Quantifying Resources in General Resource Theory with Catalysts

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Quantifying resources in general resource theory with catalysts

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A question that is commonly asked in all areas of physics is how a certain property of a physical system can be used to achieve useful tasks, and how to quantify the amount of such a property in a meaningful way. We answer this question by showing that in a general resource theoretic framework that allows the use of free states as catalysts, the amount of "resources" contained in a given state, in the asymptotic scenario, is equal to the regularized *relative entropy of a resource* of that state. While we need to place a few assumptions on our resource theoretical framework, it is still sufficiently general, and its special cases include quantum resource theories of entanglement, coherence, asymmetry, athermality, non-uniformity, and purity. As a by-product, our result also implies that the amount of noise one has to inject locally to erase all the entanglement contained in an entangled state is equal to the regularized relative entropy of entanglement.

In thermodynamics, the amount of work a system can do on its surrounding depends on the free energy of the system, highlighting the role of free energy in physically meaningful tasks. Various other physical systems have similar properties, such as the quantum entanglement of a bipartite quantum state [1] or the quantum coherence in a given quantum state [2]. While the full power of these resources is not completely understood, they have been identified as being crucial for achieving certain communication and computational tasks [3, 4] and states that possess these properties are called "resourceful states". However, there is no unified framework that operationally quantifies the amount of useful resources contained in a given state. We show that the amount of resources present in a state can be quantified in an operationally meaningful and unified way. Our result implies that the relative entropy of a resource [5] tightly captures the amount of noise required to change a resourceful state into a free state. The relative entropy of resource, $E(\rho_M)$, of a quantum state ρ_M is defined as

$$E(\rho_M) = \inf_{\sigma_M \in \mathcal{F}} \mathrm{D}(\rho_M \| \sigma_M),$$

where $\mathcal{F}$ is a collection of free states and $\mathrm{D}(\rho_M \| \sigma_M)$ is the quantum relative entropy [6], and the regularized relative entropy of resource is

$$E^{\infty}(\rho_M) = \lim_{n\to\infty} \frac{1}{n} E(\rho_M^{\otimes n}).$$

A geometric illustration of $E(\rho_M)$ is depicted in Figure 1.

The core of a resource theory rests on two system-dependent requirements: (i) the existence of a set of states that are free and inexpensive; and (ii) allowed/free operations that map the set of free states only to themselves. A resource theory emerges when quantum information theory is found to provide a unified platform for characterizing a resource [7, 8] because, in a nutshell, all resources can be viewed as inter-conversions of different system states with system-dependent constraints. Many resource theories have been developed over the past decade [9–21] which address a vast diversity of the physically meaningful properties of the natural world.

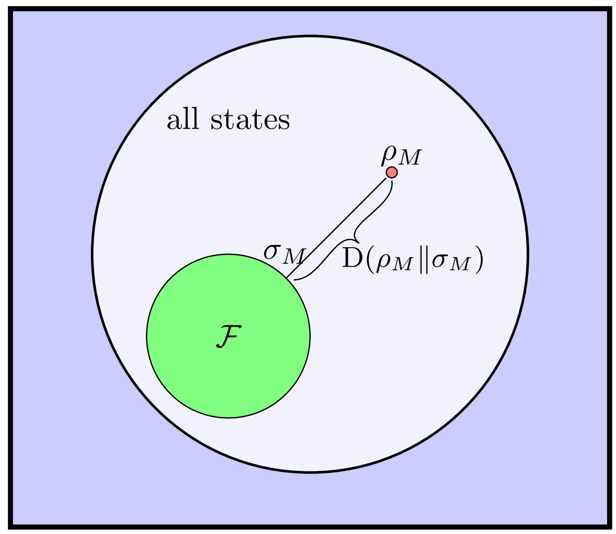

FIG. 1. A geometric illustration of the relative entropy of a resource $E(\rho_M)$.

It is well known that not every state transformation is possible, and adding a catalyst could at least make the transformation possible with a positive probability [22]. Moreover, even if a transformation is possible a priori, the addition of a catalyst often makes the process much more efficient. Therefore, individual resource theories have begun to include catalysts in their formalisms [10, 15].

