cooper pair box with the charging energy much smaller than the Josephson energy. This arrangement is chosen to suppress charge
fluctuations and increase coherence time of the qubit. In
the present work we focus on the case where the charging
energy $E_C$ is relatively small, $E_C \ll \Delta$ (here $\Delta$ is the
superconducting gap in the topological phase, it also fixes
the scale of the Josephson energy in the single-channel
junction), which is also the limit considered for a con-
ventional transmon [5]. We find that the gate-induced
charge $\epsilon N_g$ modulates the energy levels of the topologi-
ical transmon,
\[ \delta E_m(N_g) = (-1)^{m+1} \frac{\epsilon_m}{2} \cos(2\pi N_g), \tag{1} \]
where $m$ labels the energy levels, with $m = 0$ being the
ground state [17]; unlike the conventional transmon, the
modulation period is $c$. The charge sensitivity comes from
the Aharonov-Casher effect [18] in tunneling of the
phase variable $\varphi$ between the classically-equivalent min-
ima ($\varphi = 0, 4\pi$ in Fig. 2). The modulation amplitude $\epsilon_m$
is
\[ \epsilon_m = F(h) \cdot E_C \frac{2^{m+3}}{m!} \sqrt{\frac{2}{\pi}} \left( \frac{E_M}{E_C} \right)^{\frac{2m+3}{2}} e^{-4\sqrt{E_M/E_C}}. \tag{2} \]
Here $E_M = \Delta \sqrt{1-R}$ is the height of the barrier sepa-
rating the two minima of the ground-state energy in the
absence of charging, and $R$ is the reflection coefficient.
Apart from the function $F(h)$, Eq. (2) closely resembles
the respective formula [5] for a conventional transmon. It
is valid if the electron system is able to adjust to the in-
stantaneous values of $\varphi$ in the course of tunneling. Such
adiabaticity requires a sufficiently large value of the re-
fection coefficient $R$. The function $F(h)$ describes the
crossover between the diabatic and adiabatic regimes,
\[ F(h) = \frac{3^{1/6}}{2^{1/3}} \Gamma(2/3) h \approx 1.02 h, \quad h \ll 1, \tag{3} \]
\[ F(h) = 1 - \frac{\pi}{8} \cdot h^{-3} \approx 1 - 0.39 h^{-3}, \quad h \gg 1. \tag{4} \]
It depends on a single variable,
\[ h = \left( \frac{\Delta}{16E_C} \right)^{1/6} \sqrt{R}. \tag{5} \]
We first note that $F(0) = 0$, $i.e.$, in the absence of re-
fection $\delta E_m = 0$, in agreement with the general prop-
erties [2-4, 19, 20] of the Coulomb blockade effect dis-
cussed in the introduction. Below, we derive Eqs. (1)-(5)
and show that the entire crossover from $F(h) \to 0$ to
$F(h) \to 1$ occurs in a narrow region of reflection coeffi-
cients, $R \sim (16E_C/\Delta)^{1/3} \ll 1$ [21].

At zero charging energy, phase $\varphi$ across the junction
is a good quantum number. Assuming that only one
pair of helical modes propagates across a short junction,
the phase-dependent part of the ground state energy in
the sector with an even number of electrons takes the form
[6, 22]
\[ E_G(\varphi) = -\frac{1}{2} E_M \cos(\varphi/2). \tag{6} \]
Here the sign is fixed by the total parity which we as-
sume to be conserved. Furthermore, in a ballistic junc-
tion ($R = 0$), the momentum associated with the propa-
gating modes is conserved. The bound states are formed
out of states of one chirality: these are, respectively,
the right-movers at $0 < \varphi < 2\pi$ and left-movers at
$2\pi < \varphi < 4\pi$, cf. the solid (red) and bold-dashed (black)
curves in Fig. 2. The two bound states become degener-
ate with each other and with the edge of the continuum
at $\varphi = 2\pi$. In the presence of backscattering induced by
any finite $R$, both left- and right-movers participate in
the formation of the continuum and bound states. As
a result, the degeneracy is lifted, and the gap between
the ground state and continuum, $\frac{1}{2}(\Delta - E_M)$, is finite at
$\varphi = 2\pi$.

