Asymptotic Prethermalization in Periodically Driven Classical Spin Chains

Owen Howell, Phillip Weinberg, Dries Sels, Anatoli Polkovnikov, and Marin Bukov

Phys. Rev. Lett. **122**, 010602 — Published 9 January 2019

DOI: 10.1103/PhysRevLett.122.010602

Asymptotic Pre-thermalization in Periodically Driven Classical Spin Chains

Owen Howell,[1,*] Phillip Weinberg,[1] Dries Sels,[1,2,3] Anatoli Polkovnikov,[1] and Marin Bukov[4,†]

[1] *Department of Physics, Boston University, 590 Commonwealth Ave., Boston, MA 02215, USA*
[2] *Department of Physics, Harvard University, 17 Oxford st., Cambridge, MA 02138, USA*
[3] *Theory of quantum and complex systems, Universiteit Antwerpen, B-2610 Antwerpen, Belgium*
[4] *Department of Physics, University of California, Berkeley, CA 94720, USA*

(Dated: October 17, 2018)

We reveal a continuous dynamical heating transition between a prethermal and an infinite-temperature stage in a clean, chaotic periodically-driven classical spin chain. The transition time is a steep exponential function of the drive frequency, showing that the exponentially long-lived prethermal plateau, originally observed in quantum Floquet systems, survives the classical limit. Even though there is no straightforward generalization of Floquet's theorem to nonlinear systems, we present strong evidence that the prethermal physics is well-described by the inverse-frequency expansion. We relate the stability and robustness of the prethermal plateau to drive-induced synchronization not captured by the expansion. Our results set the pathway to transfer the ideas of Floquet engineering to classical many-body systems, and are directly relevant for photonic crystals and cold atom experiments in the superfluid regime.

Periodically-driven systems are currently experiencing an unprecedented revival of interest through theoretical and experimental design of novel states of matter. Commonly known as *Floquet engineering*, this approach has enjoyed success in the regime of high driving frequency, where it has been appreciated as a useful tool to ascribe novel properties to otherwise trivial static Hamiltonians [1–3]. Prominent examples include the Kapitza pendulum [4], cold-atom realisations of topological [5–12] and spin-dependent [13] bands, artificial gauge fields [14–22], spin-orbit coupling [23, 24], enhanced magnetic correlations [25], synthetic dimensions [26–28], and photonic topological insulators [29–31].

The applicability of Floquet engineering requires the ability to prepare the periodically driven system in the corresponding Floquet state [32–35], and the stability of the system to detrimental heating [36–41]. Presenting a major bottleneck at the forefront of present-date experimental research, heating processes play an important role in many-body Floquet systems, and understanding the underlying physics is expected to offer significant advances in the field. Unlike single-particle quantum systems, such as the kicked rotor [42] and weakly-interacting bosonic models [43, 44], it is believed that generic isolated clean periodically-driven quantum many-body systems heat up to an infinite-temperature state [45–50], although the debate is not fully settled [51–56]. Heating rates have been shown to be at least exponentially suppressed in the drive frequency [57, 58].

In this paper, we present a numerical study of thermalisation in a clean, globally-driven, isolated *classical* spin chain, reaching times beyond the astronomical 10^{10} driving cycles. We find that, the dynamics falls into four stages, see Fig. 1: an initial transient (i) during which the