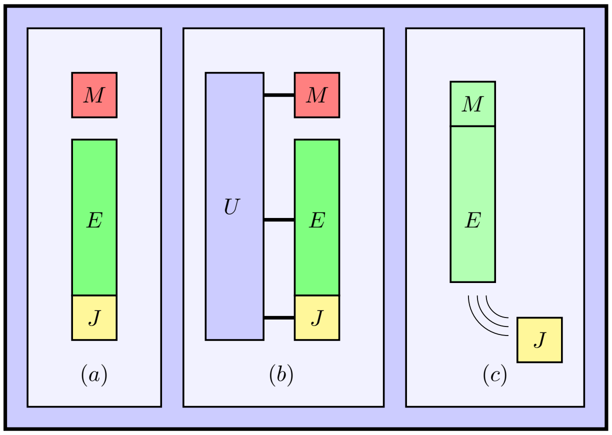

FIG. 2. Catalytic Resource Framework. (a) A free state is prepared in registers E and J. (b) Free operations U_j are performed on registers M and E, controlled by the classical register J. (c) Upon discarding the register J, the final state of registers M and E should be close to some free state.

We, therefore, consider a general resource theoretic framework that allows free states to be used as catalysts. Our main contribution is a complete characterization of the amount of resources contained in a given state ρ (relative to the free states) in the asymptotic scenario, as well as in the one-shot setting. Prior to our work, Ref. [5] also considered a general resource-theoretic framework for state transformation and showed that the asymptotic transformation rate from ρ to σ is given by the ratio $E^{\infty}(\rho)/E^{\infty}(\sigma)$. The major differences are threefolds. In [5], a catalyst was not included and the allowed free operations became resource-free only when the number of copies approached infinity. Moreover, their setting and the corresponding results do not imply our results regardless of whether catalysts are involved or not. Finally, their method cannot be used when only one copy of the resource state is involved in the transformation (i.e. the one-shot setting).

To achieve such a characterization, we begin by considering a task that provides a metric for "counting" the amount of resources present in ρ. This task, motivated by the work in Ref. [23], requires an experimenter to "destroy" all the resources present in ρ by converting them into a free state with the help of noise. The amount of randomness required to generate the noise serves as the desired measure. To further illustrate this task, we discuss the problem considered in Ref. [23], which transforms a bipartite quantum state $\rho_{AB}^{\otimes n}$ into a product state $\rho_{A^n} \otimes \rho_{B^n}$ with the aid of shared randomness and local unitaries. It was shown that the number of bits of randomness required (in other words, the randomness cost) is $\approx n \cdot \mathrm{I}(A:B)_\rho$, where $\mathrm{I}(A:B)_\rho$ is the quantum mutual information. Thus, it was observed that the total correlation contained in the state ρ_{AB} is equal to the amount of noise used to erase this correlation. This seminal result gave the first operational meaning to this entropic quantity and significantly advanced our understanding of entanglement theory. Being able to find the optimal randomness cost required to bring entangled states to separable states, thus, bears equivalent significance, if not more, since the existence of entanglement is believed to make quantum systems superior to their classical counterparts and the amount of entanglement is generally linked to its computational power [4]. Likewise, the crucial question as to the amount of valuable resources possessed by a state relative to its free states is found in every resource theory, be it quantum coherence, quantum thermodynamics, etc.

Our setting for quantifying the amount of resources is along the lines of the framework considered in the above work, but with the further freedom of allowing the use of additional free states that can aid in the transformation of the desired quantum state (Figure 2). We denote the set of all free states as $\mathcal{F}$ and the set of free operations as $\mathcal{U}$, where their formal definitions can be found in Supplemental Information. Assume that the resource state is ρ_M, defined on the register M. The experimenter can prepare a free classical-quantum state μ_{EJ}, where J corresponds to a classical register, and E denotes the corresponding quantum registers, and can perform a unitary $U_{JME} = \sum_j U_j \otimes |j\rangle\langle j|_J$ where U_j acts on registers ME and belongs to the set of free operations (operations that map a free state to a free state). The resulting quantum state Θ_{MEJ} must have the property that Θ_{ME} is close to a free state from $\mathcal{F}$ (by a distance of ε according to a suitably chosen distance measure). We call such a task an $(\varepsilon, \log|J|)$-transformation of ρ_M to $\mathcal{F}$, where $\log|J|$ is the randomness cost. If Θ_{ME} is close to $\omega_M \otimes \mu_E$ (where $\omega_M \in \mathcal{F}$ is a free state), then the quantum state μ_E is almost unaltered and, hence, serves as a catalyst. We call such a task an $(\varepsilon, \log|J|)$- catalytic transformation of ρ_M to $\mathcal{F}$ (see the Supplementary Information for a formal definition of these tasks).

