Quantum Field Theory of Nematic Transitions in Spin-Orbit-Coupled Spin-1 Polar Bosons

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Quantum field theory of nematic transitions in spin orbit coupled spin-1 polar bosons

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We theoretically study an ultra-cold gas of spin-1 polar bosons in a one dimensional continuum which are subject to linear and quadratic Zeeman fields and a Raman induced spin-orbit coupling. Concentrating on the regime in which the background fields can be treated perturbatively we analytically solve the model in its low-energy sector, i.e. we characterize the relevant phases and the quantum phase transitions between them. Depending on the sign of the effective quadratic Zeeman field $\epsilon$, two superfluid phases with distinct nematic order appear. In addition, we uncover a spin-disordered superfluid phase at strong coupling. We employ a combination of renormalization group calculations and duality transformations to access the nature of the phase transitions. At $\epsilon = 0$, a line of spin-charge separated pairs of Luttinger liquids divides the two nematic phases and the transition to the spin disordered state at strong coupling is of the Berezinskii-Kosterlitz-Thouless type. In contrast, at $\epsilon \neq 0$, the quantum critical theory separating nematic and strong coupling spin disordered phases contains a Luttinger liquid in the charge sector that is coupled to a Majorana fermion in the spin sector (i.e. the critical theory at finite $\epsilon$ maps to a quantum critical Ising model that is coupled to the charge Luttinger liquid). Due to an emergent Lorentz symmetry, both have the same, logarithmically diverging velocity. We discuss the experimental signatures of our findings that are relevant to ongoing experiments in ultra-cold atomic gases of $^{23}$Na.

The interplay of internal quantum states and strong interactions can lead to the emergence of new quantum phases of matter and criticality. For example, while spin-1/2 quantum magnets can only sustain conventional magnetic order, larger spin systems allow for order in higher angular momentum channels involving multipole moments in large spin systems [1–3]. Spinful ultra-cold atomic gases are a particularly fruitful setting to study moments in large spin systems [1–3]. Spinful ultra-cold atomic gases with a large spin (e.g. $^{52}$Cr with $S = 3$) [4, 5]. This can lead to superfluids with non-trivial magnetic structure that spontaneously break both charge conservation and spin rotation symmetries [6, 7].

Ultra cold spin-1 bosons are an ideal system to study nontrivial magnetism beyond conventional vector magnetic order parameters. A pivotal microscopic ingredient is the spin dependent interaction $g_2$ which can either be ferromagnetic ($g_2 < 0$) or polar ($g_2 > 0$) [5] and leads to different ground states displaying either non-zero or zero spin expectation value, respectively [6, 7]. In the following, we concentrate on the polar case which is readily realized with $^{23}$Na gases [5]. The condensate wavefunction can be written as a three-component spinor $\Psi_{MF} = \sqrt{2}e^{i\varphi}\hat{n}$ where the superfluid phase $\varphi$ and the unit vector $\hat{n}$ parametrize the ground state manifold. The polar condensate has nematic order signaled by non-zero eigenvalues of a rank-2 tensor order parameter [6, 7]. A quadratic Zeeman field [8] lifts the degeneracy and the ground state spinor is given by either $\hat{n} = (0, 1, 0)^T$ or a planar state $\hat{n} = (e^{i\varphi}, 0, e^{-i\varphi})^T$ depending on the sign of the quadratic Zeeman field [4]. In recent experiments, it has been demonstrated that it is possible to observe the non-trivial nematic order in $^{23}$Na [9] and that the quadratic Zeeman effect can be used to drive nematic phase transitions [10, 11]. Moreover, the nematic planar phase is interesting due to the different types of topological defects that can result from the winding of the phase $\varphi \rightarrow \varphi + 2\pi$ or the combined operation of a half-winding of the phase $\varphi \rightarrow \varphi + \pi$ and an inversion of the spinor $\hat{n} \rightarrow -\hat{n}$ that leave $\Psi_{MF}$ unchanged [12–14], which have recently been observed in $^{23}$Na [15].

