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We consider a quantum communication task between two users Alice and Bob, in which Alice and Bob exchange their respective quantum information by means of local operations and classical communication assisted by shared entanglement. Here, we assume that Alice and Bob may have quantum side information, not transferred, and classical communication is free. In this work, we derive general upper and lower bounds for the least amount of entanglement which is necessary to perfectly perform this task, called the state exchange with quantum side information. Moreover, we show that the optimal entanglement cost can be negative when Alice and Bob make use of their quantum side information. We finally provide conditions on the initial state for the state exchange with quantum side information which give the exact optimal entanglement cost.

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Introduction.— In quantum information theory, one of the most traditional research topics has been source coding problems of transmitting Alice's quantum information to Bob under various situations, with paradigmatic examples including Schumacher compression [1] and quantum teleportation [2]. A decade ago, Oppenheim and Winter devised a new type of a quantum communication task named *state exchange* [3] — in which Alice and Bob exchange their quantum information with each other by means of local operations and classical communication (LOCC) and shared entanglement — and they studied the least amount of entanglement consumed in the task when free classical communication is allowed.

In the original state exchange task, it is assumed that both Alice and Bob do not have any quantum side information (QSI) transferrable during the protocol. On the other hand, most quantum communication tasks, including state merging [4, 5] and state redistribution [6, 7], begin with the assumption that either Alice or Bob has QSI. For example, in the state merging task, Bob can make use of his QSI for merging Alice's information to himself, and the minimum amount of entanglement needed for merging turns out to be exactly given by the quantum conditional entropy [8] conditioned on Bob's QSI.

In this work we generalize in the state exchange to an exchanging task allowing Alice's and Bob's QSI, which is called the *state exchange with quantum side information*. We consider three parties, Alice, Bob, and a referee (R), sharing a pure initial state $|\psi\rangle \equiv |\psi\rangle_{AC_\mathrm{A}BC_\mathrm{B}R}$ as depicted in Fig. 1. The aim of Alice and Bob is to exchange their quantum information C_A and C_B, while the referee does nothing. To achieve their aim, Alice and Bob make use of their QSI A and B, and they have additional systems E_A^in, $E_\mathrm{A}^\mathrm{out}$ and E_B^in, $E_\mathrm{B}^\mathrm{out}$ for the use of entanglement resources.

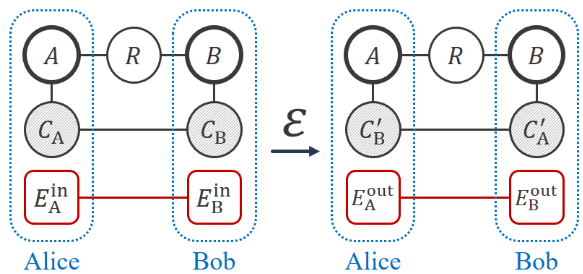

FIG. 1: Illustration of state exchange protocol $\mathcal{E}$ with QSI. Starting from an initial state $|\psi\rangle_{AC_\mathrm{A}BC_\mathrm{B}R}$ of Alice, Bob, and a referee (R), Alice and Bob exchange their parts C_A and C_B, exploiting their respective QSI A and B. The ancillary systems E_A^in and E_B^in represent an initial entanglement consumed for the exchanging task, while $E_\mathrm{A}^\mathrm{out}$ and $E_\mathrm{B}^\mathrm{out}$ indicate entanglement generated from the task.

Our main question can be formulated as follows: *"Does there exist a crucial difference in optimal strategies between the tasks of state exchange with and without QSI?"*

To answer this question we formally define the state exchange with QSI and its optimal entanglement cost in the asymptotic scenario, and then derive an upper bound for the optimal entanglement cost by conceiving a two-step strategy based on the idea mentioned in Ref. [3]. We show that in general this strategy does not provide the optimal entanglement cost of the state exchange with QSI. However for a specific initial state of the state exchange with QSI, the upper bound shows that the optimal entanglement cost for the state exchange with QSI can be *negative*, meaning that entanglement is in fact gained rather than consumed in the protocol. This result is quite remarkable since the optimal entanglement cost for the state exchange without QSI cannot be negative [3]. More importantly, this implies that the use of Alice's and Bob's QSI can significantly reduce the optimal entanglement cost of the exchanging task.