We can also define the asymptotic randomness rate of the catalytic transformation, as the per copy amount of the randomness required to transform $\rho_M^{\otimes n}$ to $\mathcal{F}$, when n is large (as formally defined in the Supplementary Information). The main result of this work is as follows, which is a consequence of Theorem B (stated in the 'Proof Techniques' section).

Theorem A. *[Informal Statement] For a quantum state ρ_M, the asymptotic randomness rate of catalytic transformation of ρ_M is given by $E^\infty(\rho_M)$.*

The achievability part of this theorem uses a simple 'controlled swap' unitary which is allowed in a large family of resource theories. However for our converse argument we also allow for quantum measurements which can be implemented via adding free ancilla followed by a free unitary followed by tracing out of ancilla subsystem. The resource cost of such protocols is counted as the total number of qubits discarded in the implementation.

Proof Techniques. It is apparent that if no limits are set on the allowed operations and free states, then it is almost impossible to obtain a useful characterization, as also noted in Ref. [5]. The postulates of the set of free states $\mathcal{F}$ in our framework are very natural. They are as follows. (i) The set of free states is a convex and closed set. (ii) If two quantum states are free states, then their tensor product is a free states as well. (iii) If a quantum state on more than one register is a free states, then we obtain a free states with a partial trace over a subset of these registers. The set of all free operations $\mathcal{U}$ is the set of unitaries that take a free state to a free state.

Since we are interested in the amount of noise required to transform a state into a free state, we naturally assume that the experimenter can apply a mixture of unitaries $\{(p_j, U_j) : U_j \in \mathcal{U}\}_{j=1}^n$. Such a setting is also found in the works [23] and [20] concerned with randomness cost. The resulting free ensemble becomes $\{p_j, U_j \sigma U_j^\dagger : \sigma \in \mathcal{F}\}_{j=1}^n$. Observe that since $\mathcal{F}$ is a convex set and U_j are free operations, the quantum state $\sum_j p_j U_j \sigma U_j^\dagger$ belongs to $\mathcal{F}$. With the help of the "Church of the Larger Hilbert Space", we can introduce a 'classical' register J and write the overall unitary operation as $U = \sum_{j=1}^n U_j \otimes |j\rangle\langle j|_J$ applied to the state $\sigma \otimes \sum_j p_j |j\rangle\langle j|_J$. We expand our set of free states $\mathcal{F}$ and free operations $\mathcal{U}$ to include such states and unitaries as well.

Now, we are in a position to formally define our task, that we call an $(\varepsilon, \log|J|)$-transformation of ρ_M to $\mathcal{F}$. We interpret $\log|J|$ as the randomness cost of the protocol and ε as the allowed error. While we have identified the register J as classical, we show in the Supplemental Information that this assumption can, in fact, be dropped to accommodate a more general transformation. This only leads to a multiplicative loss of a factor of 2 in the randomness cost. Our task is as follows.

Task 1. *An experimenter holds a resourceful quantum state ρ_M. Using a classical-quantum state $\mu_{EJ} \in \mathcal{F}$, she applies a unitary $U \in \mathcal{U}$ to obtain a joint quantum state Θ_{MEJ}:*

$$\Theta_{MEJ} = U(\rho_M \otimes \mu_{EJ})U^\dagger.$$

It is required that there exists a $\sigma_{ME} \in \mathcal{F}$ such that $\mathrm{Pur}(\Theta_{ME}, \sigma_{ME}) \leq \varepsilon$, where the chosen distance measure is the purified distance, defined as $\mathrm{Pur}(\omega, \omega') = \sqrt{1 - \mathrm{F}^2(\omega, \omega')}$ (with $\mathrm{F}(\omega, \omega')$ being the fidelity). The randomness cost is $\log|J|$.