With the latest development of artificial gauge fields, it is now possible to couple the internal spin states of the atom to their momentum using counter-propagating Raman lasers, which induces an effective spin orbit coupling (SOC) [16]. SOC’ed quantum gases can now be

FIG. 1. a) Phase diagram in the plane spanned by effective quadratic Zeeman field $\epsilon = q + \Theta^2/(2m)$ and spin-spin interaction $g_2$. For explanations on the two nematic phases and the spin liquid see the main text. The non-universal position $g_c$ of the BKT transition is marked by a star. b) Difference of the only non-zero nematicity tensor components $N_{zz} - N_{yy}$. Note that it is odd in $\epsilon$ and $N_{yy} + N_{zz} \sim 1$. The characteristic power law is non-universal $\epsilon^{|\epsilon|/(2K_s-1)}$. For $g_2 \leq g_c$, and linear for $g_2 > g_c$. c) The $m_z = 0$ component of the BEC wave function (the order parameter) scales as $\epsilon(1/\epsilon)^{1/8}$ for $g_2 > g_c$. 

\[ \langle \Psi_0 \rangle \]
realized in spinor bosons [17–21] or spinful fermions [22] with either a one or two dimensional SOC [23–26]. In bosonic gases this gives rise to “striped” superfluids [27–36] that condense at the degenerate momenta dictated by the spin orbit wave vector. While the phase diagram is now reasonably well understood, SOC’ed, polar, spin-1 gases offer an exciting platform to study the competition between different types of nematic order, and hold great promise for intriguing forms of quantum criticality. A majority of the theoretical [28, 32–35, 37, 38] and experimental [17–21] work has focused on quantum phase transitions (QPTs) that are driven by the strength of the Raman field and are accessible in both pseudospin-1/2 and spin-1 bosons. Interestingly, for polar spin-1 bosons, the phenomena and nematic QPTs that can be evoked by SOC goes beyond transverse field induced transitions, and remains largely unexplored apart from mean field (MF) [35, 39] and variational solutions [33, 36, 40]. Our work aims to fill this gap by developing a field theory description of nematic QPTs.

One major difficulty in theoretically capturing the interplay between non-perturbative topological defects, SOC, and nematic order is that it requires a strong coupling solution beyond any MF like description. Thus one of the most felicitous realms to study SOC’ed polar spinor bosons are one-dimensional (1D) systems, which represents a common setup for ultra cold atom experiments. This is due to the existence of strong analytical tools that allow for asymptotically exact low-energy solutions that take into account both the inherent strong coupling nature of 1D and topological defects [41–43]. The effective field theory of polar spin-1 bosons in the absence of a SOC is described by a spin-charge separated Lagrangian, the charge is described by a gapless Luttinger liquid (LL) and the spin sector is given by a 1D non-linear sigma model (NLσM) [41, 43]. A SOC directly couples the spin and charge degrees of freedom and therefore it is in no way obvious if spin-charge separation can still persist in SOC’ed gases.

**Summary of results and experimental predictions.** We consider a gas of 1D polar spinor bosons in the presence of a SOC (wave vector Θ) and a linear (quadratic) Zeeman field hp (g). We treat the strength of background fields perturbatively and derive the effective low energy field theory that describes a LL coupled to a NLσM in the presence of anisotropic mass terms. We solve this effective theory in the low energy limit and determine the phase diagram of the model, see Fig. 1. We uncover three distinct superfluid phases: at weak coupling, two different nematic phases depending on the sign of the effective quadratic Zeeman field ε = q + Θ2/(2m) and a spin liquid phase at strong coupling. Furthermore, we determine the nature of the QPTs between those phases, all of which are continuous. The critical state between the two nematic phases at weak coupling is a pair of spin-charge separated Luttinger liquids. In contrast, the transition from either nematic phase to the spin liquid is in the 1+1D Ising universality class with an exotic, emergent Lorentz symmetry characterized by equal, logarithmically divergent velocities in the spin and charge sector. Interestingly, a very similar QPT was discussed in the physically unrelated context of Cooper pairing near Lifshitz transitions and in topological superconductors [44–46]. Finally, Ising and LL QPT lines meet at a Berezinskii-Kosterlitz-Thouless (BKT) critical point.

