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Three-dimensional Chiral Lattice Fermion in Floquet Systems

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We show that the Nielsen-Ninomiya no-go theorem still holds on Floquet lattice: there is an equal number of right-handed and left-handed Weyl points in three-dimensional Floquet lattice. However, in the adiabatic limit, where the time evolution of low-energy subspace is decoupled from the high-energy subspace, we show that the bulk dynamics in the low-energy subspace can be described by Floquet bands with extra left/right-handed Weyl points, despite the no-go theorem. Assuming adiabatic evolution of two bands, we show that the difference of the number of right-handed and left-handed Weyl points equals twice the winding number of the adiabatic Floquet operator over the Brillouin zone. Based on these findings, we propose a realization of purely left- or right-handed Weyl particles on a 3D lattice using a Hamiltonian obtained through dimensional reduction of a four-dimensional quantum Hall system. We argue that the breakdown of the adiabatic approximation on the surface facilitates unusual closed orbits of wave packets in applied magnetic field, which traverse alternatively through the low-energy and high-energy sector of the spectrum.

Introduction.— In 1981, Nielsen and Ninomiya [1, 2] proved a theorem in 3D implying the absence of neutrinos on a lattice: there are equal number of left-handed and right-handed Weyl particles appearing in any lattice realization of the Standard Model. In solid state physics, where there is a natural lattice, generic nodes of electron bands are linearly dispersing Weyl points [3–5] (WPs), which carry a chirality ± 1, depending on the net Berry flux pierced through a sphere enclosing the node. Recent extensive studies of WPs include the research of phenomena linked to the chiral anomaly [6–18], surface Fermi-arc states [19–26], and anomalous transport properties [27, 28]. It has become an established knowledge in the field of topological semimetals that the net chirality of all the WPs must be zero.

Recently, periodically driven systems have attracted interest from condensed matter [29–69], photonics [70–81] and cold atoms [82–85] communities. In periodically driven lattices, a key concept is time-evolution operator over the period of one cycle (the *Floquet operator* $U_{\mathbf{k}}$), whose eigenvalues $\exp[-i\epsilon_i(\mathbf{k})T]$ constitute quasi-energy bands $\epsilon_i(\mathbf{k})$. Given such novel platforms, it is natural to revisit the Nielsen-Ninomiya theorem for quasi-energy bands. Especially, one of the assumptions made by Refs. [1, 2] is that the energy spectrum can be ordered at each momentum $\mathbf{k}$ as $E_1(\mathbf{k}) \leq E_2(\mathbf{k}) \leq \ldots \leq E_n(\mathbf{k})$. Such a premise does not apply to quasi-energy bands because quasi-energy is determined only up to multiples of $2\pi/T$. Furthermore, if one only considers the periodicity of Berry curvature on the Brillouin zone (BZ) boundary, one can easily find "counterexamples" of the theorem as schematically illustrated in Fig. 1(b). Even more simply, one can find a one-dimensional (1D) quasi-energy band with a single chiral mode [56, 57] as shown in Fig. 1(a), which presents a "counterexample" of the analogous no-go theorem in 1D.

In this letter, we present a topological argument proving that the Nielsen-Ninomiya no-go theorem generalizes to periodically driven lattices. However, we also show that the mentioned "counterexamples" become physically meaningful in the adiabatic limit (i.e. when the rate of changing the Hamiltonian is slow compared to the energy separation of the utilized bands of the instantaneous Hamiltonian from the rest of the spectrum.) In the latter case, the dynamics of the low-energy states (the states below the gap of the Hamiltonian at $t = 0$) is decoupled from the dynamics of the high-energy states, i.e. $U_{\mathbf{k}}$ becomes *block diagonal*. Although the spectrum of $U_{\mathbf{k}}$ obeys the no-go theorem, the spectrum of the individual low/high-energy blocks is allowed to exhibit Floquet bands with purely left- or right-handed WPs. This discovery opens an opportunity to experimentally observe the dynamics of chiral Weyl particles (neutrinos) on a lattice. For this purpose, we develop a 3D lattice model exhibiting chiral Weyl particles, which is obtained from a four dimensional (4D) quantum Hall state [86] by interpreting one momentum as the adiabatic parameter. We also infer that the adiabatic approximation breaks down on the surface due to the presence of topologically protected boundary states. In this way, the surfaces induce a circular motion of wave packets in an applied magnetic field, travelling alternatively in the low-energy and high-energy sectors of the Floquet operator.

