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Dephrasure channel and superadditivity of coherent information

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The quantum capacity of a quantum channel captures its capability for noiseless quantum communication. It lies at the heart of quantum information theory. Unfortunately, our poor understanding of nonadditivity of coherent information makes it hard to understand the quantum capacity of all but very special channels. In this paper, we consider the *dephrasure channel*, which is the concatenation of a dephasing channel and an erasure channel. This very simple channel displays remarkably rich and exotic properties: we find nonadditivity of coherent information at the two-letter level, a substantial gap between the threshold for zero quantum capacity and zero single-letter coherent information, a big gap between single-letter coherent and private informations, and positive quantum capacity for all complementary channels. Its clean form simplifies the evaluation of coherent information substantially and, as such, we hope that the dephrasure channel will provide a much-needed laboratory for the testing of new ideas about nonadditivity.

Introduction. A key goal of quantum information theory is to extend the classical theory of information, as pioneered by Shannon [1], to include quantum effects like superposition and entanglement. The capacity of a noisy communication channel plays a fundamental role in classical information theory: it is the optimal noiseless communication rate that a noisy channel can support. In the quantum setting, a noisy communication channel has multiple capacities since it can be used to accomplish different communication tasks. Thus, a quantum channel $\mathcal{N}$ has a capacity for classical communication $C(\mathcal{N})$, quantum communication $Q(\mathcal{N})$, and private classical communication $P(\mathcal{N})$. It is a central challenge of quantum information theory to evaluate these capacities, understand them, and determine their mathematical properties.

The capacity of a classical channel $\mathcal{N}\colon X \to Y$ is given by $C(\mathcal{N}) = C^{(1)}(\mathcal{N}) = \max_X I(X;Y)$, where the maximization is over input probability distributions, and the mutual information $I(X;Y) = H(X) + H(Y) - H(XY)$ quantifies the correlations between channel input and output in terms of the Shannon entropy $H(\cdot)$ [1]. This is shown in several steps: First, a random-coding argument shows that $C^{(1)}(\mathcal{N})$ is an achievable communication rate, so, $C(\mathcal{N}) \geq C^{(1)}(\mathcal{N})$, and $C(\mathcal{N}) \geq \lim_{n\to\infty}(1/n)C^{(1)}(\mathcal{N}^{\otimes n})$. Second, Fano's inequality [2] is used to show that $C(\mathcal{N}) \leq \lim_{n\to\infty}(1/n)C^{(1)}(\mathcal{N}^{\otimes n})$, so $C(\mathcal{N}) = \lim_{n\to\infty}(1/n)C^{(1)}(\mathcal{N}^{\otimes n})$. This establishes a multi-letter formula (also called a regularized formula). Third, additivity $C^{(1)}(\mathcal{N}^{\otimes n}) = nC^{(1)}(\mathcal{N})$ is proved to establish the single-letter formula $C(\mathcal{N}) = C^{(1)}(\mathcal{N})$.

Formulas for quantum capacities can be found in a similar way, but for the quantities that are achieved via random coding, additivity in the third step above typically fails. This is fantastic—it means we can achieve higher communication rates than one might naively expect. These rates can be achieved by using error-correcting codes that have more structure than random ensembles. For example, the nonadditivity of the Holevo information χ shows that entangled signal states can boost the classical capacity of a quantum channel [3], while the nonadditivity of coherent information I_c (defined in (2)) shows that structured codes can also boost the quantum communication rate over very noisy channels [4]. In the same way, the private information I_p of a quantum channel (defined in (4)) can be nonadditive, in which case the rate of private information transmission is again enhanced by considering structured private codes [5]. Quantum information transmission is necessarily private, and hence the private capacity is no less than the quantum capacity. However, there are channels showing a strict separation between the two capacities [6, 7]. This property is partly related to nonadditivity issues, as for certain channels with additive coherent and private information such a separation is not possible [8, 9].