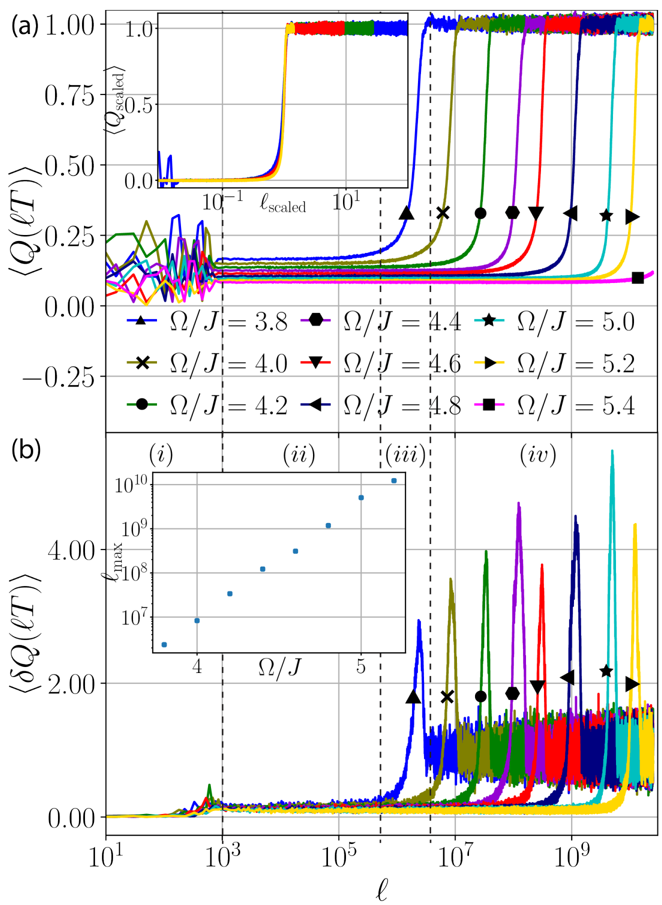

FIG. 1: Noise-averaged energy (a) and energy variance (b) as a function of the number of driving cycles ℓ. Insets: rescaled energy curves (a) by the position of the peak in the variance curves (b), reveal a dynamical heating transition from a prethermal stage to an infinite-temperature stage in the limit $\Omega, \ell \to \infty$. The dashed vertical lines mark the four stages of evolution for $\Omega/J = 3.8$ [see text]. See Fig. 2 for parameters.

*Electronic address: olh20@bu.edu
†Electronic address: mgbukov@berkeley.edu

system exhibits *constrained* thermalization to a finite energy density set by the initial ensemble, (*ii*) a frequency-controlled long-lived prethermal plateau [57, 59–61] ideally suited for Floquet engineering, followed by (*iii*) a late crossover governed by *unconstrained* thermalisation to (*iv*) a featureless infinite-temperature state. Our analysis reveals the existence of a dynamical heating transition between a prethermal plateau and an infinite-temperature stage at infinite times and infinite drive frequencies. The prethermal plateau is described by the inverse-frequency expansion originally developed for quantum systems, with improving agreement at increasing first few orders of the expansion [62]. This allows to transfer the machinery of Floquet engineering directly to classical many-body models despite the absence of a Floquet theorem for systems governed by nonlinear equations of motion. Focusing on the prethermal plateau, we demonstrate that a key role for its exponentially long duration is played by drive-induced synchronization. In particular, stopping the periodic drive and then restarting it de-synchronizes the system and strongly increases the heating rate.

We report that the time scale for a classical system to leave a small corner of phase space around the initial state and heat up to an infinite-temperature state, scales exponentially with the driving frequency [Fig. 1]. A related problem on ergodicity in classical systems was addressed in a pioneering numerical study by Fermi, Pasta, Ulam and Tsingou, and gave first evidence for a parametrically slow thermalisation in a system of coupled classical oscillators [63, 64], puzzling the community for the past few decades. For systems with finite number of degrees of freedom, theorems in Hamiltonian mechanics have been proven showing that the motion of action variables in nearly-integrable systems remains confined to a small region of phase space until exponentially long times, controlled by the integrability breaking parameter [65–68]. This behaviour is accompanied by subdiffusion, as reported for a system of periodically-kicked coupled pendula [69–72]. Chaotic many-body dynamics in periodically-kicked spin chains has been studied using a classical Loschmidt echo approach [73, 74].

Model.—Consider a classical Ising chain with periodic boundary conditions, described by the energy function

$$H(t) = \begin{cases} \sum_{j=1}^{N} J S_j^z S_{j+1}^z + h S_j^z & \text{for } t \in [0, T/2] \bmod T \\ \sum_{j=1}^{N} g S_j^x & \text{for } t \in [T/2, T] \bmod T \end{cases}$$

where J denotes the nearest-neighbour interaction strength, while h and g are the magnetic field strengths along the z and x-directions, respectively. The spin [or rotor] variable $\vec{S}_j$, $|\vec{S}_j| = 1$, on site j satisfies the Poisson bracket relation $\{S_i^\mu, S_j^\nu\} = \delta_{ij}\varepsilon^{\mu\nu\rho} S_j^\rho$, with $\varepsilon^{\mu\nu\rho}$ the fully antisymmetric tensor.

The time dependence arises due to periodic switching of two time-independent Hamilton functions, for a duration of $T/2$ each, with frequency $\Omega = 2\pi/T$. The time evolution of the system is governed by Hamilton's EOM $\dot{S}_j^\mu(t) = \{S_j^\mu, H(t)\}$. Interested in the long-time thermalisation properties, we focus on stroboscopic evolution. Integrating the EOM over one total period T, the evolved state is obtained from a successive application of a discrete map $\vec{S}_j(\ell T) = [\tau_2 \circ \tau_1]^\ell (\vec{S}_j(0))$, with $\ell \in \mathbb{N}$ counting the driving cycles. During the first half-period, the time evolution follows the *non-linear* rotation τ_1 about the z-axis:

$$\tau_1(\vec{S}_j) = \begin{bmatrix} S_j^x \cos(\kappa_j T/2) - S_j^y \sin(\kappa_j T/2) \\ S_j^x \sin(\kappa_j T/2) + S_j^y \cos(\kappa_j T/2) \\ S_j^z \end{bmatrix} \quad (1)$$

with spin-dependent natural frequency of rotation $\kappa_j = J(S_{j-1}^z + S_{j+1}^z) + h$. The dynamics in the second half-period follows the rotation τ_2 about the x-axis:

$$\tau_2(\vec{S}_j) = \begin{bmatrix} S_j^x \\ S_j^y \cos(gT/2) - S_j^y \sin(gT/2) \\ S_j^y \sin(gT/2) + S_j^z \cos(gT/2) \end{bmatrix} \quad (2)$$

The map $\tau_2 \circ \tau_1$ is the nonlinear classical analogue of the quantum Floquet unitary.

