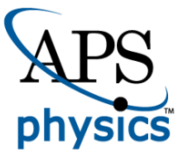

Fundamental Limitations for Measurements in Quantum Many-Body Systems

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Fundamental limitations for measurements in quantum many-body systems

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Dynamical measurement schemes are an important tool for the investigation of quantum many-body systems, especially in the age of quantum simulation. Here, we address the question whether generic measurements can be implemented efficiently if we have access to a certain set of experimentally realizable measurements and can extend it through time evolution. For the latter, two scenarios are considered (a) evolution according to unitary circuits and (b) evolution due to Hamiltonians that we can control in a time-dependent fashion. We find that the time needed to realize a certain measurement to a predefined accuracy scales in general exponentially with the system size – posing a fundamental limitation. The argument is based, on the construction of ε-packings for manifolds of observables with identical spectra and a comparison of their cardinalities to those of ε-coverings for quantum circuits and unitary time-evolution operators. The former is related to the study of Grassmann manifolds.

Introduction. – In experiments with quantum many-body systems, we usually have direct access only to a relatively small set of standard observables in measurements. For quantum computation devices, these are often Pauli measurements. In ion-trap systems, state-dependent laser-induced resonance fluorescence allows for the measurement of qubits in the computational basis [1–3]. For superconducting qubits, such projective measurements can be realized through a state-dependent shift in the resonance frequency of a dispersively coupled cavity [4, 5]. Also projection operators onto specific multi-qubit product states have been measured [6]. For ultracold atoms, the particle density can be accessed through absorption imaging [7] and more recently developed quantum gas microscopes with single-site resolution based on fluorescence imaging [8–10].

Dynamical control can be used to measure observables that are not directly accessible. This is especially important for the purpose of quantum simulation [11–14]. The design of quantum simulators is advancing rapidly [15–20]. The relevant observables for the simulated systems will often not be directly accessible in the simulating device and hence require dynamical measurement schemes. While the investigation of general abilities and limitations of such schemes has just begun, several particular incarnations are successfully used in experiments:

Measurement of Pauli-$\hat\sigma_x$ and $\hat\sigma_y$ for ion-trap qubits are realized through the application of single-qubit gates and subsequent measurement of $\hat\sigma_z$. More elaborate schemes employ two-qubit gates, spin echo, spatial shuttling of qubits or hiding in non-computational electronic states, e.g., to do Bell-state measurements [21, 22]. Similarly, for superconducting circuits, Bell-state measurements can be realized [23] through application of single-qubit rotations and controlled phase gates [24] before the standard Pauli measurements. In ultracold atom experiments, the momentum distribution is obtained in time-of-flight measurements by letting the quantum gas expand freely before absorption imaging [7, 25, 26]. Double-occupancies can be determined by rapid ramping of the lattice potential, tuning of interaction strengths, mapping double occupancy to a previously unpopulated spin state using radio-frequency pulses, and final absorption imaging [27, 28]. Nearest-neighbor correlations have been measured through an additional modulation of the lattice depth or deformation of a superlattice [29–31]. Bloch band populations can be examined by adiabatic band mapping [32–34]. Solid-state materials are studied with various scattering and microscopy techniques. The control over the Hamiltonian is naturally rather limited in this case. Nevertheless, pump-probe schemes are, for example, employed in time-resolved optical and photoemission spectroscopy [35–37], scanning tunneling microscopy [38, 39], and electron microscopy [40–42] to enlarge the set of accessible observables.

In principle, arbitrary observables can be evaluated after state tomography [43, 44] or compressed sensing procedures [45]. However, for many-body systems, the number of required measurements and classical computational resources grow exponentially with increasing system size unless additional strong constraints can be leveraged [46, 47].