The no-go theorem.— Similar to static electron bands, the generic nodal structure of a 3D Floquet lattice is still a Weyl point protected by Chern number on a sphere enclosing the node. Assuming that the translational symmetry is preserved, a Weyl point can be removed only through a pairwise annihilation with a Weyl point of opposite chirality. Therefore, even for Floquet bands, the difference of the number of right-handed and left-handed WPs $n_R - n_L$ is a topologically stable quantity, i.e. a topological invariant of $U_{\mathbf{k}}$. If we allow the unitary matrix $U_{\mathbf{k}}$ to be an arbitrary (but continuous) function of $\mathbf{k}$, then $n_R - n_L$ can indeed be nonzero. However, Floquet operators are subject to the no-go theorem for the following observation: It is possible to continuously deform all the legitimate Floquet operators $U_{\mathbf{k}}$ to the identity matrix $I_{N\times N}$ by retracting the time-evolution operator to $t = 0$ while keeping $n_R - n_L$ invariant. More explicitly, the time evolution at momentum $\mathbf{k}$: $\mathbf{k} \mapsto \mathcal{T}\exp[-i\int_0^t H_{\mathbf{k}}(t')dt']$ continuously interpolates $\mathbf{k} \mapsto I_{N\times N}$ at $t = 0$ and $\mathbf{k} \mapsto U_{\mathbf{k}}$ at $t = T$. Since

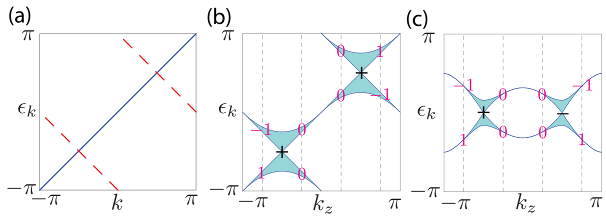

FIG. 1. (a) A chiral Floquet mode (solid blue line) inside 1D BZ can be realized by evolving a state adiabatically. The no-go theorem guarantees an additional mode with opposite chirality (dashed red line), which decouples in the adiabatic limit. (b–c) Two schemes for a pair of WPs in Floquet band structures. The panels show the spectrum along the k_z momentum, with the pale blue regions corresponding to the projected band dispersion in the k_x and k_y directions. The vertical dashed lines represent gapped two-dimensional subsystems with fixed k_z, with the Chern number of each band indicated in magenta. The black ± signs, which indicate the chirality of the corresponding WPs, can be inferred from the change in Chern number with k_z. The setting in (b), featuring two WPs of the same chirality, can be realized in the adiabatic limit.

$n_R - n_L = 0$ for the spectrum of identity matrices at $t = 0$, the same must hold for the Floquet operator $U_{\mathbf{k}}$. This topological argument has not been properly formulated and also applies to 1D proving the analogous no-go theorem.

It is germane to rephrase and generalize the observation above: assuming continuous deformations without further constraints, the Floquet operator over one cycle $U_{\mathbf{k}}$ *always* retracts to topologically trivial identity matrices . Therefore, to obtain a nontrivial topological property, one must impose certain restrictions on the admissible deformations. One choice is to permit only those that keep a finite gap in the quasi-energy spectrum. Such a choice, akin to the tenfold-way classification of static systems [87, 88], defines topological invariants of a *gap*, and usually determines a boundary state inside the gap [58, 59]. In this letter, we consider another type of constraint, namely that of the *adiabatic limit* [56]. This requires the presence of a finite gap between the low-energy and the high-energy sectors of the *instantaneous* Hamiltonian [Fig. 2(a)], and a time evolution slow relative to the energy separation of the two sectors. The argument of continuous retraction of the Floquet operator does not apply to the low-energy sector in the adiabatic limit [89], allowing us to find the "counterexamples" suggested in the introduction. Nevertheless, these "counterexamples" are consistent with the no-go theorem in the sense that there are complementary modes in the high-energy sector, which compensate the non-vanishing difference $n_R - n_L$.