We furthermore consider an idealized situation in which the referee plays a more active role and can help Alice and Bob to exchange their information [3]. By virtue of the referee's assistance, it is possible for Alice and Bob to more efficiently perform the state exchange with QSI, and this provides us with converse bounds on the optimal entanglement cost, which are lower bounds for any achievable entanglement rate. As an application of our bounds, we present conditions on the initial state for the state exchange with QSI such that the exact optimal entanglement cost can be obtained.

State exchange with quantum side information.— In the task of state exchange $\mathcal{E}$ with QSI as described in Fig. 1, the global initial state ψ_i and the global final state ψ_f are given by

$$\psi_i = \psi \otimes \Phi_{E_{\rm A}^{\rm in} E_{\rm B}^{\rm in}} \quad \text{and} \quad \psi_f = \psi' \otimes \Phi_{E_{\rm A}^{\rm out} E_{\rm B}^{\rm out}},$$

where $\psi = |\psi\rangle\langle\psi|$, $\Phi_{E_{\rm A}^{\rm in} E_{\rm B}^{\rm in}}$ and $\Phi_{E_{\rm A}^{\rm out} E_{\rm B}^{\rm out}}$ are pure maximally entangled states with Schmidt rank $e^{\rm in}(\mathcal{E})$ and $e^{\rm out}(\mathcal{E})$, respectively, $\psi' = \left(\mathbb{1}_{ABR} \otimes \mathbb{1}_{C_{\rm A}\to C'_{\rm A}} \otimes \mathbb{1}_{C_{\rm B}\to C'_{\rm B}}\right)(\psi)$, and $C'_{\rm B}$ ($C'_{\rm A}$) is Alice's system (Bob's system) with $\dim C'_{\rm B} = \dim C_{\rm B}$ ($\dim C'_{\rm A} = \dim C_{\rm A}$). Then a joint operation

$$\mathcal{E} : AC_{\rm A}E_{\rm A}^{\rm in} \otimes BC_{\rm B}E_{\rm B}^{\rm in} \longrightarrow AC'_{\rm B}E_{\rm A}^{\rm out} \otimes BC'_{\rm A}E_{\rm B}^{\rm out}$$

is called the *state exchange with quantum side information of* $|\psi\rangle$ *with error* ε, if it consists of LOCC, and satisfies

$$\left\| \left(\mathcal{E} \otimes \mathbb{1}_R\right)(\psi_i) - \psi_f \right\|_1 \le \varepsilon,$$

where $\|\cdot\|_1$ is the trace norm [8].

Let us now consider n independent and identically distributed copies of $|\psi\rangle$, say $|\psi\rangle^{\otimes n}$. If $\mathcal{E}_n$ indicates a state exchange with QSI of $|\psi\rangle^{\otimes n}$ with error ε_n, then the resource rate $\left(\log e^{\rm in}(\mathcal{E}_n) - \log e^{\rm out}(\mathcal{E}_n)\right)/n$ is called the *entanglement rate* of the protocol. If there is a sequence $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ of state exchanges $\mathcal{E}_n$ with QSI of $|\psi\rangle^{\otimes n}$ with error ε_n such that

$$\lim_{n\to\infty} \frac{\log e^{\rm in}(\mathcal{E}_n) - \log e^{\rm out}(\mathcal{E}_n)}{n} = e_{\rm r}, \qquad \lim_{n\to\infty} \varepsilon_n = 0,$$

then the real number $e_{\rm r}$ is called an *achievable* entanglement rate for the state exchange with QSI of $|\psi\rangle$. The smallest achievable entanglement rate defines the *optimal entanglement cost* $e_{\rm opt}$ for the considered task.

Note that the optimal entanglement cost only depends on the reduced state of Alice and Bob, as the Referee does not play any active part in the protocol.