Here, we assess the efficiency of dynamical measurement schemes by derivation of lower bounds on covering numbers for manifolds of observables with identical spectra and by comparing them to upper bounds on covering numbers for quantum circuits and unitary time-evolution operators. Note that ε-covering numbers $\mathcal{N}(\varepsilon)$ and ε-packing numbers $\bar{\mathcal{N}}(\varepsilon)$ of a metric space are closely related with $\bar{\mathcal{N}}(2\varepsilon) \leq \mathcal{N}(\varepsilon) \leq \bar{\mathcal{N}}(\varepsilon)$ [48, 49] (Fig. 1). The analysis shows that the time needed to realize a certain measurement to a predefined accuracy scales in general exponentially with the system size. The result holds for the spectra of all typical observables of many-body systems. In the following, we consider lattice systems consisting of L d-dimensional qudits. We use the Bachmann-Landau symbols $\mathcal{O}$ and Ω for upper and lower bounds. Variants of lemmas 1 and 3 have been stated by Szarek in Refs. [50, 51]. Proofs for the lemmas and theorem 3 are given in the Supplemental Material.

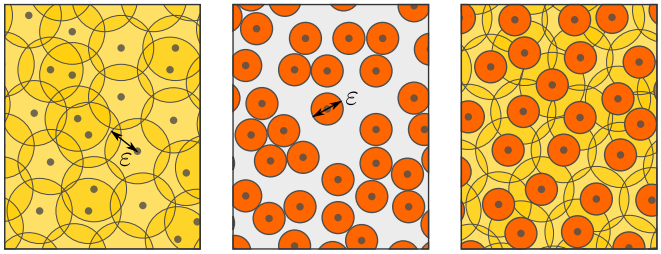

FIG. 1. Left: An *ε-covering* $\mathcal{Q}$ for a space $\mathcal{M}$ with metric d is a subset of $\mathcal{M}$ such that for every $z \in \mathcal{M}$ there is an $x \in \mathcal{Q}$ with $d(x,z) \leq \varepsilon$. The cardinality $\mathcal{N}(\mathcal{M},d,\varepsilon)$ of the smallest ε-covering is called the *ε-covering number* of $(\mathcal{M},d)$. We are comparing covering numbers for sets of evolved observables $\hat{U}^\dagger\hat{O}\hat{U}$ with covering numbers for the set of all observables with the same spectrum as $\hat{O}$. Center: An *ε-packing* $\mathcal{Q}$ is a subset of $\mathcal{M}$ such that all $x \neq y \in \mathcal{Q}$ have distance $d(x,y) > \varepsilon$. Right: An *ε-net* is an ε-covering and, at the same time, an ε-packing.

Evolution due to unitary quantum circuits. – First, let us consider the case of observables that are evolved using unitary circuits composed of arbitrary k-site gates.

Theorem 1. *Let $\hat{O} = \hat{O}^\dagger$ and consider quantum circuits of size $N_g > L$ with each gate $\hat{u}_i$ acting on at most k sites. The evolved observables $\{\hat{U}^\dagger\hat{O}\hat{U}\}$ with any such quantum circuits $\{\hat{U} = \prod_{i=1}^{N_g}\hat{u}_i\}$ are elements of*

$$\mathcal{N}_{circ} \leq L^{kN_g}\left(\frac{14N_g}{\varepsilon}\right)^{d^{2k}N_g} = e^{\mathcal{O}(N_g \ln N_g)} \tag{1}$$

balls of radius $\varepsilon w(\hat{O})$ in operator space. Here, $w(\hat{O}) := (\omega_{\max}-\omega_{\min})/2$ denotes the spectral width of $\hat{O}$, i.e., half the difference of the maximum and minimum eigenvalues of $\hat{O}$.

This can be shown by first bounding covering numbers for the quantum circuits $\hat{U}$. In similar situations, in Refs. [14, 52], the k-qudit gates $\hat{u}_i$ have been approximated by small circuits built from a finite gate library. This can be done as in practical implementations for quantum computation by first decomposing them into single-qubit and CNOT gates [53–55] and further approximating the latter according to the Solovay-Kitaev algorithm [56, 57] or alternative schemes [58]. However, one can take a more direct approach and simply employ an $\tilde{\varepsilon}$-covering for the k-qudit gates.

