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Divisibility and information flow notions of quantum Markovianity for non-invertible dynamical maps

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We analyze the relation between CP-divisibility and the lack of information backflow for an arbitrary – not necessarily invertible – dynamical map. It is well known that CP-divisibility always implies lack of information backflow. Moreover, these two notions are equivalent for invertible maps. In this letter it is shown that for a map which is not invertible the lack of information backflow always implies the existence of completely positive (CP) propagator which, however, needs not be trace-preserving. Interestingly, for a *wide class of image non-increasing dynamical maps* this propagator becomes trace-preserving as well and hence the lack of information backflow implies CP-divisibility. This result sheds new light into the structure of the time-local generators giving rise to CP-divisible evolutions. We show that if the map is not invertible then positivity of dissipation/decoherence rates is no longer necessary for CP-divisibility.

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Introduction.— Recently, the notion of non-Markovian quantum evolution received considerable attention (see review papers [1–4]). Quantum systems interacting with an environment [5, 6] are of increasing relevance due to rapidly developing modern quantum technologies like quantum communication or quantum computation [7]. It turns out that recent experimental techniques allow us to go beyond standard Markovian approximations and observe new memory effects caused by the environmental interaction [8–10]. Therefore, there is a need for the exploration of non-Markovian regime, and characterizing the genuine properties of this kind of evolution.

To this end, two main approaches which turned out to be very influential are based on the concept of CP-divisibility [11] and information flow [12]. A dynamical map $\{\Lambda_t\}_{t\geq 0}$ is a family of completely positive (CP) and trace-preserving (TP) maps acting on the space $\mathcal{B}(\mathcal{H})$ of bounded operators on the Hilbert space $\mathcal{H}$. In the present Letter we say that $\{\Lambda_t\}_{t\geq 0}$ is *divisible* if

$$\Lambda_t = V_{t,s}\Lambda_s, \tag{1}$$

where $V_{t,s} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is a linear map for every $t \geq s$. Note that since Λ_t is TP the map $V_{t,s}$ is necessarily TP on the range of Λ_s but needs not be trace-preserving on the entire $\mathcal{B}(\mathcal{H})$. However, if $V_{t,s}$ is TP on $\mathcal{B}(\mathcal{H})$, one calls $\{\Lambda_t\}_{t\geq 0}$ *P-divisible* if $V_{t,s}$ is also a positive map on the entire $\mathcal{B}(\mathcal{H})$, and *CP-divisible* if $V_{t,s}$ is CP on the entire $\mathcal{B}(\mathcal{H})$ [13]. According to [11] the evolution is considered Markovian iff the corresponding dynamical map $\{\Lambda_t\}_{t\geq 0}$ is CP-divisible. This definition is motivated by its classical limit, which is compatible with a classical Markovian process, and because such Markovian evolution can be represented as the continuous limit of sequence of discrete interactions with a memoryless environment [1, 14].

A second idea is based on a physical feature of the system-reservoir interaction. It is claimed [12] that the phenomenon of reservoir memory effects may be associated with an information backflow, that is, for any pair of density operators ρ_1 and ρ_2 one can define the information flow

$$\sigma(\rho_1, \rho_2; t) = \frac{d}{dt}||\Lambda_t\rho_1 - \Lambda_t\rho_2||_1, \tag{2}$$

where $||A||_1$ denotes the trace norm of A. Following [12] Markovian evolution is characterized by $\sigma(\rho_1, \rho_2; t) \leq 0$. Whenever $\sigma(\rho_1, \rho_2; t) > 0$ one calls it information backflow meaning that the information flows from the environment back to the system. In this case the evolution displays nontrivial memory effects and it is evidently non-Markovian.

Interestingly, both P- and CP-divisible maps have a clear mathematical characterization [15].

