Model Predictions for Time-Resolved Transport Measurements Made near the Superfluid Critical Points of Cold Atoms and K_{3}C_{60} Films

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Transport signatures of transient superfluids

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Recent advances in ultra-fast measurement in cold atoms, as well as pump-probe spectroscopy of K_3C_{60} films, have opened the possibility of rapidly quenching systems of interacting fermions to, and across, a finite temperature superfluid transition. However determining that a transient state has approached a second-order critical point is difficult, as standard equilibrium techniques are inapplicable. We show that the approach to the superfluid critical point in a transient state may be detected via time-resolved transport measurements, such as the optical conductivity. We leverage the fact that quenching to the vicinity of the critical point produces a highly time dependent density of superfluid fluctuations, which affect the conductivity in two ways. Firstly by inelastic scattering between the fermions and the fluctuations, and secondly by direct conduction through the fluctuations, with the latter providing a lower resistance current carrying channel. The competition between these two effects leads to non-monotonic behavior in the time-resolved optical conductivity, providing a signature of the critical transient state.

Measurement techniques using ultra-fast optics have enabled the study of fermionic fluids on time scales far shorter than the timescale of thermalization [1–3]. Similarly, in ultra-cold atomic systems the relaxation times are long enough that pre-thermalization behavior may be studied [4–6]. In both cases experimental techniques presently exist to study the collective behavior of fermions in the period before they have relaxed to their equilibrium or steady state behavior. As an example, a transient state with superconducting-like optical properties was produced by ultra-fast laser stimulation of K_3C_{60} films [7].

However there is a difficulty in determining whether a transient state is in any sense related to any particular phase. More precisely one may ask whether a non-equilibrium system has been taken through the vicinity of a certain second-order phase transition. For example, whether a state with "superconducting-like" optical properties is in fact related to the equilibrium superconducting phase transition, or whether it is some other transient state. Equilibrium methods for detecting such phase transitions such as the specific heat are not applicable to ultra-fast or non-equilibrium settings.

We suggest that this difficulty may be resolved by looking for signatures of time-dependent fluctuations in the experimental data. Fluctuations increase in a singular fashion in the vicinity of a second-order phase transition, and therefore may be used as a signature that such a transition has been approached. In particular we give predictions for time-resolved transport measurements of fermions quenched close to the superfluid critical point. In the process we generalize theoretical treatments for equilibrium critical systems to the strongly non-equilibrium regime.

The experimental setup proposed is a gas of fermions, initially at equilibrium at finite temperature. This may be a gas of cold atoms in an optical lattice, or electrons in a thin solid state film. An attractive interaction is turned on at a certain rate. In the case of cold atoms this may be achieved by the tuning of optical resonances [8]. For a thin film, it may be achieved by strong optical pumping of a phonon mode [9–13]. An attractive interaction causes superfluid fluctuations to develop, leading to two possibilities. If these fluctuations are sufficiently strong, then a superfluid order will spontaneously develop, as happens in equilibrium beyond the critical interaction strength. The second possibility is that superfluid fluctuations are enhanced, see Fig. 1, but spontaneous long range order does not develop. Note that the interaction strength may be instantaneously supercritical, but the system remains disordered (*i.e.*, no long range order) if there is not sufficient time for the fluctuations to develop. In this paper we consider such disordered regimes.

In order to describe the fluctuation physics we must go beyond time-dependent mean field methods [14, 15]. We develop a quantum kinetic equation, derived within the two-particle irreducible (2PI) framework [16], which describes the joint evolution of the unbound fermions and the superfluid fluctuations. These equations predict the time-resolved transport properties of the interacting system. The theory is formally controlled by a parameter $1/N$ which represents the size of the fluctuations. Our predictions depend on only a small number of parameters, and do not depend on the precise form of the fermion-fermion interaction or on the details of the band structure. Our calculation is valid for a clean system and only if the fluctuations are not very strong, as given by a non-equilibrium equivalent of the Ginzburg-Levanyuk criterion [17] specified below.