Motivated by experiments which study Floquet-engineered ordered states at high drive frequencies, we prepare the system at time $t = 0$ in the lowest-energy state (i.e. the ground state, GS) of the time-averaged Hamiltonian

$$H_{\text{ave}} \equiv H_F^{(0)} = \frac{1}{2} \sum_{j=1}^{N} J S_j^z S_{j+1}^z + h S_j^z + g S_j^x, \quad (3)$$

with energy density $E_{\text{GS}}(h/J, g/J)/N \approx -1.235J$ for $g/J = 0.9045$, $h/J = 0.809$. Whenever J, h, g have equal sign and are of the same order of magnitude, the GS features antiferromagnetic (AFM) order w.r.t. a direction in the xz-plane, parametrized by the azimuthal angle θ. Making use of translational invariance, one can determine the value of θ which minimizes the energy $E_{\text{GS}}(\theta)$. Translational invariance constrains the GS evolution to be uniquely described by two coupled spin degrees of freedom, corresponding to the AFM unit cell. To bring out the many-body character of the model, we add small noise to the azimuthal angle of each spin, independently drawn from a uniform distribution over $[-\pi/100, \pi/100]$, which breaks translational symmetry and allows for thermalisation. The quantities we consider are averaged over an ensemble of 100 noisy initial state realizations. We verified that the long-time dynamics is independent of the strength of the noise [62], provided the latter remains small enough to not significantly change the energy of the initial state. In the following, we denote by $\langle\cdot\rangle$ the average over the ensemble of noise realizations.

Heating Transition.—Compared to classical systems, studies on thermalising dynamics in quantum models feature serious deficiencies, due to significant finite-size effects inherent to state-of-the-art exact diagonalization simulations. Since energy absorption is known to happen through Floquet many-body resonances [75–77], (i), their density depends strongly on the drive frequency at any fixed many-body bandwidth. As the bandwidth scales linearly with the system size N, this puts an upper bound on Ω for the system to be in the many-body regime. This also limits the occurrence of higher-order absorption processes, reducing the overall capacity for energy absorption. (ii) Low-energy initial states, whose energy level spacing does not follow the 2^{-N} scaling of the bulk, further restrict the appearance of resonances. However, these issues are intrinsic to quantum models and none of them is problematic in periodically-driven classical systems. The classical energy manifold is continuous allowing for excitations at all energies, and one can easily reach system sizes of several hundred spins, which mitigates the constraint on the reliable upper bound for the driving frequency by a few orders of magnitude. Nevertheless, studying classical systems comes at a notable price: one cannot access the infinite-time behavior, and is thus limited to finite times.

Often, experiments in Floquet engineering are designed to study the GS of the infinite-frequency Hamiltonian (3). Therefore, $H_{\rm ave}$ constitutes a natural observable to measure the excess energy pumped into the system from the drive. Let us define the dimensionless expected energy and energy variance [55, 75], over the initial ensemble of noisy AFM states:

$$\langle Q(\ell T)\rangle \equiv \langle Q^{(0)}(\ell T)\rangle = \frac{\langle H_{\rm ave}[\{\vec{S}_j(\ell T)\}]\rangle - E_{\rm GS}}{\langle H_{\rm ave}\rangle_{\beta=0} - E_{\rm GS}} \in [0,1],$$
$$\langle \delta Q(\ell T)\rangle = \sqrt{\frac{\langle H^2_{\rm ave}[\{\vec{S}_j(\ell T)\}]\rangle - \langle H_{\rm ave}[\{\vec{S}_j(\ell T)\}]\rangle^2}{\langle H^2_{\rm ave}\rangle_{\beta=0} - \langle H_{\rm ave}\rangle^2_{\beta=0}}}. \quad (4)$$

The normalization is chosen w.r.t. an infinite-temperature ensemble, where each spin points at a random direction, and hence $\langle H_{\rm ave}\rangle_{\beta=0} = 0$ and $\langle H^2_{\rm ave}\rangle_{\beta=0} = N/3(J^2/3+h^2+g^2)$. Initializing the system in the ensemble of noisy AFM states, we have $\langle Q(\ell T)\rangle \approx 0$ if the system does not absorb energy, and $\langle Q(\ell T)\rangle = 1$ whenever the ensemble is heated to infinite temperature.