Lemma 1. *For $0 < \tilde{\varepsilon} \leq 1/10$, the $\tilde{\varepsilon}$-covering number for the unitary group $U(n)$ with respect to the operator-norm distance obeys*

$$\left(\frac{3}{4\tilde{\varepsilon}}\right)^{n^2} \leq \mathcal{N}\left(U(n),\|\cdot\|,\tilde{\varepsilon}\right) \leq \left(\frac{7}{\tilde{\varepsilon}}\right)^{n^2}. \tag{2}$$

We fix an $\tilde{\varepsilon}$-covering $\mathcal{Q}$ for the set $U(d^k)$ of all gates. For a circuit $\hat{U} = \prod_i \hat{u}_i$, let $\hat{U}_{\tilde{\varepsilon}}$ be the circuit where each of the N_g gates is replaced by the nearest element in $\mathcal{Q}$. Then, according to the triangle inequality, $\|\hat{U}_{\tilde{\varepsilon}} - \hat{U}\| \leq N_g\tilde{\varepsilon}$ and, choosing $\tilde{\varepsilon} = \varepsilon/(2N_g)$,

$$\|\hat{U}_{\tilde{\varepsilon}}^\dagger\hat{O}\hat{U}_{\tilde{\varepsilon}} - \hat{U}^\dagger\hat{O}\hat{U}\| \leq 2\|\hat{U}_{\tilde{\varepsilon}} - \hat{U}\|w(\hat{O}) \leq \varepsilon w(\hat{O}). \tag{3}$$

The upper bound in lemma 1 gives $|\mathcal{Q}| \leq (14N_g/\varepsilon)^{d^{2k}}$. With the bound L^{kN_g} on the number of possible circuit topologies and $|\mathcal{Q}|^{N_g}$ combinations for the gates in $\hat{U}_{\tilde{\varepsilon}}$, theorem 1 follows.

Evolving with time-dependent interactions. – Similarly, we can bound the volume of operators that is reachable by evolving $\hat{O}$ with respect to time-dependent Hamiltonians $\hat{H}(t)$.

Theorem 2. *For time-dependent Hamiltonians $\hat{H}(t) = \sum_{i=1}^{K}\hat{h}_i(t)$ with K terms, let interactions be k-local and norm-bounded, i.e., terms $\hat{h}_i(t)$ act on at most k sites and $|h| := \max_i \sup_{0\leq t\leq T}\|\hat{h}_i(t)\|/\hbar$ is finite. For every term $\hat{h}_i$ and all times t,s, let commutators $[\hat{h}_i(t),\hat{h}_j(s)]$ be nonzero for at most z terms $\hat{h}_j$. Observables $\{\hat{U}^\dagger(T)\hat{O}\hat{U}(T)\}$, evolved with such Hamiltonians $\{\hat{H}\}$ from $t=0$ to T, are elements of*

$$\mathcal{N}_T \leq L^{kK}\left(\frac{112T^2K^2z|h|^2}{\varepsilon^2}\right)^{4d^{2k}T^2K^2z|h|^2/\varepsilon} \tag{4a}$$

$$= L^{kK}e^{\mathcal{O}\left(T^2K^2z\ln(T^2K^2z)\right)} \tag{4b}$$

balls of radius $\varepsilon w(\hat{O})$ in operator space.

The number K of terms in the Hamiltonian is bounded by $\binom{L}{k} \leq L^k$ and we have assumed that the interaction graph (choice of k-site supports of interactions terms) is time-independent. The number of interaction graphs is hence bounded by L^{kK}. Also, z may be $\mathcal{O}(L^0)$ but can always be bounded by kL^{k-1} such that $\log\mathcal{N}_T$ is in any case polynomial in the system size L and time T. The decisive step for proving theorem 2 is a Trotter-Suzuki decomposition [59–61] of the time-evolution operator $\hat{U}(t)$ which obeys $i\hbar\partial_t\hat{U}(t) = \hat{H}(t)\hat{U}(t)$ and $\hat{U}(0) = \mathbb{1}$.