Theorem 1 *Let us assume that $\{\Lambda_t\}_{t\geq 0}$ is an invertible dynamical map, i.e. Λ_t^{-1} does exist for any $t \geq 0$. Then $\{\Lambda_t\}_{t\geq 0}$ is P-divisible iff*

$$\frac{d}{dt}||\Lambda_t X||_1 \leq 0, \tag{3}$$

for any Hermitian $X \in \mathcal{B}(\mathcal{H})$. It is CP-divisible iff

$$\frac{d}{dt}||(\mathbb{1} \otimes \Lambda_t)X||_1 \leq 0, \tag{4}$$

for any Hermitian $X \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$.

Actually, CP- (or P-) divisibility implies (4) [or (3)] for an arbitrary map. Invertibility is only essential to prove the opposite implication. Let us observe that the condition

$\sigma(\rho_1, \rho_2; t) \leq 0$ is a slightly weaker version of (3): one takes $X = \rho_1 - \rho_2$ which means that X is Hermitian but traceless. In [15] two of us proposed how to reconcile P-divisibility with information flow by noticing that any Hermitian operator X can be interpreted (up to some multiplicative constant) as a so-called Helstrom matrix [16] $X = p_1\rho_1 - p_2\rho_2$ with $p_1 + p_2 = 1$. It characterizes the error probability of discriminating between states ρ_1 or ρ_2 with prior probabilities p_1 and p_2, respectively [15].

The relation between divisibility and information flow was recently reconsidered by Bylicka *et al.* [17]. They proved

Theorem 2 *Let* $\{\Lambda_t\}_{t\geq 0}$ *be an invertible dynamical map, then it is CP-divisible iff*

$$\frac{d}{dt}||(\mathbb{1}_{d+1} \otimes \Lambda_t)(\rho_1 - \rho_2)||_1 \leq 0, \tag{5}$$

for any pair of density operators ρ_1, ρ_2 *in* $\mathcal{B}(\mathcal{H}' \otimes \mathcal{H})$ *with* $\dim(\mathcal{H}') - 1 = \dim(\mathcal{H}) = d$.

Again, invertibility is only essential to prove that (5) implies CP-divisibility. So comparing (4) with (5) one enlarges the dimension of the ancilla $d \to d + 1$, but uses only equal probabilities $p_1 = p_2$, like in the original approach to the information flow [12].

If $t = 0$ is the starting time for the system-environment interaction, any open system dynamics can be written as $\Lambda_t\rho = \mathrm{Tr}_E[U(t, 0)\rho \otimes \omega_E U^\dagger(t, 0)]$ where ω_E is a fixed state of the environment. According to the postulates of quantum mechanics $U(t, s)$ is a unitary evolution family which satisfies the Schrödinger Equation, and so it is continuous and differentiable. Since the partial trace is continuous but non-invertible, a dynamical map $\{\Lambda_t\}_{t\geq 0}$ is a continuous, differentiable family (in the parameter t), but not necessarily invertible.

Interestingly, Buscemi and Datta [18] analyzed information backflow defined in terms of the guessing probability of discriminating an ensemble of states $\{\rho_i\}$ ($i = 1, 2, \ldots$) acting on $\mathcal{H} \otimes \mathcal{H}$ with prior probabilities p_i. It was shown [18] that a discrete time evolution is CP-divisible iff the guessing probability decreases for any ensemble of states. In this approach invertibility of the maps plays no role and hence this approach is universal. However, the price one pays, is the use of ensembles containing arbitrary number of states ρ_i. Moreover, since just a discrete evolution Λ_n is considered, there is not direct relation to the problem with continuous dynamical maps. For example such maps do not satisfy time-local master equations. Anyway, [18] poses an important question *whether the assumption of invertibility in Theorem 1 may be removed.*

In this Letter we show how to generalize Theorem 1 and 2 to non-invertible dynamical maps. This result sheds new light into time-local master equations

$$\frac{d}{dt}\Lambda_t = \mathcal{L}_t\Lambda_t \ , \ \Lambda_{t=0} = \mathbb{1}. \tag{6}$$