There are four stages in the evolution of the system, see Fig. 1. Notice that the time [cf. stage (*iii*)] between the pre-thermal plateau and the infinite-temperature state, corresponding to maximum energy variance: $\ell_{\max}(\Omega) = \operatorname{argmax}_\ell \langle \delta Q(\ell T)\rangle$, scales *exponentially* [78] with the driving frequency Ω, c.f. Fig. 1b (inset). Our numerical study indicates that for $h/J = 0.0$ the heating time can be parametrized as $\ell_{\max}(\Omega) = r(g/J)\exp[-\gamma(g/J) \times \Omega/J]$, with $\gamma(g/J)$ a slowly-varying function of g/J, and $r(g/J) \propto (g/J)^\alpha$, $\alpha \approx -2.12$, within the entire range of existence of the prethermal stage [62]. Thus, we can rescale the $\langle Q(\ell T)\rangle$ curves with respect to the energy in the beginning of the prethermal plateau (*ii*):

$$\langle Q_{\rm scaled}(\ell_{\rm scaled}T)\rangle = \frac{\langle Q(\ell_{\rm scaled}T)\rangle - \langle Q\rangle_{\rm prethermal}}{\langle Q\rangle_{\beta=0} - \langle Q\rangle_{\rm prethermal}}, \quad (5)$$

where $\ell_{\rm scaled} = \ell/\ell_{\max}(\Omega)$. Figure 1a (inset) shows the collapse in the energy absorption curves with increasing frequency, whereas the width of the peak in $\delta Q(\ell_{\rm scaled})$ stays constant [62]. This behavior has remarkable similarities with the continuous phase transition observed in the 1d Ising model at zero temperature, where the squared magnetization plays the role of energy absorption, while inverse temperature and system size are analogous to drive frequency and time, respectively [62]; the correlation length corresponds to the heating time. This analogy suggests that heating may happen through a continuous phase transition in the limit $\Omega, t \to \infty$. At moderate frequencies and times, relevant for experiments, we find a sharp crossover instead.

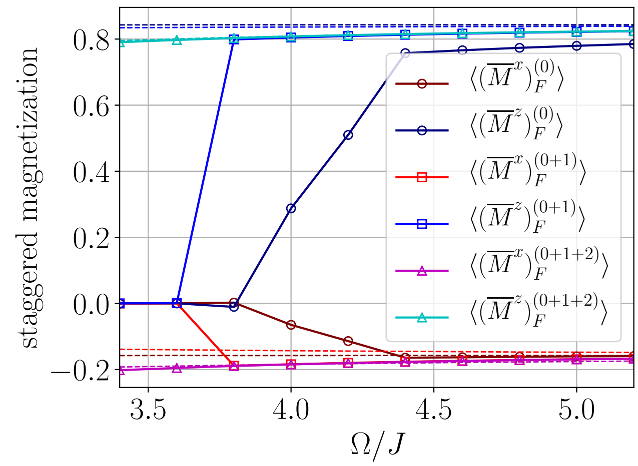

FIG. 2: Staggered magnetization $\langle(\overline{M}^\alpha)_F^{(0+1+..+m)}\rangle$ in the prethermal plateau to different order m in the ME (solid lines), compared to its value in the initial GS of the corresponding Floquet Hamiltonian (dashed). The parameters are $g/J = 0.9045$, $h/J = 0.809$, $N_T = 10^6$, and $N = 100$. Every point is averaged over an ensemble of 100 noise realisations.

Prethermal Regime.—The prethermal plateau plays a crucial role in strongly-interacting Floquet systems because it offers a stable window to experimentally realize novel many-body states of matter. We demonstrate that the inverse-frequency expansion can be used to gain a better understanding of the prethermal plateau, and present compelling evidence that this stage of evolution is captured by a *local* effective Hamilton function, amenable to Floquet engineering [79], even in chaotic classical many-body systems.

Even though Floquet theory does not apply to nonlinear EOM, a Magnus expansion (ME) can be formally defined for classical systems by replacing commutators with Poisson brackets [3, 55]. The ME approximates the exact Floquet Hamiltonian $H_F \approx H_F^{(0+\cdots+m)} \sim \mathcal{O}(\Omega^{-m})$, to a given order m in the inverse frequency [62]. However, it is an open question if and why the ME should work for classical systems. On one hand stands the notable application of the ME to the Kapitza pendulum [3, 49, 55], on the other – the recent finding that the ME does not capture resonances, which renders its convergence at most asymptotic [33, 75, 77].