Adiabatic limit.— In the adiabatic limit, the time-evolution operator $\widetilde{U}_{\mathbf{k}}$ of the low/high-energy sector over one cycle corresponds to a Wilson loop in the parameter space,

$$\widetilde{U}_{\mathbf{k}} = \mathcal{P} e^{i \oint_{\mathbf{R}(t)} \mathbf{a}_{\mathbf{k}}(\mathbf{R}) \cdot d\mathbf{R}}, \tag{1}$$

where the closed path $\mathbf{R}(t)$ represents the variation of the adiabatic parameters $\mathbf{R}$ over one cycle $t \in [0, T]$ (for simplicity, we set the cycle period to $T = 1$), and $\mathcal{P}$ indicates path-ordering.

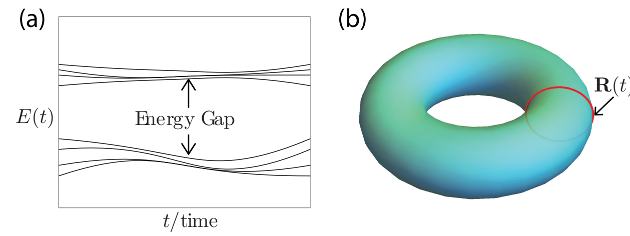

FIG. 2. (a) In the adiabatic limit of a Floquet system, the low-energy and the high-energy eigenvalues $E(t)$ of the instantaneous Hamiltonian remain separated by a large enough gap. (b) A nontrivial loop $\mathbf{R}(t)$ in the parameter space, for which the Wilson loop operator $\widetilde{U}_{\mathbf{k}}$ of Eq. (1) is not in general deformable to the identity.

Finally, $\mathbf{a}_{\mathbf{k}}(\mathbf{R})$ is the non-Abelian Berry connection [93–95]

$$[\mathbf{a}_{\mathbf{k}}(\mathbf{R})]_{mn} = i\langle \mathbf{k}, \mathbf{R}, m | \nabla_{\mathbf{R}} | \mathbf{k}, \mathbf{R}, n\rangle, \tag{2}$$

where $|m\rangle, |n\rangle$ label the low-energy (or high-energy) eigenstates of the instantaneous Hamiltonian. The Wilson loop is a geometric property of the path $\mathbf{R}(t)$. Importantly, if the path is not contractible to a point [see Fig. 2(b)] in the parameter space, then the function $\mathbf{k} \mapsto \widetilde{U}_{\mathbf{k}}$ may fail to be continuously deformable to the identity $\mathbf{k} \mapsto I_{N\times N}$, thus possibly exhibiting a nontrivial topology [89].

We first illustrate such a topological property for a 1D system with momentum k and adiabatic Floquet operator $\widetilde{U}_k = \exp(-ik)$. The eigenvalue $\exp[-i\epsilon(k)]$ has a chiral dispersion, $\epsilon(k) = k \bmod 2\pi$ [blue line in Fig. 1(a)]. Counting the number of right movers $n_R^{1\mathrm{D}}(\epsilon)$ and the number of left movers $n_L^{1\mathrm{D}}(\epsilon)$ on each quasi-energy cut ϵ reveals that $n_R^{1\mathrm{D}}(\epsilon) - n_L^{1\mathrm{D}}(\epsilon)$ does not depend on ϵ. Furthermore, this difference does not change upon continuous deformation of the dispersion, nor upon adding a trivial band [i.e. one with $n_R^{1\mathrm{D}}(\epsilon) - n_L^{1\mathrm{D}}(\epsilon) = 0$], therefore suggesting a topological character. It is easily checked [56, 89] that the difference equals to the winding number of $\widetilde{U}_k$,

$$n_R^{1\mathrm{D}}(\epsilon) - n_L^{1\mathrm{D}}(\epsilon) = \nu_1 \equiv \frac{i}{2\pi} \int_{-\pi}^{\pi} \mathrm{tr}\left[\widetilde{U}_k^{-1} \partial_k \widetilde{U}_k\right] \tag{3}$$

over the 1D BZ of the system.