Merge-and-merge strategy.— We first present a *merge-and-merge* strategy which is motivated by the *merge-and-send* protocol introduced in Ref. [3]. The idea of this strategy is as follows. Firstly, Alice's part $C_{\rm A}$ is merged from Alice to Bob by using $BC_{\rm B}$ as QSI. After finishing merging $C_{\rm A}$, Bob's part $C_{\rm B}$ is merged from Bob to Alice by using Alice's QSI A so that Alice's $C_{\rm A}$ and Bob's $C_{\rm B}$ are exchanged. By using the exact formula of the entanglement cost for merging [6, 9, 10], we have that the optimal entanglement costs of merging $C_{\rm A}$ and merging $C_{\rm B}$ are the quantum conditional entropies $H(C_{\rm A}|BC_{\rm B})$ and $H(C_{\rm B}|A)$, respectively, so that the total entanglement cost is $H(C_{\rm B}|A)+H(C_{\rm A}|BC_{\rm B})$, where the quantum conditional entropy $H(X|Y)_\rho$ of a state ρ_{XY} is defined by $H(XY)_\rho - H(Y)_\rho$, with $H(X)$ the von Neumann entropy [8] of a state ρ_X.

From the merge-and-merge strategy, we obtain the following upper bound for the optimal entanglement cost of the state exchange with QSI.

Theorem 1. *The optimal entanglement cost* $e_{\rm opt}$ *for the state exchange with QSI of* $|\psi\rangle$ *is upper bounded by*

$$e_{\rm opt} \le u(\psi) = \min\{u_1(\psi), u_2(\psi)\},$$

where $u_1(\psi) = H(C_{\rm B}|A)_\psi + H(C_{\rm A}|BC_{\rm B})_\psi$ *and* $u_2(\psi) = H(C_{\rm A}|B)_\psi + H(C_{\rm B}|AC_{\rm A})_\psi$.

Note that $u_2(\psi)$ in Theorem 1 can be obtained by firstly merging Bob's part $C_{\rm B}$ to Alice. We further refer the reader to Supplemental Material [11] for the rigorous proof of Theorem 1 which fulfills the definition of achievability.

Optimal strategy?— Since the merge-and-merge strategy is simple and intuitive, one may guess that the strategy is optimal for any initial state of the exchanging task. However, the following example shows that there can be a more effective strategy than the merge-and-merge one. Let us consider a specific form of the initial state

$$|\tilde{\psi}\rangle_{AC_{\rm A}BC_{\rm B}R} = |\tilde{\phi}\rangle_{AC_{\rm A}^1BC_{\rm B}^1R_1} \otimes |{\rm GHZ}\rangle_{C_{\rm A}^2C_{\rm B}^2R_2}\,, \qquad (1)$$

where systems $C_{\rm A} = C_{\rm A}^1C_{\rm A}^2$, $C_{\rm B} = C_{\rm B}^1C_{\rm B}^2$, $R = R_1R_2$, $|\tilde{\phi}\rangle$ is an arbitrary state on the system $AC_{\rm A}^1BC_{\rm B}^1R_1$, and

$$|{\rm GHZ}\rangle_{C_{\rm A}^2C_{\rm B}^2R_2} = \frac{1}{\sqrt{d}}\sum_{k=0}^{d-1}|kkk\rangle$$

is the Greenberger-Horne-Zeilinger state [12] with $d \ge 2$.

In order to exchange $C_{\rm A}$ and $C_{\rm B}$ in Eq. (1), it suffices for Alice and Bob to only consider the state exchange with QSI of $|\tilde{\phi}\rangle$, since the parts $C_{\rm A}^2$ and $C_{\rm B}^2$ of the state $|{\rm GHZ}\rangle$ are symmetric. Then by applying the merge-and-merge strategy on $|\tilde{\phi}\rangle$, we obtain a tighter upper bound $\min\{u_1(\tilde{\phi}), u_2(\tilde{\phi})\}$ for the optimal entanglement cost for the state $|\tilde{\psi}\rangle$ in Eq. (1) as follows:

$$\min\{u_1(\tilde{\phi}), u_2(\tilde{\phi})\} = \min\{u_1(\tilde{\psi}), u_2(\tilde{\psi})\} - \log d. \qquad (2)$$

From the relation between upper bounds in Eq. (2), we remark that there can be an arbitrarily large gap between the optimal entanglement cost and the upper bound in Theorem 1, implying that the upper bound is not optimal in the general case. This example also shows that there exist tighter upper bounds for the optimal entanglement cost. On this account, we argue that the optimal strategy for state exchange with QSI is generally nontrivial.