Lemma 2. *With the preconditions of theorem 2, the time-evolution operator can be approximated by the decomposition $\hat{U}^{\Delta t}(T) := \prod_{n=1}^{N_t}\prod_i \hat{u}_i(n)$ into N_t time steps of size $\Delta t = T/N_t$, where $\hat{u}_i(n)$ denotes the time-evolution operator from time $(n-1)\Delta t$ to $n\Delta t$, generated by $\hat{h}_i(t)$. The accuracy is*

$$\|\hat{U}^{\Delta t}(T) - \hat{U}(T)\| \leq \Delta t\, TKz|h|^2. \tag{5}$$

This can be shown following the derivation in Ref. [61] and applying the triangle inequality. Accuracy $\|\hat{U}^{\Delta t}(T) - \hat{U}(T)\| \leq \varepsilon/4$ is achieved for $N_t = 4T^2Kz|h|^2/\varepsilon$ time steps. Now we are basically back to the case of observables that are evolved by a quantum circuit and can proceed, as before, by approximating each

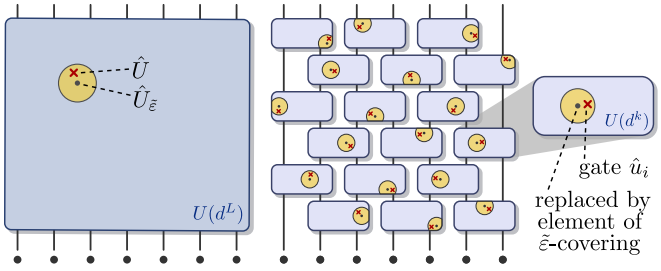

FIG. 2. Using a Trotter-Suzuki decomposition, the evolution with respect to k-local Hamiltonians with arbitrary time-dependence can be approximated by quantum circuits. Each k-site gate (here, $k = 2$) in a quantum circuit can be approximated by an element of an $\tilde{\varepsilon}$-covering for the unitary group $U(d^k)$. This construction allows us to bound covering numbers for sets of evolved observables $\{\hat{U}^\dagger \hat{O} \hat{U}\}$.

of the KN_t gates in $\hat{U}^{\Delta t}$ by the nearest element of an $\tilde{\varepsilon}$-covering $\mathcal{Q}$ for $U(d^k)$. Calling the resulting circuit $\hat{U}^{\Delta t}_{\tilde{\varepsilon}}$, we can achieve accuracy $\|\hat{U}^{\Delta t}_{\tilde{\varepsilon}}(T) - \hat{U}^{\Delta t}(T)\| \leq \varepsilon/4$ with $|\mathcal{Q}| \leq (28KN_t/\varepsilon)^{d^{2k}}$ according to lemma 1. Then,

$$\|[\hat{U}^{\Delta t}_{\tilde{\varepsilon}}(T)]^\dagger \hat{O} \hat{U}^{\Delta t}_{\tilde{\varepsilon}}(T) - \hat{U}^\dagger(T) \hat{O} \hat{U}(T)\| \leq \varepsilon w(\hat{O}) \quad (6)$$

and with $|\mathcal{Q}|^{KN_t}$ combinations for the gates in $\hat{U}^{\Delta t}_{\tilde{\varepsilon}}$, theorem 2 follows.

Efficiency for projective observables. – To quantify the efficiency with which dynamics explore the set of observables, let us first focus on the case where the accessible observable $\hat{O}$ is a projection operator. Examples for such observables are Pauli measurements that are the standard choice in quantum computing and particle densities that are typical for ultracold-atom experiments.

Let $G_{n,m}$ denote the set of all rank-n projection operators on an m-dimensional Hilbert space $\mathcal{H}$, where in our case $m = d^L$. We will bound covering numbers for $G_{n,m}$ and compare them to Eqs. (1) and (4). $G_{n,m}$ can be identified with the Grassmann manifold, the space of all n-dimensional subspaces of $\mathcal{H}$, where each such subspace corresponds to the projection onto that subspace. More useful for our purposes, $G_{n,m}$ can also be identified with the quotient group

$$G_{n,m} \cong U(m)/U(n,m), \quad (7)$$

where $U(n,m) := U(n) \times U(m-n)$ is the direct product of the unitary groups $U(n)$ and $U(m-n)$. Eq. (7) is due to the fact that every n-dimensional subspace $\tilde{\mathcal{H}}$ of $\mathcal{H}$ can be specified by a fixed reference subspace $\mathcal{H}_0$ of dimension n and an element $\hat{V}$ of $U(m)/U(n,m)$ such that