One usually says that the corresponding solution $\{\Lambda_t\}_{t\geq 0}$ is CP-divisible if the two-point propagator

$$V_{t,s} = \mathcal{T}e^{\int_s^t \mathcal{L}_\tau d\tau}, \tag{7}$$

is CPTP for any $t \geq s$, and hence one concludes that $\mathcal{L}_t$ is a time-dependent GKLS generator [19]. However, it turns out to be true only for invertible dynamics. In this Letter we show that if $\{\Lambda_t\}_{t\geq 0}$ is not invertible, it can still be CP-divisible even if the corresponding generator $\mathcal{L}_t$ does not have GKLS structure.

Divisible maps.— Interestingly, the property of divisibility is fully characterized by the following

Proposition 1 *A dynamical map* $\{\Lambda_t\}_{t\geq 0}$ *is divisible iff*

$$\mathrm{Ker}(\Lambda_t) \supseteq \mathrm{Ker}(\Lambda_s), \tag{8}$$

for any $t > s$.

Proof: If $\{\Lambda_t\}_{t\geq 0}$ is divisible and $X \in \mathrm{Ker}(\Lambda_s)$, then

$$\Lambda_t X = V_{t,s}(\Lambda_s X) = V_{t,s}0 = 0,$$

and hence $X \in \mathrm{Ker}(\Lambda_t)$.

Suppose now that (8) is satisfied. To show that $\{\Lambda_t\}_{t\geq 0}$ is divisible we provide a construction for $V_{t,s}$. This construction is highly non-unique: if $Y \in \mathrm{Im}(\Lambda_s)$, i.e. there exists X such $\Lambda_s X = Y$, we define $V_{t,s}Y = \Lambda_t X$. Suppose now that $Y \notin \mathrm{Im}(\Lambda_s)$ and let $\Pi_s : \mathcal{B}(\mathcal{H}) \to \mathrm{Im}(\Lambda_s)$ be a (Hermiticity preserving) projector onto $\mathrm{Im}(\Lambda_s)$ [14], that is, $\Pi_s\Pi_s = \Pi_s$ is an identity operation on $\mathrm{Im}(\Lambda_s)$. Define

$$V_{t,s}Y := \Lambda_t X, \tag{9}$$

where X is an arbitrary element such that $\Pi_s Y = \Lambda_s X$. It only remains to prove that this is a well-defined construction. Indeed, if $\Lambda_s X = \Lambda_s X' = \Pi_s Y$, then our construction implies $\Lambda_t X = \Lambda_t X'$ for $t > s$. Specifically, $\Delta = X - X' \in \mathrm{Ker}(\Lambda_s)$ and hence due to (8) one has $\Delta \in \mathrm{Ker}(\Lambda_t)$ which implies $\Lambda_t\Delta = \Lambda_t X - \Lambda_t X' = 0$. It should be stressed, however, that $V_{t,s}$ needs not be TP due to the fact that the projector Π_s needs not be TP. □

Note that if $\{\Lambda_t\}_{t\geq 0}$ is invertible, then it is always divisible due to $V_{t,s} = \Lambda_t\Lambda_s^{-1}$. In this case condition (8) is trivially satisfied: $\mathrm{Ker}(\Lambda_t) = \mathrm{Ker}(\Lambda_s) = 0$.

Actually, there is a simple sufficient condition for divisibility

Proposition 2 *If the dynamical map* $\{\Lambda_t\}_{t\geq 0}$ *satisfies condition* (3) *for all Hermitian* $X \in \mathcal{B}(\mathcal{H})$, *then it is divisible.*

Proof: Suppose that (3) is satisfied but $\{\Lambda_t\}_{t\geq 0}$ is not divisible, that is, there exists X such that $\Lambda_s X = 0$ but

$\Lambda_t X \neq 0$ $(t > s)$. This shows $||\Lambda_t X||_1 > 0 = ||\Lambda_s X||_1$ and hence $||\Lambda_t X||_1$ does not monotonically decrease.