Converse bounds.— As in the state exchange *without* QSI [3], we can imagine that the referee holds the reference R, and is ideally allowed to assist Alice and Bob in the following way, which is here called the *R-assisted state exchange with QSI*. The referee first divides their part R into two parts E and V by using a quantum channel $\mathcal{N}$ from R to V whose complementary channel $\mathcal{N}^c$ is from R to E [8]. Next, the referee sends the states $\rho_V = \mathcal{N}(\rho_R)$ and $\rho_E = \mathcal{N}^c(\rho_R)$ to Alice and Bob, respectively. Then the initial state $|\psi\rangle$ becomes $|\psi\rangle_{AC_\mathrm{A}VBC_\mathrm{B}E}$, where Alice and Bob hold $AC_\mathrm{A}V$ and $BC_\mathrm{B}E$, respectively. Alice and Bob now perform the state exchange with QSI of the state $|\psi\rangle_{AC_\mathrm{A}VBC_\mathrm{B}E}$.

For each n, let $\mathcal{E}_n^R$ be a state exchange with QSI of $|\psi\rangle^{\otimes n}$ with error ε_n, and E_n^bef and E_n^aft be total amounts of entanglement between Alice and Bob before and after the state exchange with QSI, respectively. Then they can be expressed as $E_n^\mathrm{bef} = nH(AC_\mathrm{A}V) + \log e^\mathrm{in}(\mathcal{E}_n^R)$ and $E_n^\mathrm{aft} = nH(AC_\mathrm{B}V) + \log e^\mathrm{out}(\mathcal{E}_n^R)$. Since the total entanglement between Alice and Bob cannot increase under LOCC [13], we have $E_n^\mathrm{bef} \geq E_n^\mathrm{aft}$, that is,

$$\log e^\mathrm{in}(\mathcal{E}_n^R) - \log e^\mathrm{out}(\mathcal{E}_n^R) \geq nH(AC_\mathrm{B}V) - nH(AC_\mathrm{A}V).$$

Let e_opt^R be the optimal entanglement cost for the R-assisted state exchange with QSI, then

$$\max_{\mathcal{N}}[H(AC_\mathrm{B}V) - H(AC_\mathrm{A}V)] \leq e_\mathrm{opt}^R.$$

Since any state exchange with QSI can be considered as an R-assisted state exchange with QSI (in which the referee trivially does nothing), it holds that $e_\mathrm{opt}^R \leq e_\mathrm{opt}$. This leads us to the following theorem.

Theorem 2. *The optimal entanglement cost e_opt for the state exchange with QSI of $|\psi\rangle$ is lower bounded by*

$$l(\psi) = \max_{\mathcal{N}}[H(AC_\mathrm{B}V)_{\mathcal{N}(\psi)} - H(AC_\mathrm{A}V)_{\mathcal{N}(\psi)}] \leq e_\mathrm{opt},$$

where the maximum is taken over all quantum channels $\mathcal{N}: R \longrightarrow V$.

In general, it is not easy to calculate the converse bound in Theorem 2, since it involves an optimization over all quantum channels. However, if the referee sends the whole part R to either Alice or Bob without dividing R in Theorem 2, then we obtain the following *computable* converse bound:

Corollary 3. *For the state exchange with QSI of $|\psi\rangle$, the optimal entanglement cost e_opt satisfies*

$$\max\{l_1(\psi), l_2(\psi)\} \leq e_\mathrm{opt},$$

where $l_1(\psi) = H(AC_\mathrm{B})_\psi - H(AC_\mathrm{A})_\psi$ and $l_2(\psi) = H(BC_\mathrm{A})_\psi - H(BC_\mathrm{B})_\psi$.