The hallmarks of our theory are as follows: (i) The described phases and fluctuation induced continuous QPTs. We emphasize, that mean field (MF) and variational theories predict a first order transition at ε = 0 and miss the spin liquid phase completely. (ii) The order parameter of the QPTs are the spin components of the condensate wave function, see Fig. 1 c). (iii) An experimentally accessible observable is the nematic tensor N_{ab} = Δ_{ab} − {S_a, S_b}/2, see Fig. 1 b). We predict a characteristic power law behavior of N_{xy}, N_{z2} with non-universal exponents. This emblematic feature of LL physics is out of reach of MF theory. For parameters in typical ultra-cold atom experiments with quasi-1D tubes of atoms at nano-Kelvin temperatures we estimate K_x ∼ O(10), and a system size and thermal length which exceed the correlation length [49]. Thus, these power-laws should be experimentally detectable. (iv) The effect of SOC is twofold: First, the condensate wave function in the nematic ε < 0 phase is heavily modulating in space. Second, SOC strongly affects the position of QPTs. However, somewhat strikingly, the universal critical behaviors are independent of the SOC. (v) Finally, the emergent Lorentz symmetry at the Ising transitions is, at least in principle, accessible via separate measurement of excitation spectra in charge and spin sectors [47, 48]. In the remainder we present the theoretical framework leading to these results and predictions.

**Model:** Continuum spin-1 bosons with mass m that are perturbed by a background helical magnetization and a constant linear Zeeman field h_0(x) = h(cos(Θx), −sin(Θx), p)^T as well as a quadratic Zeeman coupling q can be described by the normal ordered Hamiltonian density $H = \frac{1}{2} \Psi^\dagger \hat{S} \Psi + \frac{1}{2} \hat{\Phi}^\dagger \hat{H}(x) \cdot \hat{\Phi}$. We write

$$\begin{align}
H_2 &= g_0 \frac{1}{2} : (\Psi^\dagger \Psi)^2 : + g_2 \frac{1}{2} : (\Phi^\dagger \Phi)^2 : \\
H_4 &= (g_0 + g_2)/2 : (\Psi^\dagger \Phi)^2 : - g_2/2 : (\Phi^\dagger \Psi)^2 :.
\end{align}$$

We analyze the polar case g_0 > g_2 > 0 (g_0 ∼ 32 g_2 in $^{23}$Na [5]) in the semiclassical limit in which the condensate density $n_0 = \mu/\epsilon_0$ parametrically exceeds the inverse coherence length $1/\xi = \sqrt{2m\mu}$. Here, $\mu$ is the chemical potential and we set $\hbar = k_B = 1$ throughout.

The bosonic field operators $\Psi, \Phi$ are three-spinors and in the remainder we choose the adjoint representation of SU(2) as a basis of spin-1 operators $(S_a)_{bc} = -i \epsilon_{abc}$ with $a,b,c \in \{x,y,z\}$. The quartic term can be recast into the form $H_4 = (g_0 + g_2)/2 : (\Psi^\dagger \Phi)^2 : - g_2/2 :$
so that the $[U(1) \times O(3)]/\mathbb{Z}_2$ symmetry of the unperturbed action becomes manifest. Eq. (1) describes the quantum fluid in the lab frame, the frame co-rotating with the Raman field, can be accessed by $\Psi \rightarrow e^{i\hat{b}S^z} \Psi$. In this frame, Eq. (1) retains its structure, except for $\hbar \to \hbar / (1,0,0)$ and $\partial_x \to \partial_x + i\Theta S_z$ (this yields $q \to \epsilon = q + \Theta^2/(2m)$).