□

Clearly, the above condition is sufficient but not necessary, since any invertible $\{\Lambda_t\}_{t\geq 0}$ is divisible even if it does not satisfy (3).

Arbitrary dynamical maps.— Now we prove the central result which provides generalization of Theorems 1 and 2 for arbitrary, that is, not necessarily invertible, dynamical maps. Let us start with two lemmas.

Lemma 1 *Let M be a linear subspace in $\mathcal{B}(\mathcal{H})$, and consider a trace-preserving linear map $\Phi : M \to \mathcal{B}(\mathcal{H})$. If Φ is a contraction in the trace norm, then it is positive.*

Proof: take arbitrary $X \geq 0$ from M. One has $||X||_1 = \mathrm{Tr}(X)$. Now, since Φ is trace-preserving $\mathrm{Tr}(X) = \mathrm{Tr}[\Phi(X)] \leq \mathrm{Tr}|\Phi(X)| = ||\Phi(X)||_1$. Finally, since Φ is a contraction $||\Phi(X)||_1 \leq ||X||_1$ and hence it implies $||\Phi(X)||_1 = \mathrm{Tr}[\Phi(X)]$, which proves that $\Phi(X) \geq 0$. □

Note, that if $M = \mathcal{B}(\mathcal{H})$ then one recovers the well known result [20, 21] used in [15] and recently in [17].

Lemma 2 *Let M be a linear subspace in $\mathcal{B}(\mathcal{H})$ with $\dim(\mathcal{H}) = d$. If M is spanned by positive operators (density matrices), then a d-positive map $\Phi : M \to \mathcal{B}(\mathcal{H})$ can be extended to a CP map $\widetilde{\Phi} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$.*

The problem of CP extensions of a CP map $\Phi : M \to \mathcal{B}(\mathcal{H})$ is well-studied in the theory of operator algebras and was solved by Arveson [24] when M defines an operator system (see also [22, 23]). Recently the extension problem was studied in the context of quantum operations in [25, 26] beyond operator systems. In particular, Jencova proves Lemma 2 in [25]. Nevertheless, for the sake of completennes we include a explicit proof in the supplementary material [14].

Theorem 3 *If a dynamical map $\{\Lambda_t\}_{t\geq 0}$ satisfies condition (4) for any Hermitian $X \in \mathcal{B}(\mathcal{H}\otimes\mathcal{H})$, then it is divisible with CP propagators $V_{t,s}$.*

Proof: By Proposition 2 the dynamical map $\{\mathbb{1}\otimes\Lambda_t\}_{t\geq 0}$ is divisible, hence so is $\{\Lambda_t\}_{t\geq 0}$, therefore $\Lambda_t = V_{t,s}\Lambda_s$. If the map Λ_s is not invertible, the propagator $V_{t,s}$ is not uniquely defined. We show that one can find $V_{t,s}$ which is CP. Note, that (4) implies that $\mathbb{1}\otimes V_{t,s}$ is a contraction on the image of $\mathbb{1}\otimes\Lambda_s$ [15, 17]. Since $\mathbb{1}\otimes V_{t,s}$ is trace-preserving on $\mathrm{Im}(\mathbb{1}\otimes\Lambda_s)$, Lemma 1 implies that $\mathbb{1}\otimes V_{t,s}$ is positive on $\mathrm{Im}(\mathbb{1}\otimes\Lambda_s)$ or equivalently that $V_{t,s}$ is d-positive on $\mathrm{Im}(\Lambda_s)$. It should be stressed, that $V_{t,s}$ is defined on the linear subspace $\mathrm{Im}(\Lambda_s) \subset \mathcal{B}(\mathcal{H})$. Now, the question is about the extension of $V_{t,s}$ to the whole operator space $\mathcal{B}(\mathcal{H})$. However, since the subspace $\mathrm{Im}(\Lambda_s)$ is spanned by the positive operators $\Lambda_s(X)$, where X are positive operators from $\mathcal{B}(\mathcal{H})$, Lemma 2 guaranties the existence of a CP extension $\widetilde{V}_{t,s} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$. One has, therefore,