By using the continuity of the von Neumann entropy [14, 15], we can directly show that $l_1(\psi)$ and $l_2(\psi)$ in Corollary 3 are lower bounds to the optimal entanglement cost for the state exchange with QSI of $|\psi\rangle$. The proof of Corollary 3 can be found in Supplemental Material [11].

Large gap between converse bounds.— It is obvious that the lower bound presented in Corollary 3 is less tight than the one in Theorem 2. Interestingly, the gap between these two converse bounds can be arbitrarily large. To this end, let us consider the initial state

$$|\bar{\psi}\rangle_{AC_\mathrm{A}BC_\mathrm{B}R} = |\Phi\rangle_{AR_A} \otimes |\Phi\rangle_{C_\mathrm{A}R_{C_\mathrm{A}}} \otimes |\Phi\rangle_{BR_B} \otimes |\Phi\rangle_{C_\mathrm{B}R_{C_\mathrm{B}}}, \tag{3}$$

where the reference system R consists of the four subsystems R_A, R_{C_A}, R_B and R_{C_B}, and $|\Phi\rangle$ is a maximally entangled state on the corresponding bipartite system SR_S with $\dim S = \dim R_S$ for $S = A$, B, C_A and C_B. Then we can readily see that

$$l_1(\bar{\psi}) = H(C_\mathrm{B})_{\bar{\psi}} - H(C_\mathrm{A})_{\bar{\psi}} = -l_2(\bar{\psi}).$$

On the other hand, if a channel $\bar{\mathcal{N}}$ is given by $\rho_R \mapsto \rho_{R_\mathrm{A}R_{C_\mathrm{A}}}$, that is, $V = R_\mathrm{A}R_{C_\mathrm{A}}$, then we obtain

$$\begin{aligned} l(\bar{\psi}) &\geq H(AC_\mathrm{B}V)_{\bar{\mathcal{N}}(\bar{\psi})} - H(AC_\mathrm{A}V)_{\bar{\mathcal{N}}(\bar{\psi})} \\ &= H(AC_\mathrm{B}R_AR_{C_\mathrm{A}})_{\bar{\psi}} - H(AC_\mathrm{A}R_AR_{C_\mathrm{A}})_{\bar{\psi}} \\ &= H(C_\mathrm{A})_{\bar{\psi}} + H(C_\mathrm{B})_{\bar{\psi}}, \end{aligned}$$

which means that the converse bound $l(\psi)$ in Theorem 2 can be arbitrarily larger than $\max\{l_1(\psi), l_2(\psi)\}$ in Corollary 3 for the class of initial states in Eq. (3).

Optimal entanglement cost can be negative.— We finally address the crucial question: Can the optimal entanglement cost for state exchange with QSI be *negative*? First of all, let us remark that the optimal entanglement cost for state exchange *without* QSI of $|\psi\rangle_{C_\mathrm{A}C_\mathrm{B}R}$ *cannot* be negative [3]. If the optimal cost was negative, then Alice and Bob could generate as much entanglement as they need by repeatedly exchanging their state. This contradicts the basic requirement that the amount of entanglement cannot increase by LOCC [16].

However, quite remarkably, the optimal entanglement cost e_opt for the state exchange *with* QSI of $|\psi\rangle$ *can* be negative. This is readily seen since the upper bounds u_1 or u_2 in Theorem 1 can be negative. For example, e_opt is negative for the initial state

$$\begin{aligned} |\psi_\lambda\rangle_{AC_\mathrm{A}BC_\mathrm{B}R} &= \sqrt{\frac{\lambda}{2}}\,|00000\rangle + \sqrt{\frac{1-\lambda}{2}}\,|10011\rangle \\ &+ \sqrt{\frac{\lambda}{2}}\,|01100\rangle + \sqrt{\frac{1-\lambda}{2}}\,|01010\rangle \end{aligned} \tag{4}$$