Finite charging energy endows the phase with quantum
dynamics; the same-parity, classically-distinguishable
states corresponding to $\varphi = 0, 4\pi, \ldots$ may hybri-
dize. The hybridization does not occur at $R = 0$, as these
states are protected by the movers’ momentum conser-
vation, but they do hybridize at $R \neq 0$. At small charg-
ing energy, $E_C \ll \Delta$, one may view the hybridization as
the result of phase tunneling between the nearest minima
($\varphi = 0, 4\pi$ in Fig. 2).

If the gap $\frac{1}{2}(\Delta - E_M)$ is large enough, phase tunnel-
ing occurs in the adiabatic regime and is governed by
Hamiltonian
\[ H_0 = E_C (-2i\partial_\varphi - \hat{N}_g)^2 + E_G(\varphi) \tag{7} \]
acting in the space of $4\pi$-periodic functions. Here
$\hat{N} = -2i\partial/\partial\varphi$ is the operator for the electron number of
the island. To find the energy spectrum of $H_0$ as a function
of $N_g$, we map the problem onto the known one for the
conventional transmon [5] and find Eq. (2) with $F(h)$
replaced by 1 (see Sections I and VIII of [23] for details).

The adiabatic approximation fails if the gap $\frac{1}{2}(\Delta - E_M)$
is small. The corresponding quantum dynamics of the
many-body state in the topological case is very different

![Energy spectrum of a topological junction in the absence of backscattering. At $R = 0$, the bound states are degenerate at $\varphi = 2\pi$ mod $4\pi$ with the edge of continuum (shaded area).](image-url)
from that in the conventional s-wave case [4]. Disregarding for a moment the difference between driving the variable \( \varphi \) classically and allowing it to tunnel, one may say that the conventional problem is related to the Landau-Zener passage of an avoided crossing between two discrete many-body states. On the contrary, Coulomb blockade in the topological junction is related to a Demkov-Osherov process involving a discrete state and continuum [24].

We may estimate \( \mathcal{R} \) at which adiabaticity is violated by a qualitative consideration that ignores the difference between the real-time evolution and tunneling of the phase (i.e., “imaginary-time” evolution) across the \( \varphi = 2\pi \) point. The separation \( E_{\text{ex}}(\theta) \) of the bound state energy from continuum is small at \( \mathcal{R} \ll 1 \) and \( |\varphi - 2\pi| \ll 1 \); using Eq. (6), we find (hereinafter \( \varphi = 2\pi \))

\[
E_{\text{ex}}(\theta) = \frac{1}{4} \left( \mathcal{R} + \frac{\theta^2}{4} \right) \Delta. \tag{8}
\]

The energy \( E_{\text{ex}}(\theta) \) can be estimated as \( E_{\text{ex}}(\theta^*) \sim \mathcal{R}\Delta \) everywhere within the interval \( |\theta| \lesssim \theta^* \), where \( \theta^* = \sqrt{\mathcal{R}} \).

In the (imaginary) time domain, it takes time \( \tau(\theta^*) \sim \theta^*/\omega_P \) to pass this interval; here \( \omega_P = \sqrt{E_CE_M} \approx \sqrt{E_C}\Delta \) is the Josephson plasma frequency which determines the time scale for both oscillations and tunneling of the phase. The phase is passing the point \( \theta = 0 \) adiabatically if \( E_{\text{ex}}(\theta^*)\tau(\theta^*) \gg 1 \). Under that condition, the electron system adjusts to the instantaneous value of \( \varphi \) and the use of Hamiltonian (7) at any \( \varphi \) is justified.

Expressing \( E_{\text{ex}}(\theta^*) \) and \( \tau(\theta^*) \) in terms of \( \mathcal{R} \) and utilizing the definition (5), we find that the adiabaticity is violated at \( h \approx 1 \), which indeed is the crossover scale for the function \( F(h) \), cf. Eq. (2).