The benefits of quantum channels mentioned above also come with frustrations: nonadditivity effects mean that with current techniques, only multi-letter capacity formulas are available for quantum channels. Because these formulas take the form of an optimization over an infinite number of variables, at the moment we have no effective way to evaluate the capacities of a noisy quantum channel.

The main result of this paper is the discovery of a remarkably simple family of quantum channels that display the nonadditivity that makes understanding quantum capacities such a challenge. Dephrasure channels, defined below, have nonadditive coherent information that substantially pushes the threshold for non-zero quantum capacity. Compared to previous results for the depolarizing channel, the nonadditivity of coherent information observed is much larger than seen before. Perhaps more importantly, our analysis is much simpler than previous work; this allows for a clearer understanding of the effect. Moreover, these dephrasure channels show a strict separation between the coherent information and the private information, strongly suggesting that the respective capacities are strictly separated as well. Because of its simple structure and amenability to analysis, we anticipate that the dephrasure channel will become a laboratory for testing new ideas about nonadditivity and quantum channel capacities.

Quantum and private capacity. In quantum information theory, point-to-point communication between a sender and a receiver is modeled by a *quantum channel* $\mathcal{N}\colon A \to B$, a linear, completely positive, trace-preserving map between the algebras of linear operators of two Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$. The quantum capacity $Q(\mathcal{N})$ of a quantum channel $\mathcal{N}$ is defined as the highest rate at which quantum information can be faithfully transmitted through $\mathcal{N}$ (see [10] for an operational definition).

We have the following coding theorem for the quantum capacity [11–15]:

$$Q(\mathcal{N}) = \lim_{n\to\infty} \frac{1}{n} I_c(\mathcal{N}^{\otimes n}) = \sup_{n\in\mathbb{N}} \frac{1}{n} I_c(\mathcal{N}^{\otimes n}), \tag{1}$$

where the *channel coherent information* is defined as

$$\begin{aligned} I_c(\mathcal{N}) &\coloneqq \max\nolimits_\rho I_c(\rho,\mathcal{N}), \\ \text{with}\quad I_c(\rho,\mathcal{N}) &\coloneqq S(\mathcal{N}(\rho)) - S(\mathcal{N}^c(\rho)), \end{aligned} \tag{2}$$

and $S(\rho) \coloneqq -\operatorname{tr}\rho\log\rho$ is the von Neumann entropy of a state ρ (all logarithms in this paper are taken to base 2). In (2), $\mathcal{N}^c\colon A \to E$ denotes a complementary channel of $\mathcal{N}$, obtained by considering an isometric extension $V\colon \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E$ of $\mathcal{N}$ satisfying $\mathcal{N}(\rho) = \operatorname{tr}_E(V\rho V^\dagger)$ [16], and setting $\mathcal{N}^c(\rho) \coloneqq \operatorname{tr}_B(V\rho V^\dagger)$.

The optimization in (1) over an (in principle) unbounded number of channel uses n renders the quantum capacity intractable to compute in most cases. At the heart of this intractability lies the fact that the coherent information $I_c(\mathcal{N})$ can be *superadditive*: there are channels $\mathcal{N}$ and $n \in \mathbb{N}$ such that $I_c(\mathcal{N}^{\otimes n}) > nI_c(\mathcal{N})$. A notable example is the qubit depolarizing channel $\mathcal{D}_q\colon \rho \mapsto (1-q)\rho + q\frac{1}{2}\mathbb{1}$. For $q \in [0.2518, 0.255]$, it is known that $I_c(\rho_n, \mathcal{D}_q^{\otimes n}) > nI_c(\mathcal{D}_q)$ for certain input states ρ_n and appropriately chosen $n \geq 3$ [4, 17, 18]. Moreover, $I_c(\mathcal{D}_q) = 0$ for $q \geq 0.2524$, such that in the interval $q \in [0.2524, 0.255]$ the superadditivity holds in its "extreme form". There are even more exotic examples of quantum channels exhibiting superadditivity: for any given $n_0 \in \mathbb{N}$, there exists a channel $\mathcal{N}_{n_0}$ such that $I_c(\mathcal{N}_{n_0}^{\otimes n}) = 0$ for all $n \leq n_0$, but the channel still has capacity, $Q(\mathcal{N}_{n_0}) > 0$ [19].