While energy is the most natural observable to study heating, it typically cannot be measured directly in experiments. We now show that the prethermal plateau affects generic local observables. To test Floquet theory, we initialize the system in the GS of $H_F^{(0+\cdots+m)}$ with different values of m. Consider the staggered magnetization and its long-time average

$$M^\alpha(\ell T) = \frac{1}{N}\sum_{j=1}^{N}(-1)^j S_j^\alpha(\ell T); \overline{M^\alpha} = \frac{1}{10^3}\sum_{\ell=N_T}^{N_T+10^3} M^\alpha(\ell T),$$

where N_T denotes a large number of driving cycles. This observable is an order parameter for AFM correlations, and its time dependence measures how well the system retains the initial AFM structure. Figure 2 (circles) shows the components of the time-averaged magnetisation $\overline{M^\alpha}$ as a function of frequency to order $m = 0$. For $\Omega \lesssim \Omega_*$, where Ω_* is the crossover frequency, the system enters quickly the infinite-temperature stage, and all information about the initial state is lost: $\overline{M^\alpha} = 0$. However, in the long-lived prethermal plateau $\Omega \gtrsim \Omega_*$, much of the AFM correlations are preserved. The corresponding dashed lines show the staggered magnetization of the initial state, which is approached in the limit $\Omega/J \to \infty$. We verified that a similar behaviour is displayed by the spin-spin correlation function.

This raises the question whether one can Floquet-engineer expectation values of observables in the prethermal plateau. Upon increasing the order m of the ME, we find a significant improvement between the staggered magnetization of the initial ensemble and its time-averaged value in the pre-thermal plateau (squares, triangles), cf. Fig. 2. We also checked that starting from the GS of $H_F^{(0)}$, and evolving the system with the time-independent $H_F^{(0+\cdots+m)}$, results in better agreement of the magnetization dynamics with the exact stroboscopic evolution, upon increasing m [62]. Since the ME is the main theoretical tool used in Floquet engineering [1–3], this result implies that one should be able to successfully Floquet-engineer the behavior of observables in the prethermal plateau in classical systems [79]. Such a behavior likely originates from the emergent quasiconserved local integrals of motion for $\Omega \gtrsim \Omega_*$.

Floquet prethermal plateaus are defined with respect to the local approximate extensive Floquet Hamiltonian $H_F^{(0+\cdots+m)}$. They are often assumed to be featureless states which are stroboscopically equivalent to thermal equilibrium with respect to $H_F^{(0+\cdots+m)}$ for some optimal m. We now show that this assumption is incomplete: the prethermal plateau is sustained by drive-induced synchronization which is responsible for their exceptionally long stability. To demonstrate this, we compare the exact Floquet evolution, with an evolution where we repeatedly restart the dynamics from the thermal Gaussian ensemble of $H_F^{(0)}$, with mean energy and width chosen to match those of the time-evolved initial ensemble into the prethermal plateau. We call this procedure "restarting". Figure 3 shows a comparison of the expected values of $H_F^{(0)}$, following the uninterrupted (blue) and "restarting" (green) evolution. The "restarting" procedure was applied every 10^3 cycles to prevent the system from resynchronizing. As a result, the "restarted" dynamics enters the infinite-temperature stage exponentially earlier. The increase of energy caused by "restarting" suggests that the prethermal plateau contains additional very slow synchronized dynamics, probably related to Arnold diffusion or subdiffusion [70–72], which serves as a glue for the prethermal state. To argue that the prethermal plateau is a property of the time-evolved ensemble and not of the Gaussian energy ensemble used for "restarting", we fix the initial ensemble of noisy AFM states based on the GS of $H_{\text{ave}} = H_F^{(0)}$, and repeat the procedure measuring $H_F^{(0+1)}$, c.f. Fig 3. As expected, this affects the energy density of the prethermal plateau but not the duration of the stage. The restarting procedure is carried out w.r.t. a microcanonical distribution; we checked both a microcanonical and canonical distribution and found the same results. This restarting procedure shows that the pre thermal distribution cannot be characterized by energy and energy variance alone. Specifically, if this were the case then the thermilzation times for the restarting and true evolutions would be the same. This restarting procedure is analogous to dephasing of density matrix in quantum mechanics.

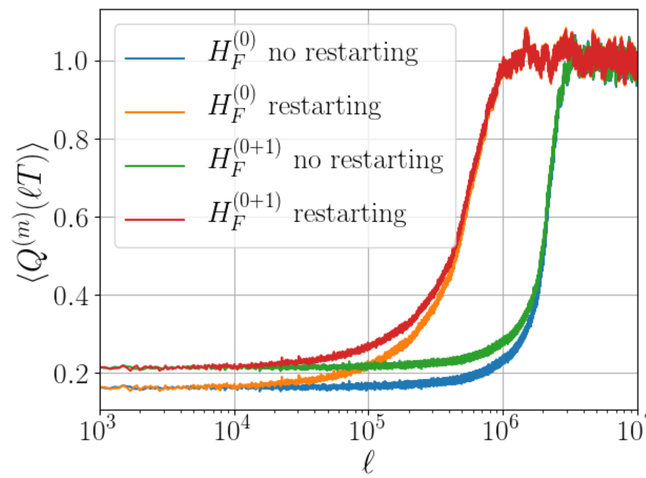

FIG. 3: Heating behavior measuring $H_F^{(0)}$ and $H_F^{(0+1)}$ with and without "restarting", for $\Omega/J = 3.8$. The restarting procedure is repeated every 10^3 cycles [see text]. We define $\langle Q^{(m)}(\ell T)\rangle$ to measure the normalized excess energy w.r.t. $H_F^{(0+\cdots+m)}$, cf. Eq. (4). The difference in energy and its variance between the new and old ensembles was chosen to be less than $0.005J$. See Fig. 2 for parameters.