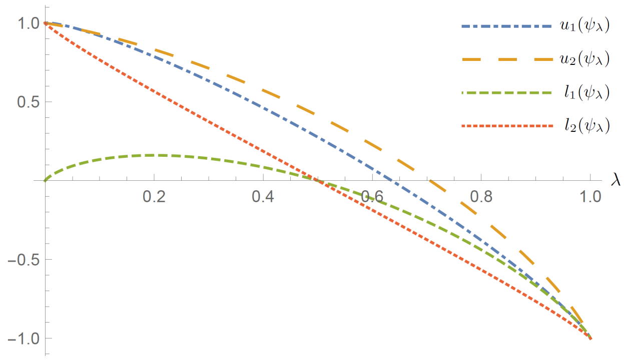

FIG. 2: Upper bounds $u_1(\psi_\lambda)$, $u_2(\psi_\lambda)$ and lower bounds $l_1(\psi_\lambda)$, $l_2(\psi_\lambda)$ to the optimal entanglement cost $e_{\rm opt}$ for the specific initial state $|\psi_\lambda\rangle$ of Eq. (4) with $0 \leq \lambda \leq 1$.

with $\lambda \geq 0.65$, as seen in Fig. 2. Furthermore, this example shows that, in the state exchange with QSI, the optimal entanglement cost can be generally reduced by exploiting the QSI AB for the exchanging task. This reveals the prominent role of the QSI for such a quantum communication primitive.

At this point we remark that the negativity of the optimal entanglement cost for the state exchange with QSI does not lead to a contradiction as follows. Let $e_{\rm opt}^{\rm 1st}$ be the optimal entanglement cost for a state exchange with QSI of the initial state $|\psi\rangle$, and let $e_{\rm opt}^{\rm 2nd}$ be the optimal entanglement cost for a state exchange with QSI of the exchanged state $|\psi'\rangle$. Then from Corollary 3,

$$e_{\rm opt}^{\rm 1st} \geq l_1(\psi) \quad \text{and} \quad e_{\rm opt}^{\rm 2nd} \geq l_1(\psi') = -l_1(\psi).$$

So in this case we have the inequality $e_{\rm opt}^{\rm 1st} + e_{\rm opt}^{\rm 2nd} \geq 0$. This shows that the total amount of entanglement generated from repeated state exchange protocols with QSI does not repeatedly increase although the entanglement cost can be negative in an individual instance of the protocol.

Optimal entanglement costs for some special cases.— We now provide several conditions which allow us to compute the exact optimal entanglement cost $e_{\rm opt}$ for the state exchange with QSI of $|\psi\rangle$. In fact, the merge-and-merge strategy is optimal under these conditions.

Corollary 4. *Let $e_{\rm opt}$ be the optimal entanglement cost of the state exchange with QSI of $|\psi\rangle \equiv |\psi\rangle_{AC_{\rm A}BC_{\rm B}R}$.*

(i) The following conditions on $|\psi\rangle$ give the exact optimal entanglement costs:

$$\begin{aligned} I(R;C_{\rm A}|A)_\psi = 0 &\iff e_{\rm opt} = u_1(\psi) = l_1(\psi), \\ I(R;C_{\rm A}|B)_\psi = 0 &\iff e_{\rm opt} = u_2(\psi) = l_1(\psi), \\ I(R;C_{\rm B}|A)_\psi = 0 &\iff e_{\rm opt} = u_1(\psi) = l_2(\psi), \\ I(R;C_{\rm B}|B)_\psi = 0 &\iff e_{\rm opt} = u_2(\psi) = l_2(\psi), \end{aligned}$$

where $I(X;Y|Z)_\rho$ indicates the quantum conditional mutual information (QCMI) of a quantum state ρ_{XYZ}, and $u_1(\psi)$, $u_2(\psi)$, $l_1(\psi)$, and $l_2(\psi)$ are given in Theorem 1 and Corollary 3.