The private capacity $P(\mathcal{N})$ of a quantum channel $\mathcal{N}$ quantifies the optimal rate of transmitting classical data with vanishing probability of error such that the joint environment state of all channel uses has vanishing dependence on the input. The private capacity can be expressed as follows [15, 20]:

$$P(\mathcal{N}) = \lim_{n\to\infty} \frac{1}{n} I_p(\mathcal{N}^{\otimes n}) = \sup_{n\in\mathbb{N}} \frac{1}{n} I_p(\mathcal{N}^{\otimes n}), \tag{3}$$

where the *private information* is defined as

$$I_p(\mathcal{N}) \coloneqq \max_{\mathfrak{E}} I_p(\mathfrak{E},\mathcal{N}), \tag{4}$$

with the maximization over quantum state ensembles $\mathfrak{E} = \{p_x, \rho_x\}$, and with

$$I_p(\mathfrak{E},\mathcal{N}) \coloneqq I(X;B)_{\mathcal{I}\otimes\mathcal{N}(\rho)} - I(X;E)_{\mathcal{I}\otimes\mathcal{N}^c(\rho)}. \tag{5}$$

The mutual information of a bipartite state σ_{AB} is defined as $I(A;B)_\sigma = S(A)_\sigma + S(B)_\sigma - S(AB)_\sigma$, and evaluated in (5) on the classical-quantum states $\mathcal{I}\otimes\mathcal{N}(\rho_{XA})$ and $\mathcal{I}\otimes\mathcal{N}^c(\rho_{XA})$, where $\rho_{XA} = \sum_x p_x|x\rangle\langle x| \otimes \rho_x$. Quantum information transmission is necessarily private, and hence $P(\mathcal{N}) \geq Q(\mathcal{N})$ for all $\mathcal{N}$. This is also true for the single-letter quantities, $I_p(\mathcal{N}) \geq I_c(\mathcal{N})$.

The private capacity exhibits similarly exotic behavior as the quantum capacity, since the private information defined in (4) is not additive [5, 21, 22]. Furthermore, there are channels with a large separation of coherent information and private information [7].

While the general situation is poorly understood, there are special classes of channels for which the quantum and private capacities can be evaluated. A channel $\mathcal{N}\colon A \to B$ with complementary channel $\mathcal{N}^c\colon A \to E$ is called *degradable*, if there is another channel $\mathcal{D}\colon B \to E$ such that $\mathcal{N}^c = \mathcal{D}\circ\mathcal{N}$. Degradable channels have additive channel coherent and private informations, $I_c(\mathcal{N}^{\otimes n}) = nI_c(\mathcal{N})$ and $I_p(\mathcal{N}^{\otimes n}) = nI_p(\mathcal{N})$ for all $n \in \mathbb{N}$, and furthermore they are equal to each other, giving $P(\mathcal{N}) = Q(\mathcal{N}) = I_c(\mathcal{N}) = I_p(\mathcal{N})$ for this class of channels [8, 23]. On the other hand, a channel is called *antidegradable*, if there exists a channel $\mathcal{A}\colon E \to B$ such that $\mathcal{N} = \mathcal{A}\circ\mathcal{N}^c$. Due to data processing, antidegradable channels have vanishing channel coherent and private information, and hence $Q(\mathcal{N}) = 0 = P(\mathcal{N})$ for antidegradable channels.