To quantify the crossover behavior, we notice that Eq. (7) determines the dynamics of the many-body state in the Born-Oppenheimer (adiabatic) approximation with \( \varphi \) being the slow variable. In that approximation, the eigenfunction of the system is factorized, \( \Psi((x_i),\varphi) \approx \Psi_x((x_i))\psi(\varphi) \). The first factor here is the many-body BCS wave function of the electron ground state at a given phase \( \varphi \). The phase-dependent part of the corresponding energy, \( E_C(\varphi) \), appears in Eqs. (6) and (7). The single-particle states comprising \( \Psi_x((x_i)) \) are defined by the Bogoliubov-de Gennes (BdG) equations where \( \varphi \) is treated as a parameter. The second factor, \( \psi(\varphi) \), is an eigenfunction of Eq. (7). If \( \mathcal{R} \gg (E_C/\Delta)^{1/3} \) (i.e., \( h \gg 1 \)), then the Born-Oppenheimer wave function is a good leading-order approximation at all \( \varphi \).

In the opposite case, \( h \ll 1 \), we use the condition \( E_{\text{ex}}(\theta)\tau(\theta) \gtrsim 1 \) to determine the range of \( \varphi \) (within the period \([0, 4\pi]\)) where the adiabatic approximation is applicable. That yields \( |\varphi - 2\pi| \gtrsim (E_C/\Delta)^{1/6} \). Our strategy is to find \( \Psi((x_i),\varphi) \) in the region \( |\varphi - 2\pi| \ll 2\pi \) by a method inspired by Demkov-Osherov approach [24] and then match the found \( \Psi((x_i),\varphi) \) with the Born-Oppenheimer wave function in the common region of applicability \((E_C/\Delta)^{1/6} \ll |\varphi - 2\pi| \ll 2\pi \). Knowing the wave functions in the entire interval \([0, 4\pi]\) allows us to find the dependence of energy spectrum on \( N_g \).

To illustrate the strategy, we concentrate on finding \( \delta E_0(0) \), cf. Eq. (1). In the vicinity of \( \varphi = 0 \), the function \( \psi(\varphi) \) is well approximated by the eigenstate of a harmonic oscillator,

\[
\psi(\varphi) = \frac{(\Delta/E_C)^{1/8}}{(8\pi)^{1/4}} \exp\left(-\frac{\varphi^2}{16\sqrt{\Delta/E_C}}\right). \tag{9}
\]

Next we extend Eq. (9) to the apex of the classically-forbidden region, \( 2\pi \gg 2\pi - \varphi \gg \max[\sqrt{\mathcal{R}}/(E_C/\Delta)^{1/6}] \), by using WKB approximation. This yields

\[
\psi(\theta) = \frac{(\Delta/E_C)^{1/8}}{(2\pi)^{1/4}} e^{-2\sqrt{\Delta/E_C}} \exp\left(-\frac{\theta - \theta^4/96}{2\sqrt{E_C/\Delta}}\right). \tag{10}
\]

Clearly, the exponentially small factor in Eq. (10) does not affect the normalization factor in Eq. (9). The extension of Eqs. (9) and (10) to arbitrary \( N_g \) and for the entire classically-forbidden region is given in Sections I, II, and III of [23].

Finding the many-body state is simplified by the observation that the phase-dependent energy \( E_C(\varphi) \) of a short junction comes from one single-particle bound state (the latter is formed by two Majorana states \( \gamma_2, \gamma_3 \) hybridized across the junction, see Fig. 1). That allows us to replace \( \{x_i\} \) by a single generalized coordinate, \( \Psi((x_i),\varphi) \rightarrow \Psi(x,\varphi) \). In the vicinity of \( \theta = 0 \), the activation energy of the bound state becomes small, see Eq. (8). That further simplifies the problem, as the relevant states are linear combinations of quasiparticle wave functions with energies close to \( \Delta \). Similar to the effective mass approximation in the theory of semiconductors [25], we construct an effective Hamiltonian [26, 27]

\[
H_{\text{eff}} = 4E_C(-i\partial_\theta - N_g/2)^2 \tag{11}
+ \frac{1}{2} \left\{ \frac{\bar{v}_F^2}{2\Delta} (i\partial_x)^2 - v_F \left( \frac{\theta}{2\sigma_z} + \sqrt{\mathcal{R}}\sigma_x \right) \delta(x) \right\} + \frac{\Delta}{2};
\]

here \( \sigma_{x,y,z} \) are Pauli matrices in the space of right/left-propagating states and \( v_F \) is the Fermi velocity (it drops out from final results). The divergent-at-the-gap density of states and energy \( E_{\text{ex}}(\theta) \) are correctly described by \( H_{\text{eff}} \), see Section IV in [23]. Note that \( [\hat{\sigma}_z, H_{\text{eff}}] = 0 \) at \( \mathcal{R} = 0 \), and the bound states at \( \theta > 0 \) and \( \theta < 0 \) belong to orthogonal sub-spaces. Therefore, at \( \mathcal{R} = 0 \) there is no tunneling between the \( \varphi = 0, 4\pi \) minima, consistent with momentum conservation.