$$\hat{V}(\mathcal{H}_0 \oplus \mathcal{H}_0^\perp) = \tilde{\mathcal{H}} \oplus \tilde{\mathcal{H}}^\perp, \quad (8)$$

where $\mathcal{H}_0^\perp$ and $\tilde{\mathcal{H}}^\perp$ are the orthogonal complements of $\mathcal{H}_0$ and $\tilde{\mathcal{H}}$ in $\mathcal{H}$. Clearly, $\tilde{\mathcal{H}} \oplus \tilde{\mathcal{H}}^\perp$ is invariant under transformations from $U(n,m)$, which explains the identification (7).

Lemma 1 bounds covering numbers for unitary groups. These can be used to bound covering numbers for the product $U(n,m)$ and, finally, the quotient group $U(m)/U(n,m)$. We obtain

$$\frac{1}{19^{m^2}} \left(\frac{7}{\varepsilon}\right)^{2n(m-n)} \leq \mathcal{N}(G_{n,m}, d', \varepsilon) \leq 38^{m^2} \left(\frac{3}{8\varepsilon}\right)^{2n(m-n)}$$

for covering numbers of the Grassmannians (7) with $\varepsilon \leq 1/20$. In this case, the induced quotient metric is $d'(\mathcal{H}_1, \mathcal{H}_2) = \inf\{\|\hat{\mathbb{1}} - \hat{V}\| \,|\, \hat{V} \in U(m) \text{ with } \mathcal{H}_2 = \hat{V}\mathcal{H}_1\}$ for all $\mathcal{H}_1, \mathcal{H}_2 \in G_{n,m}$ [50]. However, we are actually interested in $G_{n,m}$, interpreted as the set of all rank-n projection operators on $\mathcal{H}$. Then the relevant metric is not d' but the operator norm distance $\|\hat{P}_1 - \hat{P}_2\|$, where $\hat{P}_1$ and $\hat{P}_2$ project onto $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. So, in the final step, we relate covering numbers for $(G_{n,m}, \|\cdot\|)$ to those of $(G_{n,m}, d')$ using the following lemma.

Lemma 3. *Let $(\mathcal{M}_1, d_1)$ and $(\mathcal{M}_2, d_2)$ be metric spaces and $f : \mathcal{M}_1 \to M_2$ bi-Lipschitz such that $f(\mathcal{M}_1) = \mathcal{M}_2$ with*

$$d_2(f(x), f(y)) \leq K d_1(x,y) \;\forall x, y \in \mathcal{M}_1 \text{ and}$$
$$d_2(f(x), f(y)) \geq k\, d_1(x,y) \;\forall x, y \in \mathcal{M}_1 \text{ with } d_1(x,y) \leq r.$$

Then, their covering numbers obey

$$\mathcal{N}(\mathcal{M}_1, d_1, 2\varepsilon/k) \leq \mathcal{N}(\mathcal{M}_2, d_2, \varepsilon) \leq \mathcal{N}(\mathcal{M}_1, d_1, \varepsilon/K),$$

where the left inequality requires $\varepsilon \leq kr/2$.

The two relevant metrics obey the inequalities $\sqrt{2}\, d'(\mathcal{H}_1, \mathcal{H}_2)/5 \leq \|\hat{P}_1 - \hat{P}_2\| \leq 2\, d'(\mathcal{H}_1, \mathcal{H}_2)$ for subspaces $\mathcal{H}_1, \mathcal{H}_2 \in G_{n,m}$ and the projections $\hat{P}_1$ and $\hat{P}_2$ onto these subspaces. Hence, we can apply lemma 3 to $G_{n,m}$ with d_1 and d_2 being the quotient metric and operator-norm distance, respectively, where $K = 2$, $k = \sqrt{2}/5$, and $r = 2/5$. We obtain

Theorem 3. *The ε-covering numbers $\mathcal{N}_G$ for rank-n projection operators on an m-dimensional Hilbert space with respect to the operator-norm distance $\|\cdot\|$ obey*

$$\frac{1}{19^{m^2}} \left(\frac{9}{5\varepsilon}\right)^{2n(m-n)} \leq \mathcal{N}_G \leq 38^{m^2} \left(\frac{3}{4\varepsilon}\right)^{2n(m-n)}, \quad (9)$$

where the lower bound is valid for $\varepsilon \leq 1/71$ and the upper one for $\varepsilon \leq 1/10$.