Generalizing these observations, Watanabe [9] showed that $Q(\mathcal{N}) = P(\mathcal{N})$ if the complementary channel $\mathcal{N}^c$ has vanishing quantum capacity. If furthermore $P(\mathcal{N}^c) = 0$, then all information quantities above are additive, and $I_c(\mathcal{N}) = I_p(\mathcal{N}) = Q(\mathcal{N}) = P(\mathcal{N})$ [9]. In similar spirit, [24] showed that additivity of coherent information holds for the class of *informationally degradable* channels, which includes all degradable channels [24]. Moreover, building on the results in [25], we showed in [26] that for a *low-noise channel* $\mathcal{N}$ that is ε-close to the identity channel in diamond distance, both its quantum and private capacities are within $O(\varepsilon^{3/2}\log\varepsilon)$ of the channel coherent information, limiting the effect of superadditivity for such channels.

Recently, an upper bound on the quantum capacity of a general quantum channel $\mathcal{N}$ was derived based on a convex decomposition of $\mathcal{N}$ into degradable and antidegradable maps [27]. In the special case of a *flagged* channel

$$\mathcal{N} = (1-\lambda)\mathcal{D}\otimes|0\rangle\langle 0| + \lambda\mathcal{A}\otimes|1\rangle\langle 1| \tag{6}$$

with $\lambda\in[0,1]$, $\mathcal{D}$ degradable and $\mathcal{A}$ antidegradable, optimality of the bound reported in [27] seems intimately connected with whether channels of the form (6) can exhibit superadditivity of coherent information. This led us to consider the family of dephrasure channels, which we introduce next.

Dephrasure channel. The channel we consider in this paper is composed of dephasing noise followed by erasure, and simply called *dephrasure channel*. For two probabilities $p,q\in[0,1]$, it is defined as

$$\mathcal{N}_{p,q}(\rho) := (1-q)((1-p)\rho + pZ\rho Z) + q\,\mathrm{tr}(\rho)|e\rangle\langle e|, \tag{7}$$

where $Z = |0\rangle\langle 0| - |1\rangle\langle 1|$ is the Pauli Z-operator, and $|e\rangle$ is an erasure flag orthogonal to the input space. It is not difficult to see that the dephasing channel $\mathcal{Z}_p\colon \rho\mapsto(1-p)\rho+pZ\rho Z$ is degradable for any $p\in[0,1]$. Furthermore, the map $\rho\mapsto\mathrm{tr}(\rho)|e\rangle\langle e|$ is trivially antidegradable. Since $\langle e|\rho|e\rangle = 0$ for all qubit input states ρ, we can without loss of generality write $\mathcal{N}_{p,q} = (1-q)\mathcal{Z}_p\otimes|0\rangle\langle 0| + \mathrm{tr}(\cdot)|e\rangle\langle e|\otimes|1\rangle\langle 1|$, which shows that the dephrasure channel is a flagged channel of the form in (6).

In the following sections, we analyze the quantum information transmission capabilities of the dephrasure channel. Without loss of generality, we restrict the discussion to $p,q\in[0,1/2]$. Detailed derivations to all results presented in the sequel can be found in [10].

Antidegradability. The dephrasure channel $\mathcal{N}_{p,q}$ is degradable only if $q=0$ or if $p=0$ and $q\le 1/2$. For $q\ge 1/2$ the channel is trivially antidegradable due to the antidegradability of the erasure channel $\rho\mapsto(1-q)\rho+q\,\mathrm{tr}(\rho)|e\rangle\langle e|$ in this range. Furthermore, there is a non-trivial region in the (p,q)-plane in which $\mathcal{N}_{p,q}$ is antidegradable. To determine this region, we consider the following choice of complementary channel:

$$\mathcal{N}^c_{p,q}(\rho) := q\,\rho\oplus(1-q)\sum_{x=0,1}\langle x|\rho|x\rangle|\phi^x_p\rangle\langle\phi^x_p|, \tag{8}$$

where $|\phi^x_p\rangle = \sqrt{1-p}\,|0\rangle + (-1)^x\sqrt{p}\,|1\rangle$. Making use of unambiguous measurement schemes [28–30], the original channel $\mathcal{N}_{p,q}$ can be recovered from $\mathcal{N}^c_{p,q}$ (viz. $\mathcal{N}^c_{p,q}$ can be degraded to $\mathcal{N}_{p,q}$) in the region

$$\mathcal{A} := \{(p,q)\colon p\in[0,1/2], q\ge k(p)\}, \tag{9}$$

$$k(p) := \frac{1-2p}{2(1-p)}\,. \tag{10}$$

We refer to [10] for details of this calculation.

