This is the accepted manuscript made available via CHORUS. The article has been published as:

Turbulence Appearance and Nonappearance in Thin Fluid Layers
Gregory Falkovich and Natalia Vladimirova
Phys. Rev. Lett. 121, 164501 — Published 16 October 2018
DOI: 10.1103/PhysRevLett.121.164501
Turbulence appearance and non-appearance in thin fluid layers

Gregory Falkovich\textsuperscript{1,2,3} and Natalia Vladimirova\textsuperscript{4}

\textsuperscript{1}Weizmann Institute of Science, Rehovot 76100 Israel
\textsuperscript{2}Institute for Information Transmission Problems, Moscow, Russia
\textsuperscript{3}Novosibirsk State University, 630090 Russia
\textsuperscript{4}University of New Mexico, Albuquerque, USA

(Dated: September 26, 2018)

Flows in fluid layers are ubiquitous in industry, geophysics and astrophysics. Large-scale flows in thin layers can be considered two-dimensional (2d) with bottom friction added. Here we find that the properties of such flows depend dramatically on the way they are driven. We argue that wall-driven (Couette) flow cannot sustain turbulence at however small viscosity and friction. Direct numerical simulations (DNS) up to the Reynolds number $Re = 10^6$ confirm that all perturbations die in a plane Couette flow. On the contrary, for sufficiently small viscosity and friction, perturbations destroy the pressure-driven laminar (Poiseuille) flow. What appears instead is a traveling wave in the form of a jet slithering between wall vortices. For $5 \cdot 10^3 < Re < 3 \cdot 10^4$, the mean flow in most cases has remarkably simple structure: the jet is sinusoidal with a parabolic velocity profile, vorticity is constant inside vortices, while the fluctuations are small. At higher $Re$ strong fluctuations appear, yet the mean traveling wave survives. Considering the momentum flux barrier in such a flow, we derive a new scaling law for the $Re$-dependence of the friction factor and confirm it by DNS.

In contrast, for quasi-two-dimensional channel flows it is not even known if they are able to produce turbulence at all. This is despite a rapidly expanding interest motivated by the needs of industry, astrophysics, geophysics, and laboratory experiments in fluid layers and soap films (see e.g. [8, 9], the recent collection [10] and the references therein). To the best of our knowledge, in all experiments in layers and films, external forces and obstacles were needed to produce turbulence (see e.g. [11]), and it is not known if such turbulence is able to sustain itself in a channel flow past an obstacle. The reason is that 2d ideal hydrodynamics conserves energy (squared velocity) and enstrophy (squared vorticity). Force at intermediate scales can generate two-cascade turbulence with energy/enstrophy cascading respectively upscales/downscales. On the contrary, in a wall or pressure-driven flow, the input is at the largest scale so that it is apriori unclear what kind of turbulence, if any, can exist in the limit of low viscosity and friction.

Combining analytic theory and DNS, we answer here this fundamental question. We describe how turbulence appears and develops in pressure-driven flows: as “snake” traveling wave — a jet meandering between counter-rotating vortices and preserving its form even for strong fluctuations, see Fig 1. Even more remarkably, we find that wall-driven flows relax to laminar for all values of viscosity and friction used and remain laminar for as long as we were able to follow. Both findings substantially widen our fundamental perspective on turbulence and may lead to diverse practical applications.

We start our consideration by analyzing the interplay between momentum and vorticity averaged along the channel. Convection carries vorticity unchanged while viscosity diffuses it, so that any turbulence must lead to vorticity mixing and homogenization. We thus expect the mean cross-channel vorticity profile in a turbulent flow (outside viscous wall layer) to be more flat than the
laminar profile. On the other hand, turbulence transfers momentum to the walls faster than a laminar flow, thus increasing the drag and decreasing the mean velocity. Averaged along the channel, vorticity is the transverse derivative of the velocity. The requirements on momentum and vorticity profiles are compatible for pressure-driven flows where turbulence flattens both the velocity and vorticity profiles. On the contrary, the mean velocity profile is monotonous for wall-driven flows, so decreasing velocity in the bulk while keeping it at the walls would make the vorticity profile more non-uniform. We then conclude that momentum and vorticity requirements on turbulence in 2d wall-driven flows are contradictory.