In order to solve Eq. (1) in its low-energy sector, we perform a sequence of coarse graining steps which are motivated by the assumption of the hierarchy of length scales presented in Fig. 2. The meaning of each of those scales will be explained at the appropriate position of the main text. Since the dispersion relation of collective modes is linear, see Eq. (2), below, the conversion to equivalent time (energy) scales follows trivially.

**Effective low-energy theory.** As a first step towards the asymptotic solution of Eq. (1) we derive the effective long-wavelength Matsubara field theory [41, 43], for details see Ref. [49]. It is convenient to choose an Euler angle parameterization $\hat{\Psi} = e^{-i\hat{b}S^z} \hat{\Psi}_{M} e^{i\alpha S^z} \hat{\Psi}$, with $\lambda_i$ being Gell-Mann matrices. This representation separates the Goldstone modes $e^{i\theta} O = e^{i\theta^{\dagger} \lambda_i} e^{i\alpha S^z} \hat{\Psi}$ living on the manifold $[U(1) \times O(3)]/\mathbb{Z}_2$ from the massive longitudinal modes $\alpha_4$ and $\alpha_6$ from the outset. This representation of the complex unit vector $\hat{\Psi} / \sqrt{\rho}$ provides a regular Jacobian leading to the NLoM measure for the Goldstone field $\hat{n} \equiv O \hat{e}_z \in \mathbb{S}^2$. While constant $\vartheta$ and $O$ fields are zero modes of $\hat{H} - \hat{H}_s$, Eqs. (1a),(1b) ensure that the longitudinal modes take the saddle point values $\rho_M = \rho_0 - \hat{q} \& n_0 S^z \hat{n} / \rho_0$, $\alpha_{4,M} = -i \hat{e}_z O \hat{b} \cdot S \hat{O} \hat{e}_z / [2 \rho_0 g_2]$ and $\alpha_{6,M} = -i \hat{e}_z O \hat{b} \cdot S \hat{O} \hat{e}_y / [2 \rho_0 g_2]$, which are perturbative in $\rho_0 g_0$ and non-perturbative in $\hat{q}$. Fluctuations around the saddle point $\Delta \rho$ (\(\Delta \alpha_{4,6}\)) decay on the length scale $\xi_c (\xi_s = \sqrt{\rho_0 / g_2} \xi_c)$.

To access the physics at longer scales, we perform the Gaussian integration of massive modes assuming that $\vartheta$ and $\partial_x \hat{\Psi}$ are slow. We switch to the co-rotating frame and obtain the effective low-energy Lagrangian $\hat{L} = \hat{L}_0 + \hat{L}_1 + \hat{L}_2,

\begin{align*}
\hat{L}_0 &= \Delta_c \hat{n} S_z^2 \hat{n} - \Delta_h \hat{n} (S_x + p S_y^2) \hat{n}, \\
\hat{L}_1 &= -i \hat{\partial} \lambda_i \hat{n} S_z^2 \hat{n} + \lambda_\lambda \hat{n} S_x \hat{n} + i \lambda_\lambda \hat{n} S_z \hat{n}, \\
\hat{L}_2 &= \frac{K_c}{2 \pi v_c} \left[ |\hat{\theta}|^2 + v_c^2 |\hat{\theta}|^2 \right] + \frac{K_s}{2 \pi v_s} \left[ |\hat{n}|^2 + v_s^2 |\hat{n}|^2 \right].
\end{align*}