Single-letter coherent information. In order to analyze nonadditivity properties of the dephrasure channel, we first derive a formula for the single-letter coherent information $I_c(\mathcal{N}_{p,q})$ defined in (2).

The dephrasure channel is defined in terms of a Z-dephasing, and therefore one could expect that the coherent information $I_c(\rho,\mathcal{N}_{p,q})$ in (2) is maximized by states $\rho_z = (0,0,z)$ that are diagonal in the Z-eigenbasis and hence invariant under Z-dephasing. Indeed, ordinary calculus shows [10] that $I_c(\mathcal{N}_{p,q}) = \max_{\rho_z} I_c(\rho_z,\mathcal{N}_{p,q})$ in the region

$$\mathcal{R}_1 := \{(p,q)\colon p\in[0,1/2],\, 0\le q<g(p)\}\,, \tag{11}$$

$$g(p) := \frac{(1-2p)^2}{1+(1-2p)^2}. \tag{12}$$

Numerics show that $\mathcal{R}_1$ also includes the region where $I_c(\mathcal{N}_{p,q})\ge 0$ [10, Fig. 5].

For states $\rho_z = (0,0,z)$ we have the explicit formula

$$I_c(\rho_z,\mathcal{N}_{p,q}) = (1-2q)S(\rho_z) - (1-q)S(\Phi_{p,z}) \tag{13}$$

with

$$\Phi_{p,z} = \begin{pmatrix} 1-p & z\sqrt{p(1-p)} \\ z\sqrt{p(1-p)} & p \end{pmatrix}.$$

The formula (13) has a maximum at $z=0$ in the region

$$\mathcal{R}_2 := \{(p,q)\colon p\in[0,1/2],\, 0\le q<j(p)\} \tag{14}$$

$$j(p) := \frac{1-2p-2p(1-p)\ln\frac{1-p}{p}}{2-2p-2p(1-p)\ln\frac{1-p}{p}}, \tag{15}$$

that is, in this region the completely mixed state $\pi = \frac{1}{2}\mathbb{1}$ maximizes the coherent information, which evaluates to $I_c(\mathcal{N}_{p,q}) = I_c(\pi,\mathcal{N}_{p,q}) = 1-2q-(1-q)h(p)$.

To sum up, in the region $\mathcal{R}_1$ defined in (11) the coherent information $I_c(\mathcal{N}_{p,q})$ is maximized by states diagonal in the Z eigenbasis. In the subregion $\mathcal{R}_2\subseteq\mathcal{R}_1$ defined in (14), the coherent information $I_c(\mathcal{N}_{p,q})$ is maximized by the completely mixed state π, and evaluates to $I_c(\pi,\mathcal{N}_{p,q}) = 1-2q-(1-q)h(p)$. In the fish-shaped region $\mathcal{F} := \mathcal{R}_1\setminus\mathcal{R}_2 = \{(p,q)\colon p\in[0,1/2],\, j(p)<q<g(p)\}$, the coherent information is maximized by Z-diagonal states with $z\neq 0$. Furthermore, the 0-contour line of $I_c(\mathcal{N}_{p,q})$ lies in $\mathcal{F}$ [10, Fig. 5]. Fig. 1 plots the functions $g(p)$, $j(p)$ and $k(p)$ that bound these regions.