$$\Lambda_t = V_{t,s}\Lambda_s = \widetilde{V}_{t,s}\Lambda_s. \tag{10}$$

□

Clearly, if Λ_s is invertible then $\widetilde{V}_{t,s} = V_{t,s}$. It should be stressed, however, that generically $\widetilde{V}_{t,s}$ *needs not be trace-preserving.* It is always trace-preserving on $\mathrm{Im}(\Lambda_s)$. Hence, monotonicity property (4) does not imply CP-divisibility but a slightly weaker property. Examples of CP extensions which are not trace-preserving were recently provided in [26].

Image non-increasing dynamical maps.— Consider now a wide class of dynamical maps which satisfy

$$\mathrm{Im}(\Lambda_t) \subseteq \mathrm{Im}(\Lambda_s), \quad t > s. \tag{11}$$

We shall refer to these as "image non-increasing dynamical maps". Note that "kernel non-decreasing" Eq. (8) (equivalent to divisibility) only implies $\dim[\mathrm{Im}(\Lambda_t)] \leq \dim[\mathrm{Im}(\Lambda_s)]$. Leaving aside invertible maps, we can easily identify two natural instances of maps satisfying (11). The first one are normal divisible maps, i. e. $\Lambda_t\Lambda_t^\dagger = \Lambda_t^\dagger\Lambda_t$, where $\Lambda_t^\dagger$ is the dual map (Heisenberg picture), that is, $\mathrm{Tr}[\Lambda_t^\dagger(X)\rho] = \mathrm{Tr}[X\Lambda_t(\rho)]$. For normal maps the kernel is orthogonal to the image, so divisibility implies (8) and hence (11) immediately follows. The second instance are diagonalizable commutative maps (here commutative means $\Lambda_t\Lambda_s = \Lambda_s\Lambda_t$ for arbitrary t and s). In this case Λ_t is characterized by the diagonal representation $\Lambda_t\rho = \sum_\alpha \lambda_\alpha(t) F_\alpha \mathrm{Tr}(G_\alpha^\dagger\rho)$, with time independent damping basis [27] $\{F_\alpha, G_\beta\}$ such that $\mathrm{Tr}(F_\alpha^\dagger G_\beta) = \delta_{\alpha\beta}$ $(\alpha, \beta = 0, 1, \ldots, d^2 - 1)$.

Theorem 4 *If the image non-increasing dynamical map $\{\Lambda_t\}_{t\geq 0}$ satisfies condition (4) for any Hermitian $X \in \mathcal{B}(\mathcal{H}\otimes\mathcal{H})$, then it is CP-divisible.*

Proof: clearly (4) implies (Theorem 3) that $\{\Lambda_t\}_{t\geq 0}$ is divisible with $V_{t,s}$ which is CPTP on $\mathrm{Im}(\Lambda_s)$. Since $\Lambda_{t=0} = \mathbb{1}$, continuity implies that there exists some small ϵ such that Λ_ϵ is invertible. Let us take t_1 the smallest time instant where the dynamics becomes non-invertible, i. e. $\{\Lambda_t\}_{t_1 > t\geq 0}$ is invertible. Then we can write $\Lambda_{t_1} = V_{t_1,t_1-\epsilon}\Lambda_{t_1-\epsilon}$, where $V_{t_1,t_1-\epsilon}$ is CPTP [on the entire $\mathcal{B}(\mathcal{H})$] for $\epsilon \in (0, t_1)$. Consider now the operator