For a more formal argument, consider 2d Navier-Stokes equation with unit density and a uniform friction rate \( \alpha \):

\[
\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \nu \Delta \mathbf{v} - \nabla p - \alpha \mathbf{v}, \quad \nabla \cdot \mathbf{v} = 0. \tag{1}
\]

Already for the frictionless case (relevant e.g. for flows on superhydrophobic surfaces [12] or soap films under low air pressure) dramatic difference from the 3d case follows from the relation between momentum and vorticity fluxes, unique for two dimensions. Denote \( u, v \) the fluctuating velocity components respectively parallel and perpendicular to the walls, which are placed at \( y = \pm L/2 \) and move with \( \pm V/2 \). Let us average \( x \)-component of (1) with \( \alpha = 0 \) both over time and over \( x \) (zонально). The result can be written using vorticity, \( \omega = \nabla \times \mathbf{v} \):

\[
\partial_t (\nu \Omega + \langle uv \rangle) = \nu \Omega_y - \langle v \omega \rangle = \langle \nabla p \rangle. \tag{2}
\]

Here \( \Omega(y) \) is the vorticity averaged over time and \( x \). Turbulence adds extra fluxes of \( x \)-momentum and vorticity, related by the Taylor theorem: \( \partial_y \langle uv \rangle = -\langle v \omega \rangle \). When \( \langle \nabla p \rangle = 0 \), the first part of (2) gives the constancy of the cross-flow momentum flux, whose direction is set by vorticity at the wall. On the other hand, the second part of (2) gives \( \langle v \omega \rangle = \nu \partial_y \Omega_y \), that is existence of turbulence would absurdly mean that the vorticity flux is directed along the mean vorticity gradient. In other words, the right direction of the momentum flux (from large to small values) requires the wrong direction of the vorticity flux (from small to large) in 2d wall-driven flows. In essence, laminar profile \( U = V y / L \) already has a constant vorticity; one cannot excite turbulence to make it more flat. If we add bottom friction then the laminar profile gets an inflection point, but it is a vorticity minimum so that the flow is getting more stable according to the Fjortoft criterium [13]. Indeed, adding to the viscous flow extra dissipation due to bottom friction could diminish fluctuations but cannot create them.

These non-rigorous but plausible arguments suggest that a wall-driven flow must relax to the laminar state, \( U(y) = V \sinh(y / \sqrt{\alpha / \nu}) / 2 \sinh(L / \sqrt{\alpha / V / 2}) \), for any \( \nu \) and \( \alpha \). This is supported by DNS whose details are described in [32]. Starting from different multi-vortex configurations, we observe different transients and eventual relaxation to the laminar flow in all cases. These results strongly suggest that the laminar wall-driven flow is the global attractor in two dimensions at however small viscosity and friction. To the best of our knowledge, this is the first such example in the whole fluid mechanics.

From another perspective, impossibility of turbulence in a 2d wall-driven flow can be related to sign-definite mean vorticity and shear. Even when we initially create vortices of both signs, the vorticity of the sign opposite to the mean is destroyed by the shear, while the same-sign vorticity is homogenized back into the laminar profile. On the contrary, for pressure-driven flows, the mean vorticity has opposite signs at opposite walls, so that turbulence cannot homogenize vorticity back to the laminar profile.

We turn now to pressure-driven flows and define the dimensionless control parameters \( Re_A = A^{1/2} L^{3/2} / \nu \) and \( Ru_A = A^{1/2} / \alpha L^{1/2} \). Here \( A = \langle \nabla p \rangle \) is either the mean pressure gradient divided by density or the gravity acceleration for soap films. There is a rich history of modeling 2d Navier-Stokes channel flows, see, e.g. [3, 16, 20] and the references therein. In particular, extensively studied was subcritical instability of the laminar flow at \( Re_c = 5772 \) (where \( Re = 3L U / \nu \) and \( U \) is the flow rate) [21] and streamwise localization of traveling wave at \( Re < Re_c \) [20]. To the best of our knowledge, the largest \( Re = 10^4 \) was achieved in [16, 17], where transitional turbulence was observed and fully developed turbulence was estimated to appear around \( 2 \cdot 10^5 \) which were beyond computer resources back then. Here we explore higher \( Re \) never treated before; we also add uniform friction to relate to real fluid layers.

We focus on a solution that appear from a generic initial condition in a wide interval of \( Re \). To much surprise we find that in relatively short channels with periodic boundary conditions a periodic traveling wave is a long-time attractor at intermediate \( Re \), and even strongly turbulent state at high \( Re \) has the mean profile of such a form (Fig. 1). The transition to the steady-state is slow and can be non-monotonic. In all cases, pressure-driven flows relax to either of two states: the laminar uni-directional flow or a “snake” traveling wave. In the latter case, most of the flux occurs along a sinusoidal jet meandering between two sets of counter-rotating vortices rolling along the walls. An example of quite turbulent evolution which results in a remarkably simple flow is shown in Fig 2. While we cannot rule out switches between the states (as in 3d pipe flow) on an astronomical time scale, we have not seen them once the statistical steady state is established.