Outlook.—We revealed a dynamical heating transition between a prethermal and an infinite-temperature stage in the limit of infinite times and drive frequencies. Its existence influences strongly the evolution of periodically-driven many-body spin chains even at the experimentally-relevant moderate frequencies and times. This becomes manifest in a long-lived prethermal plateau, which can be modeled in a controllable fashion by an approximate effective nonlinear Floquet-Hamilton function derived within the limitations of the inverse-frequency expansion. Contrary to naïve expectations, the prethermal plateau is fused by drive-induced synchronization, and is not a featureless thermal state.

Even though a detailed comparison of thermalization in classical and quantum Floquet systems would be desirable, our analysis already presents compelling evidence that the prethermal plateau, observed in a variety of quantum models, survives in the classical limit [55, 72]. This suggests that studies in cold atomic Floquet systems aiming to explain the contribution to heating due to higher bands or preparation of states under periodic driving, can be done (semi-)classically to reduce finite-size effects. In fact, our study directly relates to experimental platforms, such as shaken superfluid ultracold gases, where the physics is governed by a classical spin model [17], or photonic topological insulators [29–31], described by the nonlinear wave equation.

Note added: While this manuscript was under peer review, a related work proving prethermalization in classical periodically-driven spin systems appeared [80].

Acknowledgements.— We thank Pankaj Mehta for interesting and insightful discussions. OH was supported by BU UROP student funding and the Simmons Foundation Investigator grant MMLS to Pankaj Mehta. DS acknowledges support from the FWO as post-doctoral fellow of the Research Foundation – Flanders and CMTV. AP and PW were supported by NSF DMR-1813499, AFOSR FA9550-16-1-0334 and ARO W911NF1410540. MB acknowledges support from the Emergent Phenomena in Quantum Systems initiative (EPiQS) of the Gordon and Betty Moore Foundation and ERC synergy grant UQUAM. We used QuSpin for simulating the nonlinear EOM in the case of monochromatic drive [81, 82]. The authors are pleased to acknowledge that the computational work reported on in this paper was performed on the Shared Computing Cluster which is administered by Boston University's Research Computing Services. The authors also acknowledge the Research Computing Services group for providing consulting support which has contributed to the results reported within this paper.

[1] N. Goldman and J. Dalibard, Phys. Rev. X **4**, 031027 (2014).

[2] A. Eckardt, Rev. Mod. Phys. **89**, 011004 (2017).

[3] M. Bukov, L. D'Alessio, and A. Polkovnikov, Advances in Physics **64**, 139 (2015).

[4] P. Kapitza, Soviet Phys. JETP **21**, 588 ((1951)).

[5] T. Oka and H. Aoki, Phys. Rev. B **79**, 081406 (2009).

[6] T. Kitagawa, T. Oka, A. Brataas, L. Fu, and E. Demler, Phys. Rev. B **84**, 235108 (2011).

[7] G. Jotzu, M. Messer, R. Desbuquois, M. Lebrat, T. Uehlinger, D. Greif, and T. Esslinger, Nature **515**, 237 (2014).

[8] M. Aidelsburger, M. Lohse, C. Schweizer, M. Atala, J. T. Barreiro, S. Nascimbène, N. R. Cooper, I. Bloch, and N. Goldman, Nature Physics **11**, 162 (2015).

[9] N. Fläschner, B. S. Rem, M. Tarnowski, D. Vogel, D.-S. Lühmann, K. Sengstock, and C. Weitenberg, Science **352**, 1091 (2016).

[10] M. Tarnowski, F. N. Ünal, N. Fläschner, B. S. Rem, A. Eckardt, K. Sengstock, and C. Weitenberg, arXiv preprint arXiv:1709.01046 (2017).

[11] M. Tarnowski, M. Nuske, N. Fläschner, B. Rem, D. Vogel, L. Freystatzky, K. Sengstock, L. Mathey, and C. Weitenberg, Phys. Rev. Lett. **118**, 240403 (2017).

[12] M. Aidelsburger, S. Nascimbene, and N. Goldman, arXiv preprint arXiv:1710.00851 (2017).

[13] G. Jotzu, M. Messer, F. Görg, D. Greif, R. Desbuquois, and T. Esslinger, Phys. Rev. Lett. **115**, 073002 (2015).

[14] J. Struck, C. Ölschläger, R. Le Targatn, P. Soltan-Panahi, A. Eckardt, M. Lewenstein, P. Windpassinger, and K. Sengstock, Science **333 (6045)**, 996 (2011).