A complete proof is given in the Supplemental Material. The Hilbert space dimension of our many-body systems grows exponentially in the system size, $m = \dim \mathcal{H} = d^L$. For the case of interest, where n and $m - n$ are finite fractions of m, i.e., projection operators as those of Pauli measurements ($n = D/2$), and sufficiently small $\varepsilon = \mathcal{O}(1)$, theorem 3 states that covering numbers for $G_{n,m}$ grow *double-exponentially* with L, $\mathcal{N}_G = \exp\left[\Omega(m^2)\right] = \exp\left[\Omega(d^{2L})\right]$. In contrast, theorems 1 and 2 show that, even with full control over the

system dynamics, covering numbers for the set of evolved observables grow only *exponentially* as a function of the circuit depth N_g or evolution time T, respectively. Hence, generic projections can only be realized by implementing exponential-depth quantum circuits or evolving the system for a time T that grows exponentially with the system size L, i.e., most projections can not be reached with polynomial cost: If one evolves for a time T that is polynomial in the system size, the evolved observable necessarily ends up in one of $\mathcal{N}_T$ balls of radius ε in operator space [Eq. (4)] while we can pack an exponentially larger number $\mathcal{N}_G$ of disjoint ε-balls into the relevant operator space [Eq. (9)]. So most observables have a distance larger than ε from those that can be generated by polynomial-time dynamics.

Efficiency for generic observables. – So far, we have only considered observables being projection operators ($w(\hat{O}) = 1/2$) and found that measuring them is in general inefficient with respect to growing system size L. We can easily extend this result to observables $\hat{O}$ that have only two eigenvalues $\omega_1 < \omega_2$ with exponential degeneracies n and $m-n = e^{\Omega(L)}$. As long as the spectral width is polynomial in L, $w(\hat{O}) = \frac{\omega_2-\omega_1}{2} = \mathcal{O}(L^\alpha)$ for some constant $\alpha \geq 0$, theorems 1 and 2 with polynomial N_g and T still yield exponential upper bounds on ε-covering numbers for observables that can be reached through evolution of a predefined reference observable. And, as long as $\omega_2-\omega_1$ has an L-independent lower bound [is $\Omega(L^0)$], theorem 3 with sufficiently small ε ($\varepsilon \to \varepsilon/|\omega_2-\omega_1|$) yields double-exponential lower bounds on ε-covering numbers for the set of observables with the given spectrum.

In fact, we can generalize much further:

Theorem 4. *For a fixed $\alpha \geq 0$, sufficiently small $\varepsilon > 0$, and every system size L, let G_ω be the set of observables with some spectrum $\{\omega_k\}$ of polynomial width $w = \mathcal{O}(L^\alpha)$. For some $\bar{\omega}_1 < \bar{\omega}_2$ with $\bar{\omega}_2 - \bar{\omega}_1 = \Omega(L^0)$, let the $\varepsilon/2$-neighborhoods of $\bar{\omega}_1$ and $\bar{\omega}_2$ contain exponentially many eigenvalues ω_k, i.e., $\left|\{\omega_k \text{ with } |\omega_k - \bar{\omega}_i| \leq \varepsilon/2\}\right| = e^{\Omega(L)}$. Then ε-covering numbers for G_ω grow double-exponentially in L and, generally, elements of G_ω cannot be reached through application of polynomial-depth quantum circuits or polynomial-time evolution with Hamiltonians as characterized in theorems 1 and 2.*

This is because, for every observable $\hat{O} \in G_\omega$, we can define $\hat{O}'$ by replacing all eigenvalues in the $\varepsilon/2$-neighborhood of $\bar{\omega}_i$ by $\bar{\omega}_i$. So, eigenvalues $\bar{\omega}_1$ and $\bar{\omega}_2$ of $\hat{O}'$ have exponential degeneracies $n, m-n = e^{\Omega(L)}$. For sufficiently small ε, theorem 3 now yields double-exponential lower bounds on $\varepsilon/2$-covering numbers for the set of operators that differ from $\hat{O}'$ only in terms of the $\bar{\omega}_i$-eigenvectors. As $\|\hat{O}-\hat{O}'\| \leq \varepsilon/2$, it follows that ε-covering numbers for G_ω also grow double-exponentially in L.