The kinetic part of the action, Eq. (2c), which we denote as $\hat{L}_2 = \hat{L}_{\text{LL}}[\hat{\theta}] + \hat{L}_{\text{LLoM}}[\hat{n}]$, contains bare coupling constants $K_{c,s} = \sqrt{2 \pi \rho_0 \varepsilon_0} \xi_c,s$ and velocities $v_c = \sqrt{\rho_0 g_0 / m}$ and $v_s = \sqrt{\rho_0 g_2 / m}$. We omitted anisotropic corrections to kinetic terms due to $\vartheta$, $\Theta$ and $\hat{q}$, because they are small and will renormalize to zero quickly. In addition to the known kinetic term $\hat{L}_2$, Eq. (2) contains symmetry breaking terms with no derivatives $\Delta_c = \rho_0 \varepsilon_c, \Delta_h = 2 \rho_0 g_2$ and one derivative $\lambda_\lambda = \varepsilon_0 g_0, \lambda_\lambda = \lambda_\lambda g_2, \lambda_\lambda = \Theta / \rho_0 m$, which are the focus of this letter. In Ref. [49] we treat a weak trapping frequency $\omega_t \leq m g_0$ via the replacement $\rho_0 \rightarrow \rho_0 / (1 - x^2 / x_{\text{trap}}^2)$. We find that this introduces the largest finite length scale $t_{\text{trap}} = \sqrt{2 / \rho_0 m x_{\text{trap}}^2}$ into the problem, which is less restrictive then the presence of finite temperature $(T = v_s / T)$, and their combined effect rounds out the observable critical properties (see Fig. 2).

**Characterization of phases.** We begin the asymptotic solution of Eq. (2) by determining all phases and their characteristics, see Fig. 1 a). Groundstates which are also accessible to variational [33, 36, 40] and MF [35, 39] treatments follow from the consideration of the potential term $\Delta_c S_x^2 = \Delta_h (S_x + p S_y^2)$ which independently of $p$ predicts a first order transition at $\epsilon = 0$ [49]. For $p = 0$ it has eigenvalues $\Delta_c, -\Delta_c, -\Delta_h$ with eigenstates $\hat{e}_z, \hat{e}_y, \hat{e}_x$, respectively (for $p \neq 0$ see [49]). At finite $\hat{q}$, the groundstate at $\epsilon = 0$ ($\epsilon < 0$) is $\hat{\Psi}_M \approx \sqrt{\rho_0} e^{i \vartheta} \hat{e}_z + \hat{h}_y / (2 \rho_0 g_2)$ ($\hat{\Psi}_M \approx \sqrt{\rho_0} e^{i \vartheta} \hat{e}_z + \hat{h}_y / (2 \rho_0 g_2)$), where the $\hat{h}_y$ corrections stem from $\alpha_{4,6}$. This state is denoted $\hat{U}_\perp$ (UN$_\parallel$ + XY spiral) because at MF level it displays uniaxial nematic order ($\langle N_{zz} \rangle = \rho_0 + O(\hat{q}^2)$) ($\langle N_{yy} \rangle = \rho_0 + O(\hat{q}^2)$). Both states show weak magnetization ($\langle S_x \rangle = -h / g_2$). In the lab frame the magnetization follows the helical magnetic field and for $\epsilon < 0$ there is a strong modulation of the superfluid wavefunction because bosons condense at finite momentum $k = \Theta$ producing a stripe superfluid [49]. MF theory predicts a first order transition at $\epsilon = 0$: the ground state in the spin sector becomes degenerate and the order parameter $\langle N_{ab} \rangle$ changes discontinuously. Finally, there is a third phase in which the spin sector is quantum disordered, i.e. a spin liquid [43]. This occurs when $K_s \rightarrow 0$, a scenario that is not captured by the bare parameters entering Eq. (2) but can be reached upon RG transformations.