Superadditivity of coherent information. In this section we show that the dephrasure channel $\mathcal{N}_{p,q}$ exhibits superadditivity of the coherent information within the region $\mathcal{F}$: there are $(p,q)\in\mathcal{F}$ for which $I_c(\mathcal{N}^{\otimes n}_{p,q}) > nI_c(\mathcal{N}_{p,q})$. This also holds in the 'extreme' form that there are (p,q) for which $\frac{1}{n}I_c(\mathcal{N}^{\otimes n}_{p,q}) > I_c(\mathcal{N}_{p,q}) = 0$.

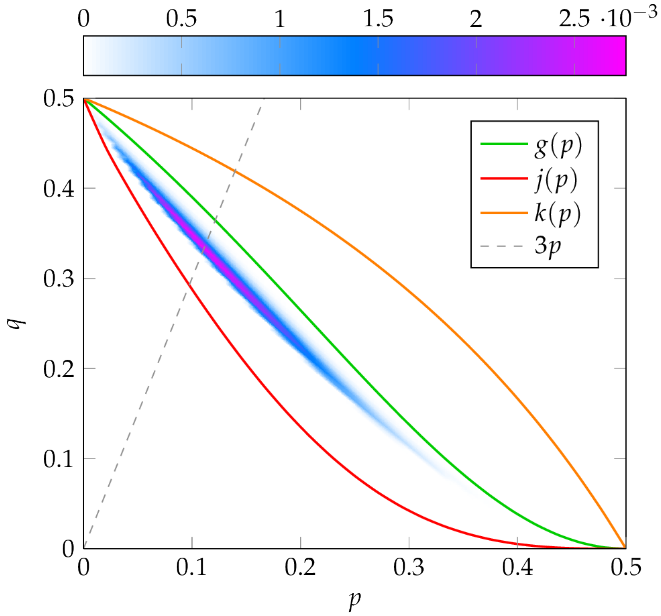

FIG. 1. Heat map of the quantity $\max_\lambda \frac{1}{2} I_c(\rho_2, \mathcal{N}_{p,q}^{\otimes 2}) - I_c(\mathcal{N}_{p,q})$. The repetition code ρ_2 is defined in (16). The functions $g(p)$ (green) and $j(p)$ (red) are defined in (12) and (15), respectively. The function $k(p)$ (orange) defined in (10) bounds the region $\mathcal{A}$ of antidegradability of $\mathcal{N}_{p,q}$ defined in (9).

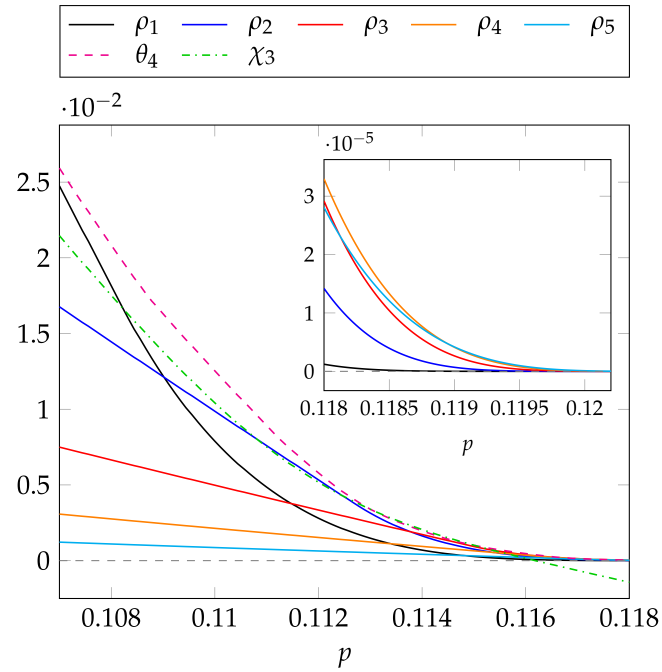

FIG. 2. Plot of the coherent information $I_c(\cdot, \mathcal{N}_{p,3p}^{\otimes n})/n$ in the interval $p \in [0.107, 0.118]$ for the repetition code ρ_n for $n = 1, \ldots, 5$ defined in (16) (solid lines), the generalized Z-diagonal code θ_4 defined in eq. (36) in [10] for $k = 4$ (dashed line), and the non-diagonal code χ_3 defined in eq. (37) in [10] (dash-dotted lines). The zero line is plotted as a dashed gray line for reference. The inset plot shows the repetition codes ρ_n in the interval $p \in [0.118, 0.1202]$, showing that repetition codes increase the threshold of the dephrasure channel.