The kinetic equations incorporate two scattering processes. The first is incoherent Andreev reflection of fermions off superfluid fluctuations. This leads to an aging effect where the time-resolved optical conductivity decreases after the quench with a characteristic power law. The second scattering process is the binding and unbinding of fermions into Cooper pairs, leading to the

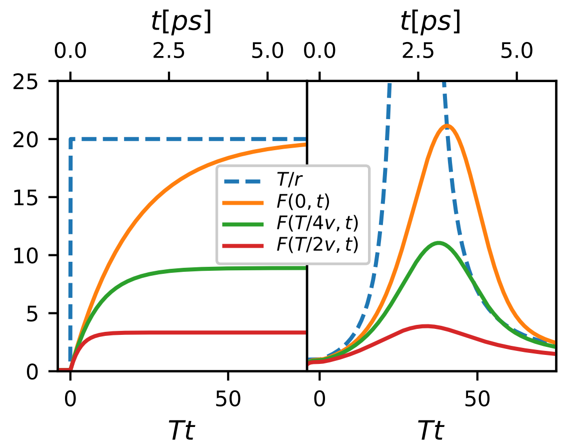

FIG. 1. Growth of superfluid fluctuations $F(q,t)$ following an interaction quench. Fluctuations at several different momenta q are shown. The times are measured in units of T^{-1} (lower axis) and in terms of ps (upper axis) for $T \sim 100\mathrm{K}$. The quantity T/r is the inverse of the detuning from the critical point, which at equilibrium is equal to $F(q=0)$. Left: hard quench from the normal state to $T/r = 20$. Right: Soft quench, with $r(t>0) = T[1-(t/t_*)e\exp(-t/t_*)]$, $t_*T = 30$.

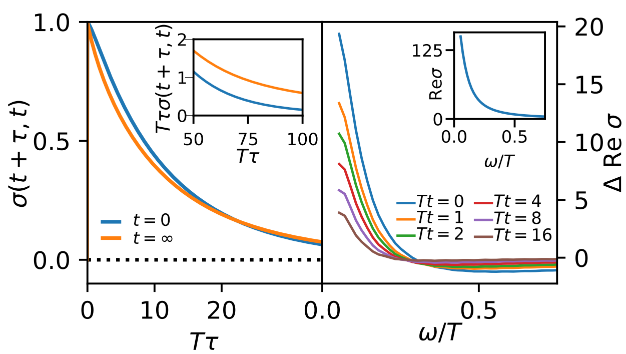

FIG. 2. Conductivity for the hard quench. Left: Conductivity $\sigma(t+\tau,t)$ as a function of $T\tau$ for $t=0$ (blue) and $t \to \infty$ (orange). The parameters are $T\tau_r = 5$, $\alpha = 0.5$ and detuning $r=0$. In the inset are the tails of $\sigma(t+\tau,t)$, $T\tau > 30$, plotted as $T\tau\sigma(t+\tau,t)$ to improve visibility. The curves asymptote to a constant as explained in the main text. Right: $\mathrm{Re}\,[\sigma(\omega,t) - \sigma(\omega, t\to\infty)]\,(e^2/\hbar)$ for different times since the quench. Note the times increase in a geometric fashion. Inset: $\mathrm{Re}\sigma(\omega, t\to\infty)$. This diverges as $\log\omega$ as $\omega \to 0$ so all curves are clipped at $\omega = .01T$

growth in fluctuations after the quench, and to a non-equilibrium Azlamazov-Larkin-like effect [17], where the Cooper-pair fluctuations serve as an additional low resistance channel for the current.

The kinetic equations describe the conductivity for arbitrary dependence of the interaction on the time. We give results for two different quench protocols, "hard" and "soft". In the hard quench (see left panel Fig. 1) the interaction is instantaneously changed to be in the vicinity of the equilibrium phase transition. We show that there are power-law corrections to the conductivity at times close to the quench.