**Characterization of phase transitions.** Having identified the three phases of the problem, we now characterize the nature of the QPTs between them. We first discuss the RG flow close to the repulsive fixed point $K_s = \infty$ at small $\Delta_c, \Delta_h, \lambda_\lambda, \lambda_\lambda$. It is well known that $d K_s / d \bar{b} = -1 / 2 + O(1 / K_s, \Delta_c, \Delta_h, \lambda_\lambda, \lambda_\lambda)$. As usual, $b$ denotes the running logarithmic scale. The unperturbed weak coupling theory suggests that the spin liquid is approached at the length scale $\xi_{SL} \sim \xi_s \exp(\sqrt{2} \rho_0 \xi_s)$. However, the scaling dimensions of $\Delta_c, \Delta_h, \lambda_\lambda$ and $\lambda_\lambda, \lambda_\lambda$ are $[2 - 3 / (2 K_s)], [1 - 3 / (2 K_s)]$, and $[1 - 1 / (2 K_s)]$, i.e. RG
relevant at weak coupling. We define the length scales $\xi_{a,b}$ self consistently as the scale when the couplings $\Delta_{a,b}(b)$ hit the running scale, by assumption $\xi_{a,b} < \xi_{c}$. Beyond $\xi_{a,b}$ the NLrM field is locked to the easy plane $\tilde{n} = (0, \sin(\phi), \cos(\phi))^T$ perpendicular to the background magnetization realizing a spin-flop like phase of itinerant polar bosons. Following Fig. 2 a sine Gordon theory emerges. The coupling to the charge Luttinger liquid is characterized by $\mathcal{L}_{EP} = \mathcal{L}_{LL}[\eta] + \mathcal{L}_F$.

$$\mathcal{L}_F = \frac{K_s}{2\pi v_s} \left[ (\phi)^2 + \frac{1}{2} \sin^2(\phi) \right] + \sqrt{\Delta} - i\tilde{v} \lambda \sin(\phi). \quad (3)$$

All coupling constants in Eq. (3) are evaluated at the scale $\xi_{a,b}$ and we absorbed a factor of $1/(1 + p^2)$ into $\Delta_e, \lambda_e$. Note that, while $K_s$ is large only if $\xi_{a,b} \ll \xi_{SL}$ and may be renormalized to values of the order of unity or even smaller otherwise. In terms of Eq. (3), the phase UN$_{\parallel}$ (UN$_{\parallel}$ + XY spiral) is characterized by $\langle \phi \rangle = 0 \mod \pi$ ($\langle \phi \rangle = \pi/2 \mod \pi$).

The fields entering Eq. (3) allow for various topological defects: 2$\pi$ phase slips in $\tilde{v}$ and $\phi$ fields as well as $\pi$ phase slips in $\tilde{v}$ accompanied with a $\pm \pi$ phase slip in $\phi$ [12]. The scaling dimensions [13, 49, 50] of the associated fugacities (Boltzmann weights) are $(2 - K_c), (2 - K_s)$ and $[2 - (K_e + K_s)/4]$, respectively. Therefore, in the given parameter regime ($K_c \gg 1$), only the fugacity $y$ of $2\pi$ phase slips in the spin field $\phi$ may be relevant. We incorporate the associated operator into Eq. (3) and derive [49] the weak coupling RG equations to second order in $\lambda_e, \Delta_e y$ and to zeroth order in $1/K_e$ extending the previously reported [51] results to the case of finite $\lambda_e$:

$$\frac{d\Delta_e}{db} = (2 - 1/K_s)\Delta_e, \quad \frac{dy}{db} = (2 - K_s)y, \quad \frac{dK_s}{db} = \Delta_e^2 - K_s^2 y^2, \quad \frac{d\lambda_e}{db} = (1 - 1/K_s)\lambda_e, \quad \frac{d(K_e v_c)}{db} = \lambda_s^2 K_e v_c, \quad \frac{dK_e}{db} = \frac{v_s}{K_e v_c}.$$ (4)

Regularization dependent factors were absorbed into a redefinition of $\lambda_e, \Delta_e y$. Figure 3 a) displays the RG flow in the plane $(\Delta_e y, K_e)$ and illustrates that (i) the MF first order transition at $\epsilon = 0$ for $K_s \geq 2$ is actually continuous and described by a line of spin-charge separated LL critical points with enhanced symmetry, (ii) the phase transition to the spin disordered phase is BKT at $\epsilon = 0$, and (iii) the quantum critical point at $\epsilon \neq 0$ occurs at $K_e = 1$, but at strong coupling $\Delta_e y \rightarrow \infty$. At this fixed point, the spin charge coupling $\lambda_e$, which is relevant (irrelevant) for $K_e > 1$ ($K_e < 1$), becomes marginal. To determine the relevance of $\lambda_e$ and the nature of the strong coupling phase transition, Eq. (3) is fermionized [49, 52] on the $K_s = 1$ hyperplane leading to $\mathcal{L}_{EP,K_s=1} = \mathcal{L}_{LL}[\eta] + \mathcal{L}_F$.