We first demonstrate superadditivity of $I_c(\mathcal{N}_{p,q})$ using a simple (weighted) n-repetition code

$$\rho_n := \lambda |0\rangle\langle 0|^{\otimes n} + (1-\lambda)|1\rangle\langle 1|^{\otimes n}, \tag{16}$$

where $\lambda \in [0,1]$. Observe that

$$I_c(\rho_n, \mathcal{N}_{p,q}^{\otimes n}) = S\left(\mathcal{N}_{p,q}^{\otimes n}(\rho_n)\right) - S\left(\mathcal{N}_{p,q}^{\otimes n}(\phi_n)\right), \tag{17}$$

where in the second term, the output entropy of the complementary channel in (2) is rephrased in terms of the entropy of the purification of ρ_n, $\phi_n \equiv |\phi_n\rangle\langle\phi_n|$ with $|\phi_n\rangle := \sqrt{\lambda}|0\rangle^{\otimes n+1} + \sqrt{1-\lambda}|1\rangle^{\otimes n+1}$, and $\mathcal{N}_{p,q}^{\otimes n}$ acts on all but the first (purifying) system of ϕ_n.

The expression (17) is independent of the particular purification of ρ_n. To evaluate it, note that $\mathcal{N}_{p,q}^{\otimes n}$ is a sum of channels involving i erasures and $n-i$ dephasing errors for $i = 0, \ldots, n$. Any two erasure patterns differing in at least one position yield orthogonal output states, and hence (17) splits up into a sum over the different erasure patterns. Moreover, for a fixed erasure pattern with $1 \le i \le n-1$ erasures the two entropy terms on the right-hand side of (17) yield the same value $h(\lambda) := -\lambda \log \lambda - (1-\lambda)\log(1-\lambda)$, the binary entropy of λ. Hence, we only need to evaluate (17) in the cases of n dephasing erorrs and n erasures. Our calculation in [10] yields

$$I_c(\rho_n, \mathcal{N}_{p,q}^{\otimes n}) = ((1-q)^n - q^n)\, h(\lambda) \\ -(1-q)^n \left(1 - \frac{u}{2}\log\frac{1+u}{1-u} - \frac{1}{2}\log\left(1-u^2\right)\right), \tag{18}$$

for $u = u(\lambda, p, n) = \sqrt{1 - 4\lambda(1-\lambda)(1-(1-2p)^{2n})}$.

The formula (18) provides examples of superadditivity of the coherent information of $\mathcal{N}_{p,q}$. This is demonstrated in Fig. 1, where we plot a heat map of the quantity $\max_\lambda \frac{1}{2} I_c(\rho_2, \mathcal{N}_{p,q}^{\otimes 2}) - I_c(\mathcal{N}_{p,q})$. The region with the largest values of this quantity is colored in purple in Fig. 1, and crossed by the $(p, 3p)$-diagonal (dashed line). We therefore further investigate the optimized coherent information of the repetition code (16) along this diagonal for $n = 1, \ldots, 5$ [31]. In Fig. 2, we plot $\max_\lambda I_c(\rho_n, \mathcal{N}_{p,3p}^{\otimes n})/n$ for $1 \le n \le 5$ in the intervals $p \in [0.107, 0.118]$ and $p \in [0.118, 0.1202]$. In the latter interval, there are examples of the extreme form of superadditivity of the coherent information, i.e., $\frac{1}{n}\max_\lambda I_c(\rho_n, \mathcal{N}_{p,3p}^{\otimes n}) > I_c(\mathcal{N}_{p,3p}) = 0$, increasing the threshold of the dephrasure channel in this regime.