As we are interested in states with energy \( E \approx -\Delta/2 \) (see Fig. 2), the problem can be further simplified by factoring out the leading (linear in \( \theta \)) exponential term in the wave function and replacing \( x \) and \( \theta \) by dimensionless variables \( y \) and \( z \):

\[
\Psi(x,\theta) = \exp \left(-\sqrt{-\Delta/4E_C}\theta \right) \Psi(y, z), \tag{12}
\]

\[
x = 2^{-2/3}(\Delta/E_C)^{1/6}v_F/\Delta y, \quad \theta = 2^{5/3}(E_C/\Delta)^{1/6} z.
\]
In the new variables, the Schrödinger equation for $\Psi(y,z)$ at $\mathcal{N}_g = 0$ depends on a single parameter $\hbar$ given by Eq. (5):

$$\left( \partial_z - \frac{1}{2} \partial_y^2 - (z\hat{\sigma}_z + \hbar \hat{\sigma}_x)\delta(y) \right) \Psi(y,z) = 0. \quad (13)$$

Its solution in the Born-Oppenheimer approximation,

$$\Psi^{(0)}(y,z) = \psi^{(0)}(y)g^{(0)}(z)\tilde{U}(z)\chi, \quad (14)$$

$$\psi^{(0)}(y) = 2^{1/3} \frac{(E_C/\Delta)^{1/12}}{\left| \frac{v_F}{\kappa_z} \right|^{1/2}} e^{-\kappa_z |y|}, \quad (15)$$

$$g^{(0)}(z) = \frac{(\Delta/E_C)^{1/8}}{(2\pi)^{1/4}} e^{-2\sqrt{\Delta/E_C}} \exp \left( \frac{1}{2} \int_0^z dz' \kappa_z^2 \right),$$

rotates it to align with the $z$-dependent quantization axis. The solution in the Born-Oppenheimer approximation, Eq. (10), reproduces Eq. (7) in its region of validity [upon returning from $g^{(0)}(z)$ to $g(y)$. Here $\kappa_z = (z^2 + \hbar^2)^{1/2}$, pseudo-spinor $\chi$ is an eigenvector, $\hat{\sigma}_z \chi = \chi$, and the unitary operator

$$\hat{U}(z) = \exp \left[ -\frac{iz}{\hbar} \cot^{-1} \left( \frac{z}{\hbar} \right) \right] \quad (15)$$

represents the rotation of the Hamiltonian, $\hat{H}$, by $\pi / 4$.

The rotation rate in Eq. (15) scales as $1/\hbar$; obviously, the adiabatic approximation fails at $\hbar \ll 1$. We develop perturbation theory in $\hbar$ to find the energy eigenvalues in this limit. At $\hbar = 0$, we can take advantage [24] of the linear $z$-dependence of a coefficient in Eq. (13) and solve the partial differential equations for $\sigma_z = \pm 1$ analytically. For that, we apply the Fourier transformation to Eq. (13),

$$\psi^{(0)}(y) = 2^{1/3} \frac{(E_C/\Delta)^{1/12}}{\left| \frac{v_F}{\kappa_z} \right|^{1/2}} e^{-\kappa_z |y|}, \quad (16)$$

$$F_{\sigma_z}(p) \equiv \int_{-\infty}^{\infty} \frac{dk}{2\pi} \psi^{(0)}(k,p),$$

which allows us to obtain a closed first-order differential equation for $F_{\sigma_z}(p)$,

$$-i\sigma_z [e^{-ix/4}/(2p)^{1/2}] \partial_p F_{\sigma_z}(p) = F_{\sigma_z}(p) \quad (17)$$

where $p^{1/2} > 0$ for $p > 0$. Solution of Eq. (17) followed by inserting the Fourier transform $\psi^{(0)}(k,p)$ of Eq. (16) yields

$$\psi_{-1}(y,-z) = \psi_1(y,z) \approx 2^{7/12} \frac{\pi^{1/4}}{\sqrt{\Delta/E_C}} (\Delta/E_C)^{1/24} (\Delta/v_F)^{1/2} \times \int_{-\infty}^{\infty} \frac{dp}{2\pi} \exp \left[ ipz - (2ip)^{1/2} |y| + \frac{2}{3} i(1+p)^3/2 \right]. \quad (18)$$