Note that in many cases, it is desirable that the free state μ_E be returned in as close to its original form as possible, that is, act as a catalyst. Our achievability result shall belong to such a class of transformations. Hence, Task 1 is said to be an $(\varepsilon, \log|J|)$-*catalytic transformation* of ρ_M if $\mu_{EJ} = \mu_E \otimes \mu_J$ and $\sigma_{ME} = \sigma_M \otimes \mu_E$ for some $\sigma_M \in \mathcal{F}$.

Finally, we say that the asymptotic randomness *rate* of the catalytic transformation of ρ_M is R, if, for every $\varepsilon > 0$, there exists an integer $n_0(\varepsilon)$ such that for all $n \geq n_0(\varepsilon)$, there exists a (ε, nR)-catalytic transformation of $\rho_M^{\otimes n}$ to $\mathcal{F}$.

We provide a near optimal characterization of the randomness cost of Task 1 in Theorem B below. We obtain matching upper and lower bounds of the randomness cost even if one only has a single copy of a given state, i.e., the one-shot scenario [24]. Our one-shot bounds are given in terms of the smooth max-relative entropy [25], which is defined as

$$\mathrm{D}_{\max}^{\varepsilon}(\rho\|\sigma) = \min_{\rho' : \mathrm{Pur}(\rho', \rho) \leq \varepsilon} \min\{\lambda : \rho' \preceq 2^\lambda \sigma\}.$$

This is a one-shot analogue of the quantum relative entropy. Hence, our result also provides a new operational meaning for this quantity in the resource theoretic framework. The upper bound, or the achievability result, is as follows. The experimenter possesses the quantum state ρ_M. Let σ'_M be the free state that minimizes the quantity $\min_{\sigma_M \in \mathcal{F}} \mathrm{D}^{\varepsilon}_{\max}(\rho_M \| \sigma_M)$. Let k be an integer such that $\log k = \mathrm{D}^{\varepsilon}_{\max}(\rho_M \| \sigma'_M) + 2\log\frac{1}{\delta}$ (for an error parameter δ). The experimenter introduces the free state $\sigma'_{M_1} \otimes \sigma'_{M_2} \otimes \ldots \otimes \sigma'_{M_k}$ (as a catalyst, where $M_1, M_2, \ldots, M_k$ are registers equivalent to M) and the maximally mixed state $\frac{\mathrm{I}_J}{k}$ in register J of dimension k. The registers $M_1, M_2, \ldots, M_k$ are collectively viewed as the register E introduced in Task 1. Controlled on the classical value j in register J, the experimenter swaps the registers M and M_j. The quantum state in register M is now σ'_M. Upon discarding the classical register J, the quantum state in registers $M_1, M_2, \ldots, M_k$ is

$$\frac{1}{k}\sum_{j=1}^{k} \sigma'_{M_1} \otimes \ldots \otimes \sigma'_{M_{j-1}} \otimes \rho_{M_j} \otimes \sigma'_{M_{j+1}} \otimes \ldots \sigma'_{M_k}.$$

From the convex-split lemma [26], this quantum state is close to the original state $\sigma'_{M_1} \otimes \sigma'_{M_2} \otimes \ldots \otimes \sigma'_{M_k}$, with purified distance at most $\varepsilon + \delta$. Thus, we have the following theorem.

Theorem B. *Fix $\epsilon, \delta > 0$, and a quantum state ρ_M.*

- *Achievability: There exists an $(\varepsilon + \delta, \log k)$-catalytic transformation of ρ_M to $\mathcal{F}$, where $\log k := \min_{\sigma_M \in \mathcal{F}} \mathrm{D}^{\varepsilon}_{\max}(\rho_M \| \sigma_M) + 2\log\frac{1}{\delta}$.*
- *Converse: For every $(\varepsilon, \log|J|)$- transformation of ρ_M to $\mathcal{F}$, it holds that*

$$\log|J| \geq \min_{\sigma_M \in \mathcal{F}} \mathrm{D}^{\varepsilon}_{\max}(\rho_M \| \sigma_M).$$

The proof of this theorem is given in the Supplemental Material, which includes the aforementioned argument for achievability and a converse proof. The proof of Theorem A follows from an asymptotic and i.i.d. analysis of this result and the continuity of relative entropy of resource [27].