$$\mathcal{L}_F = \frac{1}{2}\eta^T \left[ \partial_{\tau} + v_s \tilde{p} \sigma_z + (M_e + i\lambda \tilde{v}) \sigma_y K_e + M_e \sigma_y \right] \eta. \quad (5)$$

The Majorana four spinor $\eta$ is subject to masses $M_e \sim \Delta_e \xi_e, M_c \sim y \xi_c$. Pauli matrices in left-right (Nambu) space are denoted $\sigma_a (\kappa_a)$. At $\lambda = 0$, two Ising transitions occur at $M_c = \pm M_{c0}$, corresponding to the turquoise discs in Fig. 3 a). The effective theory, Eq. (5), at the critical point corresponds to a single gapless Majorana mode coupled to a gapless boson by a Lorentz symmetry breaking term. This effective theory is related to the problem studied in Refs. [44–46] by means of a Lorentz boost $(v_s x, v_t \eta, v_c y) \rightarrow (x - v_s \eta, v_t \eta)$ and an analytical continuation $\lambda \rightarrow i\lambda$. In that case, an attractive weak coupling fixed point $\lambda \rightarrow 0$ with emergent Lorentz symmetry and vanishing velocity $v_c = v_t \rightarrow 0$ was uncovered along with a putative phase separated region at strong coupling. Returning to our theory, it is useful to present the one-loop RG equations in terms of $G = |\lambda|/\sqrt{K_c}, u = v_c/v_s, \tilde{v} = \sqrt{v_c v_s}$.

$$\frac{dG}{db} = \frac{ug^3}{8} \frac{(1 - u)^2}{(1 + u)^2}, \quad \frac{du}{db} = -\frac{u^2 g^2}{4} \frac{1}{(1 + u)^2}, \quad \frac{dv}{db} = 2u v g^2 \frac{10 u - u^2 - 1}{(1 + u)^2}, \quad \frac{dK_e}{db} = \frac{ug^2}{4} K_c. \quad (6)$$

The mass has scaling dimension 1 + $u g^2 (u + 1)/2(1 + u)^2$. Due to the imaginary coupling in our model, the flow is reversed as compared to Refs. [44–46], hence $\tilde{v}$ increases near $u = 1$. The first two RG equations in Eq. (6) decouple and are plotted in Fig. 3, b). The assumption $g_0 > g_2$ implies starting values $v_c > v_s$, therefore the effective theory (5) resides in the basin of attraction of the weak coupling fixed point $(\lambda, v_c/v_s) = (0, 1)$. By consequence the critical theory separating the spin disordered from the nematic phases at finite $|\epsilon|$ is a theory with central charge $c = 3/2$, emergent Lorentz symmetry $v_c = v_s$, and logarithmically divergent velocity.

This concludes the derivation of the quantum critical theories. The zero temperature scaling of the order parameter and nematic tensor, Fig. 1, is weakly rounded at
finite temperature in the center of a harmonic trapping potential and obtained via a semiclassical evaluation using renormalized coupling constants [49]. In particular, the semiclassically expected first order jump is washed out by the strong quantum fluctuations at ϵ = 0 which corroborates the significance of the quantum field theoretical analysis. It will be interesting to study the predicted QPT numerically using the density matrix renormalization group to solve the SOC spin-1 Bose-Hubbard model [34]. Despite the SOC removing any spin conserving quantum numbers [37], we expect a numerical solution remains tractable in the superfluid regime provided that the truncation of the bosonic Hilbert space is treated carefully [38].

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[49] Supplementary Materials to this letter.