Inspired by the 1D case summarized by Eq. (3), we speculate that the difference $n_R - n_L$ between the number of right-handed and left-handed WPs in a 3D system is related to the winding number ν_3 of $\widetilde{U}(\mathbf{k})$ over a 3D BZ,

$$\begin{aligned} \nu_3 = & \frac{1}{24\pi^2} \int d^3\mathbf{k}\, \varepsilon^{\alpha\beta\gamma} \times \\ & \mathrm{tr}\left[(\widetilde{U}_{\mathbf{k}}^{-1} \partial_{k_\alpha} \widetilde{U}_{\mathbf{k}})(\widetilde{U}_{\mathbf{k}}^{-1} \partial_{k_\beta} \widetilde{U}_{\mathbf{k}})(\widetilde{U}_{\mathbf{k}}^{-1} \partial_{k_\gamma} \widetilde{U}_{\mathbf{k}})\right], \end{aligned} \tag{4}$$

where $\varepsilon^{\alpha\beta\gamma}$ is the anti-symmetric tensor and $\alpha, \beta, \gamma \in \{x, y, z\}$ are spatial indices. In the next section, we inspect the relation between topological quantities ν_3 and $n_R - n_L$ for a class of two-band models.

Two-band model.— The presence of a WP requires at least two bands. We thus consider a pair of bands in the adiabatic

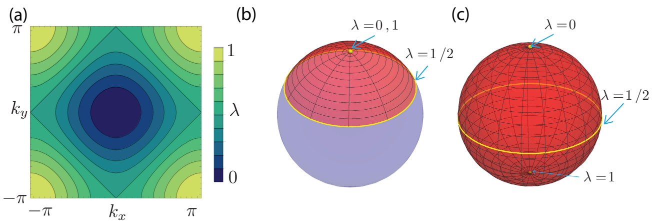

FIG. 3. Oriented covering of a three-dimensional sphere S^3 by the image of the BZ. We visualize the discussion using 2D manifolds, without changing the conceptual part of the argument. (a) We partition BZ into a family of submanifolds labelled by $\lambda \in [0,1]$. The submanifold with $\lambda \in \{0,1\}$ are pointlike, while all the intermediate ones are "slices" of co-dimension one. (b) A map BZ $\to S^3$ with *trivial* $\nu_3 = 0$. For $\lambda \in [0, \frac{1}{2}]$, the image of the BZ slices descends down from certain point (here chosen to be the "north pole"), leading to positive integrand in Eq. (7) ("covering"), while for $\lambda \in [\frac{1}{2}, 1]$ the image of the BZ slices rises back to the original point, leading to negative integrand ("uncovering"). The total oriented covering is zero. (c) A map BZ $\to S^3$ with *nontrivial* $\nu_3 = 1$. The BZ slices descend from the origin (at $\lambda = 0$) all the way to the antipodal point (at $\lambda = 1$). The compensation with negative integrand does not occur.

limit, and decompose the Floquet operator into

$$\widetilde{U}_{\mathbf{k}} \in \mathrm{U}(2) \cong S^1 \times \mathrm{SU}(2), \tag{5}$$

where the $S^1 \cong \mathrm{U}(1)$ part refers to matrices of the form $\mathrm{diag}[\det(\widetilde{U}_{\mathbf{k}}), 1]$, while the SU(2) part has unit determinant. The ν_3 invariant comes from a nontrivial third homotopy group, which is independent of the S^1 part. For simplicity, we narrow our discussion to systems with $\nu_1 = 0$ on all closed paths inside the BZ, such that the image in the S^1 component can be continuosly deformed to identity. We decompose

$$\widetilde{U}_{\mathbf{k}} = n_0(\mathbf{k})\sigma_0 + i[n_1(\mathbf{k})\sigma_1 + n_2(\mathbf{k})\sigma_2 + n_3(\mathbf{k})\sigma_3], \tag{6}$$

where σ_0 is the identity and $\sigma_{1,2,3}$ are the Pauli matrices. The condition on unit determinant requires $\hat{\mathbf{n}}(\mathbf{k}) = (n_0(\mathbf{k}), n_1(\mathbf{k}), n_2(\mathbf{k}), n_3(\mathbf{k}))$ to be a real unit vector on a three-dimensional sphere S^3. The number of times that the image of T^3 "wraps" around the S^3 is given by the winding number

$$\nu_3 = \frac{1}{2\pi^2} \int d^3\mathbf{k} \varepsilon^{abcd} n_a(\partial_{k_x} n_b)(\partial_{k_y} n_c)(\partial_{k_z} n_d), \tag{7}$$