The constant of integration here is found by matching the $|z| \gg 1, z < 0$ asymptote of Eq. (18) with the Born-Oppenheimer limit, Eqs. (14). Knowing the wave functions (18) at $h = 0$, we may express the first-order correction to energy in terms of the matrix element of perturbation, $\langle \psi_{-1}(y,z) | h \hat{\sigma}_x \delta(y) | \psi_1(y,z) \rangle$,

$$\epsilon_0 = 2^{3/4} v_F \sqrt{\mathcal{R}} (E_C/\Delta)^{1/6} \int_{-\infty}^{\infty} dz \psi_1^*(0,z) \psi_{-1}(0,z). \quad (19)$$

Performing the integration with the help of Eq. (18), we arrive at the asymptote (3), see also Section VI of [23].

In the opposite case, $h \gg 1$, we find correction (4) by perturbing away from the adiabatic limit, Eqs. (14). The correction stems from the perturbations $\partial_z \hat{U}(z), \partial_z \psi_{(0)} \propto 1/\hbar$ appearing in Eq. (13) upon substitution of Eqs. (14) and (15) in it. We are interested in the correction which vanishes at $z \to -\infty$ and modifies the asymptote of the adiabatic, localized in $y$, solution at $z \gg 1$. The perturbations, effective in the region $|z| \ll \hbar$, mix the localized state with the itinerant ones, differing in energy by $\sim \hbar^2$. Therefore the modification of the localized state $\psi^{(0)}(y,z)$ appears in the second-order perturbation theory. The power counting thus gives +1 from the term in the Hamiltonian, $-2$ from the second order perturbation theory, and $-2$ from the energy cost giving the correction $\propto 1/\hbar^3$. The evaluation of the numerical coefficient appearing in Eq. (4) is presented in Section VII of [23].

The interpolation between the diabatic and adiabatic asymptotes of $F(h)$ is shown in Fig. 3. It is obtained by generalizing $\mathcal{R}_{\text{eff}}$ to arbitrary phases with the help of substitution $\theta/2 \to 2 \sin(\theta/4)$ in Eq. (11). The generalized Hamiltonian, being projected at $\mathcal{R} \ll 1$ on its low-energy sector, reproduces Eq. (7) in the region of phases $|\theta| \gg (E_C/\Delta)^{1/6}$. By finding numerically the energy spectrum of that Hamiltonian, we get the relative amplitude of the gate modulation, $F$, as a function of two parameters $\mathcal{R}$ and $E_C/\Delta$ (see details in Section IX of [23]). The results at the lowest values of $E_C/\Delta$ are compatible with $F$ depending on a single parameter, $\sqrt{\mathcal{R}(E_C/\Delta)}^{1/6} \propto \hbar$, and having asymptotes (3) and (4).

To conclude, we addressed the problem of the crossover from a pronounced charging effect to its full absence in a topological superconducting junction upon reduction of the reflection coefficient $\mathcal{R}$. The many-body prob-
the superconducting coherence length). Here we assume \( \delta E \) is negligible.


[21] To describe \( \delta E_m \) at larger \( R \), still assuming \( \Delta \gg E_C \), one may set \( F(h) = 1 \) and replace Eqs. (1) and (2) by the proper Mathieu characteristic value at given \( E_M/E_C \), see Section VIII of [23]. In the opposite limit, \( \Delta \ll E_C \), gate charge modulation of the ground state energy at \( R \ll 1 \) is hardly sensitive to superconductivity and was addressed in Ref. [3].

[22] For definiteness, hereinafter we assume the device is in an even total-parity state; the sign of the right-hand side is opposite in the odd-parity state.

[23] More details can also be found in the supplementary information.