$$\Pi_{t_1} := \lim_{\epsilon\to 0^+} V_{t_1,t_1-\epsilon}. \tag{12}$$

It turns out that Π_{t_1} is a CPTP projection onto $\mathrm{Im}(\Lambda_{t_1})$. We provide a detailed proof of this in the supplementary material [14]. Hence $\widetilde{V}_{t,t_1} = V_{t,t_1}\Pi_{t_1}$ is CPTP on the entire $\mathcal{B}(\mathcal{H})$. Consider now the smallest $t_2 > t_1$ such that $\dim[\mathrm{Im}(\Lambda_{t_2})] < \dim[\mathrm{Im}(\Lambda_{t_1})]$ and $\dim[\mathrm{Im}(\Lambda_t)] = \dim[\mathrm{Im}(\Lambda_{t_1})]$ for $t_1 \leq t < t_2$. For image non-increasing dynamical maps it means that $\mathrm{Im}(\Lambda_{t_2}) \subset \mathrm{Im}(\Lambda_{t_1})$, and $\mathrm{Im}(\Lambda_t) = \mathrm{Im}(\Lambda_{t_1})$ for $t_1 \leq t < t_2$. Then considering $\{V_{t,s}\}_{t_2>t>s\geq t_1}$ as a bijective family of maps on the space $\mathrm{Im}(\Lambda_{t_1})$, the same argument as before, with the role of Λ_t played now by V_{t,t_1}, applies to show that Π_{t_2} is a CPTP on $\mathrm{Im}(\Lambda_{t_1})$, which projects onto $\mathrm{Im}(\Lambda_{t_2})$. Finally, let $\{t_1, \ldots, t_k\}$ be a set such that $\dim[\mathrm{Im}(\Lambda_t)]$ is discontinuous, that is,

$$\dim[\mathrm{Im}(\Lambda_{t_1})] > \dim[\mathrm{Im}(\Lambda_{t_2})] > \ldots > \dim[\mathrm{Im}(\Lambda_{t_k})].$$

Note, that for $t \in [t_i, t_{i+1})$ one has

$$\mathrm{Im}(\Lambda_{t_i}) = \mathrm{Im}(\Lambda_t) \supsetneq \mathrm{Im}(\Lambda_{t_{i+1}}). \tag{13}$$

Hence, for $s \in [t_i, t_{i+1})$ one defines

$$\widetilde{V}_{t,s} = V_{t,s}\Pi_{t_i} \ldots \Pi_{t_1}, \tag{14}$$

which is CPTP on the entire $\mathcal{B}(\mathcal{H})$. □

Note that a parallel argument applies to show the equivalence between Eq. (3) and P-divisibility in the case of image non-increasing maps.

Theorem 5 *: A dynamical map $\{\Lambda_t\}_{t\geq 0}$ satisfying condition (5) for any pair of density operators ρ_1, ρ_2 in $\mathcal{B}(\mathcal{H}' \otimes \mathcal{H})$ with $\dim(\mathcal{H}') - 1 = \dim(\mathcal{H}) = d$ is divisible with CP propagators $V_{t,s}$. In addition, if the map is image non-increasing, it is CP-divisible.*

The proof of this theorem follows from Theorems 3 and 4, and a similar argument as in [17]. We leave it as a supplementary material [14].

CP-divisibility vs. master equation – Any differentiable Λ_t satisfies a time-local master equation of the form of (6), so that $V_{t,s} = \mathcal{T}e^{\int_s^t \mathcal{L}_\tau d\tau}$. Then CP-divisibility implies that $V_{s,s}$ is a CPTP identity map on some subspace M, such that $\mathrm{Im}(\Lambda_s) \subseteq M \subseteq \mathcal{B}(\mathcal{H})$. Moreover, if $\{\mathbb{1} \otimes \Lambda_t\}_{t\geq 0}$ is image non-increasing and contracting, there exists a CPTP projector Π_s onto $\mathrm{Im}(\Lambda_s)$.