where ε^{abcd} is the anti-symmetric tensor and $a, b, c, d \in \{0, 1, 2, 3\}$ index components of $\hat{\mathbf{n}}$. Geometrically, the winding number density (i.e. the integrand) represents the oriented area that $\hat{\mathbf{n}}(\mathbf{k})$ swipes when we vary $\mathbf{k}$ over an infinitesimal cube ($d^3\mathbf{k}$) in BZ. A heuristic picture is that the image of $d^3\mathbf{k}$ is "covering" the S^3 at $\mathbf{k}$ if the oriented area is positive, while it is "uncovering" the S^3 if the oriented area is negative. We illustrate this concept on a pair of simple examples in Fig. 3, where we partition BZ into a family of submanifolds labelled by $\lambda \in [0, 1]$ for easier visualization.

A generic point of S^3 is covered (uncovered) n_+ (n_-) times by $\widetilde{U}_{\mathbf{k}}$. The geometric meaning implies that for all points

$$\nu_3 = n_+ - n_-. \tag{8}$$

Especially, Eq. (8) also applies to the "north pole" and "south pole", $\pm\sigma_0 \in S^3$, which correspond to degeneracies of the Floquet bands at quasi-energy 0 vs. π. The Floquet operator in the vicinity of a right-handed (+) and left-handed (−) WP takes the form $\widetilde{U}_{\mathbf{k}} = e^{\pm i(\mathbf{k}-\mathbf{Q}_N)\cdot\boldsymbol{\sigma}}$ at the north pole ($\widetilde{U}_{\mathbf{k}} = e^{i[\pi \pm (\mathbf{k}-\mathbf{Q}_S)\cdot\boldsymbol{\sigma}]}$ at the south pole), where $\mathbf{Q}_{N/S}$ is the momentum of the WP. The integrand of Eq. (7) is positive at right-handed WPs, and negative at left-handed WPs. Therefore, we find using Eq. (8) that

$$\nu_3 = n_R^N - n_L^N = n_R^S - n_L^S, \tag{9}$$

where the superscript indicates the quasi-energy of the WPs (i.e. the corresponding pole of the S^3). This implies that for two bands in the adiabatic limit, $n_R - n_L = 2\nu_3$. Especially, the value $\nu_3 = 1$ vs. 0 distinguishes the situations of Fig. 1(b–c). The result in Eq. (9) further means that WPs of opposite chirality but corresponding to opposite poles are not able to annihilate. Finally, the number of WPs has to be even for the adiabatic evolution of two bands. (More generally, we conjecture that ν_3 counts the number of Berry phase quanta flowing through the Floquet bands in the quasi-energy direction and for $N \geq 2$ bands to exhibit a minimum of $N\nu_3$ WPs.)

4D quantum Hall model.— A Floquet lattice with a nontrivial winding number ν_3 is related to 4D quantum Hall system [96, 97] if we identify the adiabatic parameter as the momentum k_w along the fourth dimension. It was shown by Ref. [56] that ν_3 of a Floquet operator of the occupied bands in the adiabatic evolution is *equal* to the second Chern number of the corresponding 4D model. This relation provides a practical way for developing Floquet models with a nontrivial ν_3 and thus, according to Eq. (9), with nonzero $n_R - n_L$. For example, one such a simple Hamiltonian [89, 96] is

$$\begin{aligned} H(\mathbf{k}, k_w) = {} & A(\sin k_x\Gamma_1 + \sin k_y\Gamma_2 + \sin k_z\Gamma_3 + \sin k_w\Gamma_4) \\ & + (\cos k_x + \cos k_y + \cos k_z + \cos k_w + m)\Gamma_5, \end{aligned} \tag{10}$$

where the Dirac matrices Γ_i obey the anti-commutation relation $\left\{\Gamma_i, \Gamma_j\right\} = 2\delta_{ij}$. One can adiabatic evolve k_w as a function of time from 0 to 2π periodically and the evolution of the lower two instantaneous bands can be described by two Floquet bands with nonzero net chirality of WPs.