We also discuss a soft quench where the interaction is smoothly switched on and off (see right panel Fig. 1), which is more applicable to K_3C_{60} films where the transient state survives for 2-10ps [7]. We find that the optical conductivity evolves non-monotonically in frequency, providing a strong signal for time resolved fluctuation effects and the existence of a transient lower resistance current carrying channel.

Model and 2PI Formalism

We consider a model with spinful fermions in a potential (optical or crystal) lattice, without disorder. The electrons interact via an on-site attractive interaction $U(t)$, which is allowed to vary with time. In equilibrium, at temperature T, the critical interaction strength $U_c(T)$ separates the disordered and superfluid phases.

We employ the 2PI [18] formalism, which generates equations of motion for two-operator correlations, such as the fermion Green's functions. The formalism takes a single generating functional $\Gamma[\cdot]$, which we take corresponding to the random phase approximation (RPA) justified by a large-N approximation [19]. The resulting equations are [20]

$$g^{-1}\circ G = \frac{i}{N}(D\cdot G)\circ G; \quad D^{-1}[G] \equiv U^{-1}(t) - \Pi[G], \quad (1)$$

where $\Pi[G] \equiv iG\cdot G$, $\circ$ implies convolution, g and G are the non-interacting and interacting Green's functions respectively, while D is the propagator of the superfluid fluctuations, with the equal time Keldysh [21] component representing the size of the fluctuations $F(q,t)$ (see Fig. 1).

The functionals D, Π depend self-consistently on G, thus Eq. (1) represents a highly non-linear set of equations. We consider an initial state of free fermions at a non-zero temperature T.

Fluctuations

In an equilibrium system of fermions, with an attractive interaction below the critical interaction, the expectation value $\langle\Delta(\vec{q}=0)\rangle$ vanishes, where $\Delta^\dagger(\vec{q}) \equiv \sum_k c^\dagger_k c^\dagger_{q-k}$ is the operator that creates a Cooper pair of momentum q, and $c^\dagger_k$ creates a single fermion. However, superfluid fluctuations $F_{\mathrm{eq}}(q)$ defined by $F_{\mathrm{eq}}(q) \propto \langle|\Delta^\dagger(\vec{q})|^2\rangle$ increase as the critical point is approached (setting $k_B = \hbar = 1$),

$$F_{\mathrm{eq}}(q) = \frac{T}{v^2|q|^2/T + r}; \quad vq \ll T, \quad r \ll T, \quad (2)$$

v being the average Fermi velocity, and the detuning $r \propto U_c(T)-U$. Thus the long wavelength fluctuations become pronounced as $r \to 0$.

The non-equilibrium dynamics are governed by two energy scales: the temperature T and the detuning from the critical point $r \propto U - U_c(T)$. However, for a system close to the critical point with $T \gg r$ the dynamics are essentially classical, as quantum fluctuations are averaged out on the time scale T^{-1}, but the dynamics occur on the scale r. Thus the quantum kinetic equation can be reduced to a joint kinetic equation for the evolution of the occupation numbers of the fermions and the distribution of fluctuations [20].

The dynamics for the fluctuations are,

$$\left(\partial_t + r(t) + \frac{v^2|q|^2}{T}\right) F(q,t) = T, \tag{3}$$

We emphasize that $F(q,t)$ is not given by the instantaneous value of $r(t)$, but rather by the full history of $r(t)$. This yields non-trivial dynamics. Note that if $vq \gtrsim T$, then the $F(q,t)$ will equilibrate on the short time $1/T$. Thus the non-trivial time dependence is concentrated in the modes $qv \ll T$ and it is sufficient to work with them. Eq. (3) is valid only when $F(q=0,t) \ll E_F/T$ [20]. This reduces to the Ginzburg-Levanyuk criterion $r \gg T^2/E_F$ in equilibrium.