Implications and Applications. Our first contribution is a unified resource-theoretic framework for quantifying the amount of resources contained in a resourceful state, and connecting this amount to the regularized relative entropy of a resource. Our general resource framework includes the resource theories of entanglement, coherence, thermodynamics, non-uniformity, purity and asymmetry (The full discussion of these special cases can be found in Supplemental Information). In particular, our result implies that the amount of "entanglement" contained in ρ_M is equal to the regularized relative entropy of entanglement [28, 29] asymptotically, yielding a direct operational meaning for this quantity. Hence, our work resolves an open question posted in Ref. [23], where only gapped upper and lower bounds were provided. In addition, our result also recovers the *symmetrization cost* given by the relative entropy of frameness [20]. While both Refs. [20, 23] employ resource-destroying maps, our framework that allows the use of free states as catalysts is more general and results in stronger matching one-shot bounds (Theorem B in the 'Proof Techniques' section) that were not possible previously.

Second, our result directly yields that the regularized relative entropy of a resource is an upper bound for distilling the aforementioned resources, since it is impossible to distill more of a resource than originally contained in a state. Interestingly, it is possible to distill the maximal amount of resources for the various resources described below.

In the resource theory of entanglement [1, 23, 28–31], the set of free states is the collection of separable states and the free operations contain the local quantum operations and classical communications (LOCC). The authors of Ref. [30] showed that the amount of maximally entangled states that one can distill from infinitely many copies of a given state is equal to the regularized relative entropy of entanglement when non-entangling maps are allowed as free operations. When this is combined with our work, the role of regularized relative entropy of entanglement is set on a firm footing since the amount of distillable entanglement should intuitively be equal to the "amount of entanglement" possessed by the given bipartite state (if a reversible entanglement theory holds true). Note that interpreting our achievability proof of Theorem A in this context reveals that the resource-destroying controlled unitaries can be implemented by LOCC. Furthermore, our converse proof shows that the randomness cost is optimal even when non-entangling operations (the operations that do not change separable states to entangled states) are used. Finally, we emphasize that there are fundamental differences between our entanglement-erasing framework and that in Ref. [32]. In the latter, the total correlation (both classical and quantum correlations) is erased with the help of catalysts, and the number of qubits that have to be discarded is given in terms of the smooth max-mutual information. However, in our case, the convex-split lemma allows us to erase *just* the quantum correlation, and leads to a characterization of noise in terms of the smooth max-relative entropy.

In the resource theory of coherence [2, 31, 33, 34], the set of free states is the collection of diagonal states on a pre-determined basis. It was shown that the amount of distillable coherence is also maximal and is equal to the relative entropy of coherence under the set of strictly incoherent operations (SIO) [34]. This, again, coincides with our result; however, the unitary operation required in our achievable proof is permutation unitary, which corresponds to the smaller class of physically incoherent operations (PIO) [35]. Finally, in the resource theory of nonuniformity [13] and purity [16], the only free state is the completely mixed state. The maximally distillable nonuniformity is again given by relative entropy of a resource when noisy operations are used [13].

Before ending this part of discussion, we remark that in the resource theory of quantum thermodynamics [9–14], the free states are Gibbs quantum states, that are states of the form $\rho_\beta(H) = \frac{e^{-\beta H}}{\mathrm{Tr}(e^{-\beta H})}$, for an arbitrary Hamiltonian $H > 0$. It has been shown that the amount of a pure excited state that can be distilled is in terms of min-relative entropy in the one-shot setting [11]. While this quantity yields the relative entropy of a resource in the asymptotic setting, it is smaller than our one-shot randomness cost, given in terms of max-relative entropy.

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