Corollary 1 *If the image non-increasing dynamical map $\{\Lambda_t\}_{t\geq 0}$ satisfies a time-local master equation (6), then it is CP-divisible iff $\mathbb{1} \otimes \mathcal{T}e^{\int_s^t \mathcal{L}_\tau d\tau}$ is a TP contraction on $\mathcal{B}(\mathcal{H}) \otimes \mathrm{Im}(\Lambda_s)$ for all pairs $t \geq s$.*

In the following examples we will show that this does not require a time dependent GKLS form for all times (another example can be found in [14]).

Example 1 (Amplitude damping channel) *The dynamics of a single amplitude-damped qubit is governed by a single function $G(t)$ which depends on the form of the reservoir spectral density $J(\omega)$ [5]:*

$$\Lambda_t\rho = \begin{pmatrix} |G(t)|^2\rho_{11} & G(t)\rho_{12} \\ G^*(t)\rho_{21} & (1-|G(t)|^2)\rho_{11} + \rho_{22} \end{pmatrix}, \tag{15}$$

This evolution is generated by the following time-local generator

$$\mathcal{L}_t\rho = -\frac{is(t)}{2}[\sigma_+\sigma_-, \rho] + \gamma(t)(\sigma_-\rho\sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-, \rho\}),$$

where $\sigma_\pm$ are the spin lowering and rising operators together with $s(t) = -2\mathrm{Im}\frac{\dot{G}(t)}{G(t)}$, and $\gamma(t) = -2\mathrm{Re}\frac{\dot{G}(t)}{G(t)}$. This generator is commutative and diagonalizable. Now, the dynamical map is invertible whenever $G(t) \neq 0$. Suppose now that $G(t_*) = 0$ and $G(t) \neq 0$ for $t < t_*$ (note that $G(0) = 1$). The image of Λ_{t_*} is just proportional to the ground state $P_0 = \sigma_-\sigma_+$, and so a CPTP projector onto $\mathrm{Im}(\Lambda_{t_*})$ reads

$$\Pi_{t_*}X = P_0\mathrm{Tr}X. \tag{16}$$

It is, therefore clear that $\{\Lambda_t\}_{t\geq 0}$ is divisible iff $G(t) = 0$ for $t \geq t_*$. Hence, finally, the map $\{\Lambda_t\}_{t\geq 0}$ is CP-divisible iff it is divisible and $\gamma(t) \geq 0$ for $t < t_*$. Note that $\gamma(t)$ is arbitrary for $t \geq t_*$. The only constraint is $G(t) = 0$: $\gamma(t)$ blows up to $+\infty$ at $t = t_*$, and then is arbitrary provided $\int_0^t \gamma(\tau)d\tau = \infty$ for all $t \geq t_*$. Hence, positivity of $\gamma(t)$ is sufficient but not necessary for CP-divisibility. It is necessary only if $\gamma(t)$ is finite for finite times, that is, the generator $\mathcal{L}_t$ is regular and the map Λ_t is invertible. Note that divisibility means that if the system relaxed to the ground state (at time t_*) it stays in that state forever. In addition, CP-divisibility means that the relaxation to the ground state was monotonic $\frac{d}{dt}|G(t)| \leq 0$.

Example 2 (Random unitary evolution) *Consider the qubit evolution governed by the following time-local generator*

$$\mathcal{L}_t\rho = \frac{1}{2}\sum_{k=1}^{3}\gamma_k(t)(\sigma_k\rho\sigma_k - \rho), \tag{17}$$

which leads to the unital dynamical map (time-dependent Pauli channel): $\Lambda_t\rho = \sum_{\alpha=0}^3 p_\alpha(t)\sigma_\alpha\rho\sigma_\alpha$. The map is invertible if its corresponding eigenvalues