The nonlinear 4D quantum Hall response implies that chiral Floquet systems produce a current $\mathbf{j} \propto \nu_3(\partial_t k_w)\mathbf{B}$ in an applied magnetic field, where $\partial_t k_w$ is analogous to electric field in the w direction. Taking the case of Fig. 1(b) as an example, the appearance of current follows easily by sketching the Landau level spectrum, which contains chiral modes [see Fig. 4(a)] traveling in the direction of the applied magnetic field. If the material has no boundary in that direction, this phenomenon corresponds to *chiral magnetic effect* (CME) [6, 98, 99]. In usual Weyl semimetals, CME is produced by creating a non-equilibrium state with chiral imbalance [100]. In our Floquet system, the chiral imbalance naturally arises in the adiabatic limit, since the evolution of the low-energy subspace is described by Floquet bands with nonzero net chirality of WPs.

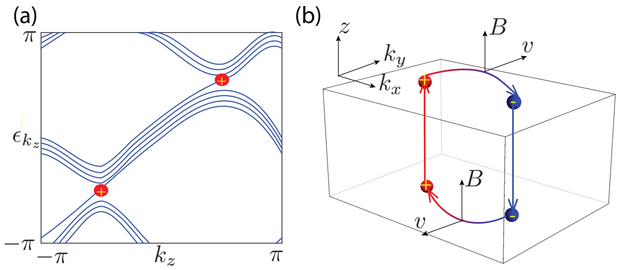

FIG. 4. (a) Landau levels exhibiting a chiral mode along the magnetic field direction for Floquet bands from Fig. 1(b). The chiral mode carries a current (chiral magnetic effect) in the absence of a boundary. (b) At the boundary, the chiral mode in the low-energy sector (red) evolves to the high-energy sector (blue) through a Fermi-arc connecting the WPs, and then travels back along the blue mode [101–103].

On the other hand, we expect the presence of a boundary to facilitate a circular motion of a wave packet through the system. To understand this phenomenon, first note that the adiabatic approximation breaks down on the boundary since 4D quantum Hall Hamiltonian exhibits gapless boundary states for certain k_w. This allows the low-energy and the high-energy sectors to couple at the boundary and one must consider the whole Floquet bands, which have zero net chirality. Since each sector has a nonvanishing (mutually opposite) chirality, we expect the coupling to take the form of Fermi arcs connecting the two sectors. To complete the argument, we consider a wave packet with momentum near the WP of the low-energy sector. In an applied magnetic field, the wave packet moves upward along the system via the bulk chiral Landau level, until it reaches the system boundary. Then it evolves along the surface Fermi arc under the influence of Lorentz force while reaching the high-energy sector. The new setting allows the wave packet to descend through the system along the Landau level of opposite chirality, until it finally completes the cycle by returning to the low-energy sector along the Fermi arc on the bottom of the system [see Fig. 4(b)].

Experimantal realization.— Here, we propose to simulate the dynamics of chiral Weyl particles in a 3D Floquet system with nontrivial topological invariant ν_3 in the adiabatic limit. Due to the high controllability and tunability, ultra-cold atoms and photonic waveguides have been proposed and realized as ideal platforms for studying topological physics. Following the present techniques, in Supplemental Material [89], we discuss the feasibility of constructing such a Floquet model using a 3D array of ring resonators, where the modulation of the rings serve as the adiabatic parameter k_w from 0 to 2π [89] in one cycle. We remark that the chiral Floquet spectrum [e.g. Fig. 1(b)] contains more experimentally probable information than that of the adiabatic response [104–111], such as the dynamics of each Floquet mode, the existence of Fermi-arc states and the resulting circular motion of wave-packet dynamics as shown in Fig. 4(b).

Conclusion.— We have shown the validation of Nielsen-Ninomiya no-go theorem in Floquet lattice and demonstrated the possibility of having purely left/right-handed WPs in the adiabatic limit. We have proven for the adiabatic evolution of two bands that the sum of the chirality of WPs is equal to twice the 3D winding number of the Floquet operator. We have made analogy of such a system to 4D quantum Hall system and proposed circular motion of wave packet as a signature. Our work will serve as a theoretical groundwork and shed light on experimental simulation of chiral Weyl particles.

Note added.— After finishing this manuscript, we became aware of a related preprint by Higashikawa et al. [112], where a Floquet band with nonvanishing total chirality of WPs is constructed without the analogy of 4D quantum Hall system and the argument that no-go theorem still holds.

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