[15] J. Struck, C. Ölschläger, M. Weinberg, P. Hauke, J. Simonet, A. Eckardt, M. Lewenstein, K. Sengstock, and P. Windpassinger, Phys. Rev. Lett. **108**, 225304 (2012).

[16] P. Hauke, O. Tieleman, A. Celi, C. Ölschläger, J. Simonet, J. Struck, M. Weinberg, P. Windpassinger, K. Sengstock, M. Lewenstein, and A. Eckardt, Phys. Rev. Lett. **109**, 145301 (2012).

[17] J. Struck, M. Weinberg, C. Ölschläger, P. Windpassinger, J. Simonet, K. Sengstock, R. Höppner, P. Hauke, A. Eckardt, M. Lewenstein, and L. Mathey, Nature Physics **9**, 738 (2013).

[18] M. Aidelsburger, M. Atala, M. Lohse, J. T. Barreiro, B. Paredes, and I. Bloch, Phys. Rev. Lett. **111**, 185301 (2013).

[19] H. Miyake, G. A. Siviloglou, C. J. Kennedy, W. C. Burton, and W. Ketterle, Phys. Rev. Lett. **111**, 185302 (2013).

[20] M. Atala, M. Aidelsburger, M. Lohse, J. T. Barreiro, B. Paredes, and I. Bloch, Nature Physics **10**, 588 (2014).

[21] C. J. Kennedy, W. C. Burton, W. C. Chung, and W. Ketterle, Nature Physics **11**, 859 (2015).

[22] M. Bukov, M. Kolodrubetz, and A. Polkovnikov, Phys. Rev. Lett. **116**, 125301 (2016).

[23] V. Galitski and I. B. Spielman, Nature **494**, 49 (2013).

[24] K. Jiménez-García, L. J. LeBlanc, R. A. Williams, M. C. Beeler, C. Qu, M. Gong, C. Zhang, and I. B. Spielman, Phys. Rev. Lett. **114**, 125301 (2015).

[25] F. Görg, M. Messer, K. Sandholzer, G. Jotzu, R. Desbuquois, and T. Esslinger, arXiv preprint arXiv:1708.06751 (2017).

[26] A. Celi, P. Massignan, J. Ruseckas, N. Goldman, I. B. Spielman, G. Juzeliūnas, and M. Lewenstein, Phys. Rev. Lett. **112**, 043001 (2014).

[27] B. Stuhl, H.-I. Lu, L. Aycock, D. Genkina, and I. Spielman, Science **349**, 1514 (2015).

[28] M. Mancini, G. Pagano, G. Cappellini, L. Livi, M. Rider, J. Catani, C. Sias, P. Zoller, M. Inguscio, M. Dalmonte, *et al.*, Science **349**, 1510 (2015).

[29] M. C. Rechtsman, J. M. Zeuner, Y. Plotnik, Y. Lumer, D. Podolsky, F. Dreisow, S. Nolte, M. Segev, and A. Szameit, Nature **496**, 196 (2013).

[30] M. Hafezi, Phys. Rev. Lett. **112**, 210405 (2014).

[31] S. Mittal, J. Fan, S. Faez, A. Migdall, J. M. Taylor, and M. Hafezi, Phys. Rev. Lett. **113**, 087403 (2014).

[32] R. Desbuquois, M. Messer, F. Görg, K. Sandholzer, G. Jotzu, and T. Esslinger, Phys. Rev. A **96**, 053602 (2017).

[33] P. Weinberg, M. Bukov, L. DAlessio, A. Polkovnikov, S. Vajna, and M. Kolodrubetz, Physics Reports (2017).

[34] V. Novičenko, E. Anisimovas, and G. Juzeliūnas, Phys. Rev. A **95**, 023615 (2017).

[35] M. Bukov, arXiv preprint arXiv:1808.08910 (2018).

[36] T. Bilitewski and N. R. Cooper, Phys. Rev. A **91**, 033601 (2015).

[37] T. Bilitewski and N. R. Cooper, Phys. Rev. A **91**, 063611 (2015).

[38] M. Weinberg, C. Ölschläger, C. Sträter, S. Prelle, A. Eckardt, K. Sengstock, and J. Simonet, Phys. Rev. A **92**, 043621 (2015).

[39] M. Reitter, J. Näger, K. Wintersperger, C. Sträter, I. Bloch, A. Eckardt, and U. Schneider, Phys. Rev. Lett. **119**, 200402 (2017).

[40] J. Näger, K. Wintersperger, M. Bukov, S. Lellouch, E. Demler, U. Schneider, I. Bloch, N. Goldman, and M. Aidelsburger, arXiv preprint arXiv:1808.07462 (2018).

[41] T. Boulier, J. Maslek, M. Bukov, C. Bracamontes, E. Magnan, S. Lellouch, E. Demler, N. Goldman, and J. Porto, arXiv preprint arXiv:1808.07637 (2018).