$$\lambda_i(t) = e^{-\Gamma_j(t)-\Gamma_k(t)} \;\; ; \;\; \Gamma_j = \int_0^t \gamma_j(\tau)d\tau$$

where $\{i, j, k\}$ is a permutation of $\{1, 2, 3\}$, are different from zero (note, that the remaining eigenvalue $\lambda_0(t) = 1$). Now, if for example $\Gamma_3(t_1) = \infty$ at finite time t_1, then $\lambda_1(t_1) = \lambda_2(t_1) = 0$, and hence divisibility implies $\lambda_1(t) = \lambda_2(t) = 0$ for $t \geq t_1$. One finds the corresponding CPTP projector

$$\Pi_{t_1}X = \frac{1}{2}(X + \sigma_3X\sigma_3).$$

Note that $\Pi_{t_1}\sigma_1 = \Pi_{t_1}\sigma_2 = 0$. Now, if at $t_2 > t_1$ one has $\Gamma_2(t_2) = \infty$ (or equivalently $\Gamma_1(t_2) = \infty$), then $\lambda_3(t_2)$

vanishes as well and hence divisibility implies $\lambda_i(t) = 0$ for $t \geq t_2$ ($i = 1, 2, 3$). One finds the corresponding CPTP projector

$$\Pi_{t_2} X = \frac{1}{2}\mathbb{I}\,\mathrm{Tr}(X),$$

that is, it fully depolarizes an arbitrary state of the system. To summarize: the evolution is CP-divisible iff all $\gamma_\alpha(t) \geq 0$ for $t < t_1$, and $\gamma_3(t)$ continues to be nonegative up to t_2. From t_2 on the system stays at the maximally mixed state.

Conclusions. — In this Letter we analyzed the relation between monotonicity of the trace norm (4) and CP-divisibility of the dynamical map $\{\Lambda_t\}_{t\geq 0}$. While CP-divisibility always implies (4), it is well known that for invertible maps the converse is also true, that is, these two notions are equivalent. For maps which are not invertible the situation is much more subtle (as was recently noticed in [17]). We proved that in this case, Eq. (4) implies a slightly weaker property — there exists a family of completely positive maps $V_{t,s}$ on $\mathcal{B}(\mathcal{H})$ which are trace-preserving on the image of Λ_s [but not on the entire $\mathcal{B}(\mathcal{H})$]. Interestingly, for maps which are image non-increasing trace-preservation is guarantied on $\mathcal{B}(\mathcal{H})$ and hence they are CP-divisible. This result sheds new light into the structure of the time-local generator $\mathcal{L}_t$ which gives rise to CP-divisible evolution. For invertible maps, $\mathcal{L}_t$ has a structure of time-dependent GKLS generator, in particular all dissipation rates $\gamma_k(t) \geq 0$ for all $t \geq 0$ [28]. It is no longer true for dynamical maps which are not invertible, that is, they correspond to singular generators [29]. In this case $\gamma_k(t) \geq 0$ but only for $t \in [0, t_*)$, where t_* is the first moment of time where Λ_t becomes non-invertible. For $t \geq t_*$ some $\gamma_k(t)$ might be temporally negative, and still the evolution might be CP-divisible. The point t_* at which some $\gamma_k(t)$ becomes singular, provides an analog of the event horizon: the future behavior of a set of $\gamma_k(t)$ does not effect the evolution of the system. A typical example is the evolution reaching equilibrium state at finite time t_*. Then the system stays at equilibrium forever irrespective of the future ($t > t_*$) time dependence of the generator.

Finally, we note that the relation between the result by Buscemi and Datta [18] on guessing probabilities for discrete evolution Λ_n and our results on continuous evolution is not evident and deserves further analysis. On the other hand, the general problem of finding a CPTP extension of a CPTP propagator $V_{t,s}$ on a subspace remains open. If possible, it would ensure the complete equivalence of CP-divisibility and complete contractivity.

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