We were also able to find codes that outperform the weighted repetition code (16). These include more general Z-diagonal codes (such as θ_4 in Fig. 2, defined in eq. (36) in [10]), as well as certain *non-diagonal* codes (such as χ_3 in Fig. 2, defined in eq. (37) in [10]). Furthermore, other interesting non-diagonal codes can be found using a neural network state ansatz [32].

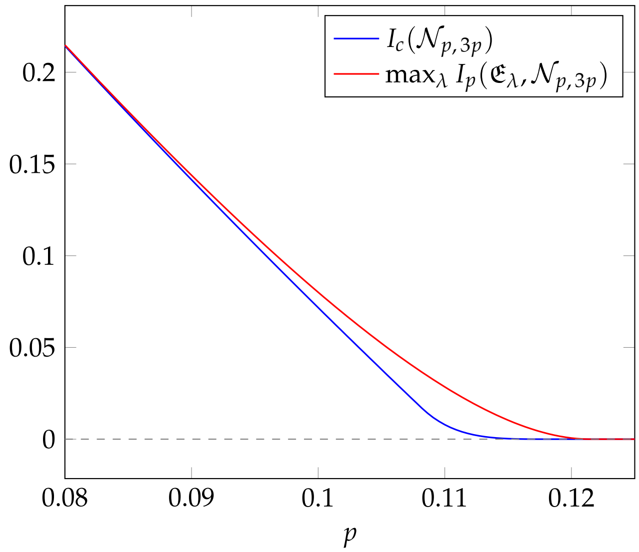

FIG. 3. Plot of the optimal single-letter coherent information $I_c(\mathcal{N}_{p,3p})$ for $p \in [0.08, 0.125]$ (blue), and a lower bound to the single-letter private information, $\max_\lambda I_p(\mathfrak{E}_\lambda, \mathcal{N}_{p,3p})$ (red), where the private code $\mathfrak{E}_\lambda$ is defined in (19).

Separation of private information and coherent information. Finally, we investigate the capabilities of private information transmission of the dephrasure channel. Numerical investigations suggest [10] that the following ensemble $\mathfrak{E}_\lambda = \{p_1, \rho_1; p_2, \rho_2\}$ maximizes the single-letter private information $I_p(\mathcal{N}_{p,3p})$:

$$\begin{aligned} p_1 &= \frac{1}{2}, \quad \rho_1 = \lambda|+\rangle\langle+| + (1-\lambda)|-\rangle\langle-| \\ p_2 &= \frac{1}{2}, \quad \rho_2 = (1-\lambda)|+\rangle\langle+| + \lambda|-\rangle\langle-|, \end{aligned} \tag{19}$$

where $\lambda \in [0,1]$ and $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$. This private code shows a strict separation between $I_c(\mathcal{N}_{p,3p})$ and $I_p(\mathcal{N}_{p,3p})$ in the interval $p \in [0.08, 0.125]$, plotted in Fig. 3. We note that the private information remains positive up to $p \simeq 0.12145$, which is exactly where the diagonal $(p, 3p)$ meets the curve $g(p)$ defined in (12). Interestingly, the latter marks the border where Z-diagonal codes optimize the single-letter coherent information.

It is an interesting open question whether the dephrasure channel also exhibits superadditivity of private information. However, to demonstrate this effect one first needs to determine the optimal single-letter private information. We conjecture the private code in (19) to be optimal for the dephrasure channel $\mathcal{N}_{p,3p}$.

Finally, we note that the complementary channel $\mathcal{N}^c_{p,q}$ has positive coherent information for all $p, q \in (0, 1/2]$ (see [10]), which implies that $P(\mathcal{N}^c_{p,q}) \geq Q(\mathcal{N}^c_{p,q}) > 0$ for all $p, q \in (0, 1/2]$. Similarly as for the depolarizing channel [33], this indicates that Watanabe's results [9] cannot be applied to the dephrasure channel.

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