Theorem 4 accounts for all typical classes of observables: (a) projection operators as, for example, occurring in Pauli measurements, (b) observables that act in a finite-size subspace like single-site observables $\hat{O}_i$ or two-site operators $\hat{O}_i\hat{O}_j$ for two-point correlation functions, (c) extensive observables like energy etc. As a matter of fact, observables $\hat{O}$ with a polynomial spectral width $w(\hat{O})$, usually obey the preconditions of theorem 4: Due to the exponential growth of the Hilbert space with L, the density of eigenstates for such observables grows exponentially in the bulk of the spectrum and, hence, points $\bar{\omega}_1$ and $\bar{\omega}_2$ with the required properties generally exist.

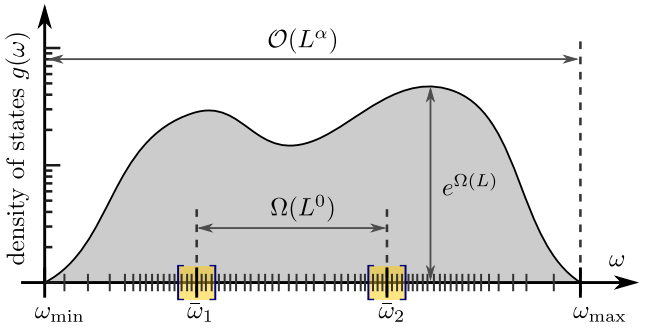

FIG. 3. The results on limitations for measurements of projection operators are generalized in Theorem 4. It covers observables with a spectral width that grows polynomially with increasing system size L, i.e., $w = \mathcal{O}(L^\alpha)$ with $\alpha \geq 0$. There necessarily exist points with an exponential density of states $g(\omega) = e^{\Omega(L)}$. Theorem 4 applies to observables which have two such points $\bar{\omega}_{1,2}$ with distance $\bar{\omega}_2 - \bar{\omega}_1 = \Omega(L^0)$.

Discussion. – We have found that dynamical measurement schemes for observables that are not directly accessible, i.e., a controlled time evolution and subsequent measurement of directly accessible observables, are in general inefficient. For a predefined accuracy the required evolution time generally increases exponentially with the system size. Quantitative estimates can be obtained by comparing Eqs. (1) and (4) with Eq. (9). So it is a question of clever design to allow for the measurement of observables of interest through efficient dynamical schemes and a suitable encoding of models in quantum simulation protocols. This is a fruitful field for future research.

A few examples concerning quantum information where, according to the presented results, it will be difficult to realize required measurements efficiently for many-body systems are the following. The goal of Helstrom measurements [62, 63] is to distinguish two quantum states $\hat{\varrho}_1$ and $\hat{\varrho}_2$. The optimal observable to be measured is the projection onto the support of the positive spectral component of $\hat{\varrho}_1 - \hat{\varrho}_2$. Another example is quantum data compression which can be implemented by measuring projection operators for so-called typical subspaces when the source distribution is known [64] or projection operators for Csiszár-Körner subspaces when the source is unknown [65]. Compression schemes with a cost that is polynomial in the number of copies have been devised [66, 67], but according to our results they will still be inefficient with respect to the system size, even with full control over system dynamics. Projections onto such typical subspaces are also employed for entan-

glement distillation and dilution [68]. Similarly, we are limited in the types of error correcting codes [69–71] and, in particular, error syndrome measurements that can be realized efficiently for purposes of fault-tolerant quantum computation. Hence, some classes of errors and decoherence effects (see, e.g., Refs. [72–76]) relevant for quantum computation are harder to correct for.

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