(ii) There exists a quantum channel $\mathcal{N} : R \longrightarrow V$ such that $I(C_{\rm B} : V|A)_{\mathcal{N}(\psi)} = I(C_{\rm A} : E|AV)_{\mathcal{N}(\psi)} = 0$ if and only if $e_{\rm opt} = u_1(\psi) = l(\psi)$, where $l(\psi)$ is in Theorem 2. Similarly, there exists $\mathcal{N} : R \longrightarrow V$ such that $I(C_{\rm A} : E|B)_{\mathcal{N}(\psi)} = I(C_{\rm B} : V|BE)_{\mathcal{N}(\psi)} = 0$ if and only if $e_{\rm opt} = u_2(\psi) = l(\psi)$.

(iii) Let $|\hat{\psi}\rangle_{AC_{\rm A}BC_{\rm B}}$ be a pure initial state shared by Alice and Bob (with no referee), then for the state exchange with QSI of $|\hat{\psi}\rangle_{AC_{\rm A}BC_{\rm B}}$ one has $e_{\rm opt} = H(AC_{\rm B})_{\hat{\psi}} - H(AC_{\rm A})_{\hat{\psi}}$.

By combining the aforementioned upper and lower bounds, the conditions for the exact optimal cost in Corollary 4 are directly obtained. We remark that there are no general implications among the four QCMI conditions in Corollary 4 (i), that is, there exists an initial state which only satisfies one of these QCMI conditions. We presents related examples in Supplemental Material [11].

Conclusion.— In this work, we have considered the state exchange with QSI as a fundamental quantum communication task, and have provided the formal descriptions for the protocol and its optimal entanglement cost. We have derived upper and lower bounds to the optimal entanglement cost. From these bounds, we have exactly evaluated the optimal entanglement cost for several special classes of states, including all pure bipartite states. Furthermore, we have shown that the optimal entanglement cost for the state exchange with QSI can be negative. This is at striking variance with the state exchange without QSI, whose entanglement cost is always nonnegative.

By replacing classical communication with quantum communication, we can consider a fully quantum version of the state exchange with QSI of $|\psi\rangle_{AC_{\rm A}BC_{\rm B}R}$. Similar to the idea of Theorem 1, this task can be performed by applying the state redistribution protocol [6, 7] twice. For example, if the part $C_{\rm A}$ is firstly redistributed from Alice to Bob in this strategy, then its achievable rates $E_{\rm r}$ and $Q_{\rm r}$ for ebits and qubit channels are given by

$$\begin{aligned} E_{\rm r} &= \frac{1}{2}[l_1(\psi) + l_2(\psi)], \\ Q_{\rm r} &= \frac{1}{2}u_1(\psi) + \frac{1}{2}[H(C_{\rm A}|A)_\psi + H(C_{\rm B}|BC_{\rm A})_\psi], \end{aligned}$$

where $u_1(\psi)$, $l_1(\psi)$, and $l_2(\psi)$ are in Theorem 1 and Corollary 3. However, in this case the achievable region of a resource pair $(E_{\rm r}, Q_{\rm r})$ is completely unknown.

To the best of our knowledge, a protocol exchanging Alice's and Bob's information in a single step has not been known, and so in this work we have considered the merge-and-merge strategy, in order to obtain achievable entanglement rates. Hence it would be very meaningful to devise such a direct exchanging protocol. Moreover, recent results for one-shot quantum state merging [17] and implementing bipartite unitaries [18] may be useful to figure out novel strategies which can provide tighter achievable bounds than those in Theorem 1.

As potential applications of the state exchange, our task can be considered as a simple and fundamental situation in distributed quantum computation [18–21], in which multiple quantum devices connected by quantum communication are exploited. Moreover, it turns out that swap gates play a crucial role in universal quantum computation [22]. Since the state exchange is conceptually nothing but a swap operation between two remote users, our results would be useful to realize the swap gates in distributed quantum computation with reduced entanglement costs.

Finally, we expect that studying variations on the state exchange with QSI makes quantum information theory richer. For example, one can assume that Alice and Bob can consume noisy resources [23, 24] instead of noiseless resources, or that Alice or Bob is additionally allowed to make use of a local resource, such as maximally coherent states [25–27], as in the incoherent state merging [27] and the incoherent state redistribution [28]. Exploring these avenues deserves further investigation.

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