[42] S. Fishman, D. R. Grempel, and R. E. Prange, Phys. Rev. Lett. **49**, 509 (1982).

[43] S. Lellouch, M. Bukov, E. Demler, and N. Goldman, Phys. Rev. X **7**, 021015 (2017).

[44] S. Lellouch and N. Goldman, "Parametric instabilities in resonantly-driven bose-einstein condensates," (2017), arXiv:1711.08832 .

[45] L. D'Alessio and M. Rigol, Phys. Rev. X **4**, 041048 (2014).

[46] A. Lazarides, A. Das, and R. Moessner, Phys. Rev. E **90**, 012110 (2014).

[47] Y. Bar Lev, D. J. Luitz, and A. Lazarides, SciPost Physics **3**, 029 (2017).

[48] R. Moessner and S. Sondhi, Nature Physics **13**, 424 (2017).

[49] R. Citro, E. G. Dalla Torre, L. DAlessio, A. Polkovnikov, M. Babadi, T. Oka, and E. Demler, Annals of Physics **360**, 694 (2015).

[50] K. Seetharam, P. Titum, M. Kolodrubetz, and G. Refael, Phys. Rev. B **97**, 014311 (2018).

[51] T. Prosen, Phys. Rev. Lett. **80**, 1808 (1998).

[52] T. Prosen, J. Phys. A: Math. Gen. **31**, L645 (1998).

[53] T. Prosen, Phys. Rev. E **60**, 3949 (1999).

[54] T. Prosen, Phys. Rev. E **65**, 036208 (2002).

[55] L. DAlessio and A. Polkovnikov, Annals of Physics **333**, 19 (2013).

[56] A. Haldar, R. Moessner, and A. Das, Phys. Rev. B **97**, 245122 (2018).

[57] D. A. Abanin, W. De Roeck, and F. Huveneers, Phys. Rev. Lett. **115**, 256803 (2015).

[58] T. Mori, T. Kuwahara, and K. Saito, Phys. Rev. Lett. **116**, 120401 (2016).

[59] M. Bukov, S. Gopalakrishnan, M. Knap, and E. Demler, Phys. Rev. Lett. **115**, 205301 (2015).

[60] S. A. Weidinger and M. Knap, Scientific reports **7**, 45382 (2017).

[61] F. Peronaci, M. Schiró, and O. Parcollet, arXiv preprint arXiv:1711.07889 (2017).

[62] See Supplemental Material.

[63] E. Fermi, J. Pasta, S. Ulam, and M. Tsingou, Los Alamos National Laboratory **Document LA-1940** (1955).

[64] S. F. C. Danieli, DK Campbell, "Intermittent fpu dynamics at equilibrium," (2016), arXiv:1611.00434 .

[65] N. N. Nekhoroshev, Functional Analysis and Its Applications **5**, 338 (1971).

[66] J. Moser, Math. Phys. K1 IIa **nr.6 (1955)**, 87120 (1955).

[67] J. Littlewood, Proceedings of the London Mathematical Society **3**, 343 (1959).

[68] J. Pöschel, Mathematische Zeitschrift **213**, 187 (1993).

[69] M. Pettini and M. Landolfi, Phys. Rev. A **41**, 768 (1990).

[70] T. Konishi and K. Kaneko, Journal of Physics A: Mathematical and General **23**, L715 (1990).

[71] S. Notarnicola, F. Iemini, D. Rossini, R. Fazio, A. Silva, and A. Russomanno, arXiv preprint arXiv:1709.05657 (2017).

[72] A. Rajak, R. Citro, and E. G. D. Torre, arXiv preprint arXiv:1801.01142 (2018).

[73] G. Veble and T. Prosen, Phys. Rev. Lett. **92**, 034101 (2004).

[74] G. Veble and T. Prosen, Phys. Rev. E **72**, 025202 (2005).

[75] M. Bukov, M. Heyl, D. A. Huse, and A. Polkovnikov, Phys. Rev. B **93**, 155132 (2016).

[76] P. W. Claeys and J.-S. Caux, arXiv preprint arXiv:1708.07324 (2017).

[77] P. W. Claeys, S. De Baerdemacker, O. E. Araby, and J.-S. Caux, arXiv preprint arXiv:1712.03117 (2017).

[78] F. Machado, G. D. Meyer, D. V. Else, C. Nayak, and N. Y. Yao, arXiv preprint arXiv:1708.01620 (2017).

[79] F. H. Higashikawa, Sho and M. Sato, Axriv preprint arXiv:1810.01103 (2018).

[80] T. Mori, Phys. Rev. B **98**, 104303 (2018).

[81] P. Weinberg and M. Bukov, SciPost Phys. **2**, 003 (2017).

[82] P. Weinberg and M. Bukov, arXiv preprint arXiv:1804.06782 (2018).