The time of transients can be reduced by starting with a low-amplitude perturbation to mimic naturally developing instability of the laminar flow [2, 21]. Then, the early evolution shows well-defined exponential growth. Modeling a 12L channel, we applied perturbations with the wavelengths \( \lambda_{pert} = 3, 4, 6, 12L \) for \( Re_A = 894 \) and \( Ru_A = 179 \). The largest growth rate \( \gamma \) was found for \( \lambda_{pert} = 4L \) and is shown in Fig 3a. In \( Re_A - Ru_A \) plane, \( \gamma = 0 \) line separates laminar and sinuous flows in Fig. 3b. The inset in Fig. 3 shows the Reynolds number as a func-
Re runs (Re = 14200, Re_A = 894 and Re = 6320, Re_A = 447) saturate at channel-filling periodic trains, which fit respectively six and five wavelengths. Closer to laminar threshold, for Re = 4620 (Re_A = 316), we have observed streamwise localization or train breakdown [19, 20], where one out of four pairs of counter-rotating vortices was periodically disappearing (see [32] for details).

For 400 \lesssim Re_A \lesssim 1500, the spatially periodic mean flow in a co-moving reference frame has a beautifully simple structure: The jet is sinusoidal with approximately parabolic velocity profile, while vorticity is essentially constant across each vortex, which appears as a plateau in the vorticity cross-sections in Fig. 4a. Constant vorticity inside the vortices can be explained in the spirit of [23] as a consequence of viscosity being very small and yet finite: The former means that vorticity must be constant along the (closed) streamlines, while the latter means constancy across the streamlines in a stationary flow. The same argument suggests the vorticity flux constancy across the jet, so that vorticity changes linearly between opposite values at the separatrices.

At Re_A \approx 1500 the flow becomes turbulent. Chaotic small vortices are created at the walls and swept into a big vortex of the same sign thus feeding the large-scale flow. For Re_A \gtrsim 2000 and up to 8000 (Re = 2.94 \cdot 10^5) the relative level of velocity and vorticity fluctuations remains constant within our accuracy. All turbulent flows observed in 4L channel have a pronounced large-scale structure of a jet and 2L-periodic chain of vortices, similar to the sinuous flow. This is seen from comparison of Fig. 1a with Fig. 1c averaged in the frame of the stronger negative vortex (the other vortices are blurred by fluctuations). While horizontally averaged $\bar{U}$, $\bar{\Omega}$ for sinuous and turbulent flows are of similar shape, time averaging exposes qualitative difference: in the turbulent vortex, mean vorticity has a peak rather than a plateau, see Fig. 4 and Fig. 1.

From the topology of the mean flow, seen in Fig. 1a, we
now derive the relation between the applied force $A$ and the flow rate $\bar{U}$ in the limit of large $Re_A$. The transfer of momentum (or equivalently vorticity) from the center to the walls encounters two separatrices: separating the vortex from the jet and from the wall layer. Vorticity is diffused by viscosity across the separatrix, then is carried fast by advection inside the vortex, and then transferred by viscosity towards the wall. There are thus two viscous bottlenecks (transport barriers) in this transfer: on the jet-vortex separatrix and on the wall boundary layer. The width $\ell$ of the separatrix layer can be estimated requiring the diffusion time $\ell^2/\nu$ to be comparable to the turnover time $L/U$, which gives $\ell \simeq (\nu L/U)^{1/2}$ and the effective viscosity $\nu_e \simeq U\ell \simeq \sqrt{\nu UL}$ (we do not distinguish $U$ and $\bar{U}$ in the estimates). The momentum flux due to force must be carried by the viscosity towards the walls, $A \simeq \nu_e U/L^2$, which gives the flow rate and turbulent viscosity:

$$Re \simeq \frac{UL}{\nu} \simeq \frac{L^2 A^{2/3}}{\nu^{1/3}} = Re_A^{4/3}, \quad \nu_e \simeq \nu Re_A^{2/3}. \quad (3)$$

To describe the wall layer, note that $v \equiv 0$ at a wall. Integrating (2) over $y$ from wall to wall, we obtain that $\Omega(L/2) = -U'(L/2) = AL/2\nu$ is always equal to the laminar value, see the inset in Fig. 4b. Now we estimate the width of the wall layer, $U/U'(L) \simeq LA^{2/3} \nu^{-1/3} / AL \nu^{-1} \simeq \nu/3 A^{1/3} \simeq \ell$, which confirms that (3) is self-consistent. Alternatively, one derives (3) stating that the momentum flux is proportional to the velocity difference across the layer: $AL \simeq UU_x \simeq U^2 \ell/L$.

Appearance of thin boundary layers at large $Re$ (revealed in details in [32]) must lead to a sharp maximum of the vorticity derivative: $\max \Omega_y \simeq (L/2) / \ell \simeq LA^{4/3} \nu^{-5/3}$, which is much larger than $\Omega_y(L/2) = A/\nu$, derived from (2) at a wall. Away from the wall layer, turbulence suppresses the mean vorticity gradient, as seen in the insets in Figs 4b and 5b.

Numerical simulations support (3), see Figure 5. The scaling $Re \propto Re_A^{4/3}$ continues through both weakly and strongly fluctuating regimes, even though the proportionality constant slightly changes at the transition (inset in Figure 5a). The mean vorticity at the boundary layer also follows the scaling (3), as seen in the inset in Fig 5b plotted for the rescaled quantity $\tilde{\Omega}_y = \Omega_y(y)/\max \tilde{\Omega}_y = \nu \Omega_y/ALRe_A^{2/3}$. Since (3) follows from the spanwise structure, it holds approximately even for the broken train (the leftmost cross in Figure 5a); the breakdown increases the flow rate a bit, apparently by widening the jet.

Let us discuss the role of the turbulent fluctuations. Flow dissipates energy and enstrophy, and the viscous dissipation rate of the former is proportional to the latter: $\nu \langle |\nabla \omega|^2 \rangle = \nu \langle \omega^2 \rangle$. Law (3) gives the same estimate for the pressure work and the dissipation rate: $\nu \tilde{\Omega}^2 \simeq \nu U^2 / AL \simeq A^{5/3} L^{-1/3} \simeq AL$, that is the mean flow is able to dissipate energy by itself. Indeed, the DNS data in Fig. 4 show that turbulent enstrophy fluctuations are smaller than mean, while velocity fluctuations are negligible. Enstrophy dissipation is determined by the vorticity gradients shown in Figure 5b. The mean gradient follows (3), while variance (and the enstrophy dissipation) near wall is much larger and grows with $Re$ faster than the mean. The numerics thus confirm that the enstrophy is dissipated by turbulence rather than by the mean flow.

According to (3) the friction factor of 2d channel, $AL/\nu^2$, decays as $Re_A^{-2/3} \sim Re^{-1/2}$, faster than in 3d, where one finds the empirical Blasius law $Re^{-1/4}$ for moderate $Re$ and the logarithmic decay for large $Re$. Such $Re^{-1/2}$ scaling was actually observed in decaying grid-generated turbulence in soap film experiments and hypothesized to be related to enstrophy cascade [24]. Here we have shown that this law is quite universal.

The law (3) is expected to hold when the time of the momentum transfer to the wall, $U/A \simeq L(A\nu)^{-1/3}$, is shorter than the friction time $\alpha^{-1}$, otherwise friction imposes linear regime with $U \propto A$. This requires force exceeding both viscous and friction thresholds: $A \gg \nu^2 L^{-3} (A\nu^3)/\nu$. Discuss briefly the role of the third dimension and the layer thickness $h$. For planar flows with open top and no-slip bottom, $\alpha = 3\nu/h^2$, while vertical motions invalidate the very notion of $\alpha$. An ability of moving walls to excite turbulence must depend on $h$: we expect turbulence when the wall Reynolds number $Vh/\nu$ becomes large. How wall-generated 3d turbulence will be distributed over a wide thin channel deserves future studies, particularly on account of strong planar flows suppressing vertical motions [15]. For pressure-driven flows, the validity of (3) requires $\ell \simeq LRe_A^{-2/3} = LRe^{-1/2} > h$. As $Re$ approaches $(L/h)^2$, we expect the decay of the friction factor with $Re$ to slow down and eventually converge to the 3d values observed in rectangular ducts [25].

Traveling wave pattern thus enhances effective viscosity and suppresses the flow rate compared to the laminar regime. It is instructive to compare (3) with the en-
hancement of diffusivity $\kappa$ by the factors $Pe^{1/2}$ for cellular [26, 27] and $Pe^{1/3}$ for wall-attached flows [28], where $Pe = UL/\kappa$. That enhancement leads to acceleration of flame fronts [29] and other phenomena. Similar to (3), interplay between small noise and advection universally leads to the $1/3$-scaling with noise amplitude: for tumbling frequency of a polymer in a shear flow [30], for the Lyapunov exponent of an integrable system under stochastic perturbation [31].

To conclude, wall-driven 2d flows relax to laminar at all values of viscosity and friction used. We described the traveling wave which replaces the pressure-driven laminar flow. In distinction from 3d, as the Reynolds number grows, the fluctuations increase yet the mean flow preserves its traveling-wave “snake” form. A remarkable property of 2d snake are separatrices, which modify momentum transport to the walls leading to a new scaling law for the friction factor.

Acknowledgments

We thank A. Obabko for help in using Nek5000. The work was supported by the grants of the Minerva Foundation, ISF (882), RSF (14-22-00259) and NSF (DMS-1412140). Simulations are performed at Texas ACC using XSEDE, supported by NSF grant ACI-1548562 through allocation TG-DMS140028.

[23] G. K. Batchelor, J Fluid Mech 1,2, 177-190 (1956)
[32] See Supplemental Material [url], which includes movies, for more details on the numerical procedure, transient evolution and flow structure.