Conductivity

We now discuss methods to detect the aging of superfluid fluctuations. A signature in photo-emission discussed elsewhere [19], showed that growing fluctuations lead to a decreasing fermion lifetime, given by a universal scaling form. Here we discuss the signatures in time-resolved transport experiments. The transport is studied by varying the 2PI action, or the quantum kinetic equations, in response to an external electric field [20].

We begin from the definition of the conductivity $J(t) = \int dt' \sigma(t,t') E(t')$, where all quantities are constant in space. There is no single notion of a Fourier transform in a non time-translation invariant setting, but for the purposes of demonstration, we study the behavior of the transform $\sigma(\omega,t) = \int d\tau\, \sigma(t+\tau,t) \exp(-i\omega\tau)$. Note that $\sigma(\omega,t)$ depends on the state at times $> t$, especially at small ω. In order to quantify the departure from simple ohmic behavior, we plot the frequency dependent relaxation time $\tau_{\mathrm{Dr}}(\omega) = -\mathrm{Im}\,[\sigma] \,/\, [\omega \mathrm{Re}\sigma]$. This is a frequency independent constant when the conductivity is governed by the Drude law $\partial_t J(t) = -\tau_{\mathrm{Dr}}^{-1} J(t)$.

In a fluid without a lattice potential or disorder, Galilean invariance and conservation of momentum forces $\sigma(t,t') \propto \theta(t-t')$, so that the current never relaxes. However the combination of a lattice potential and interaction causes a fraction of the current to relax, even without disorder or Umpklapp scattering. The dynamics of the momentum decouple from the relaxing current J [20] which obeys the following equation in d spatial dimensions [20],

$$\partial_t J(t) - \frac{\rho}{\bar{m}} E(t) = -\frac{1}{\tau_r}\Bigg(A(t) J(t) - \alpha \int_0^t dt' \left[B(t,t') J(t') + \frac{\rho}{2\bar{m}} C(t,t') E(t') \right] \Bigg), \tag{4}$$

$$A(t) \equiv T^{-\frac{d}{2}} \int_0^t ds\, \frac{e^{-\int_s^t du\, r(u)}}{(t-s)^{d/2+1}},\ C(t,s) \equiv \int_0^s ds' B(t,s'), \tag{5}$$

$$B(t,s) \equiv T^{-\frac{d}{2}} \int_0^s dt' \frac{\left(1+\frac{d}{2}\right) + r(t)(t-t')}{(t-t')^{d/2+2}} e^{-\int_{t'}^t du\, r(u)}. \tag{6}$$

Above ρ is the density of fermions, $\bar{m}$ is the average effective mass, and α, τ_r are material dependent parameters related to the extent to which Galilean invariance is broken. Without loss of generality we set $\rho/\bar{m} = 1$.

These equations are the central result of this paper. They may be solved numerically for an arbitrary $r(t)$ and electric field, giving the conductivity $\sigma(t,t')$. We spend the remainder of the paper analyzing the implications of these equations for $d=2$.

The equation for the relaxational current J in Eq. (4) has two components. The $A(t)$ term represents incoherent small angle scattering of fermions off of superfluid fluctuations. The second term, proportional to α, is a "memory" term, which is conceptually similar to the Azlamazov-Larkin term studied in the theory of equilibrium fluctuation superconductivity. It can be understood as coming from the binding of fermions into long-lived fluctuating Cooper pairs, which decay at time $\gg T^{-1}$. This term is non-local in time because the dynamics of the fluctuations are governed by the time scale r^{-1}, which diverges at the critical point.

An electric field at time t' directly affects the fluctuations altering the distribution of the Cooper pairs, which then affect the current at a much later time t. This is the origin of term C, which only appears because the fluctuations are charged. Thus the kinetic equation is quite different than for other kinds of critical points, such as magnetic critical points, where the order parameter is neutral.

Hard quench and aging

We now seek signatures of criticality in the transient regime, first considering the hard quench where $r(t) \gtrsim T$ for $t < 0$ and $r(t) = r$ for $t > 0$. We begin by studying the fully critical case, $r = 0$, where the functions A,B,C,

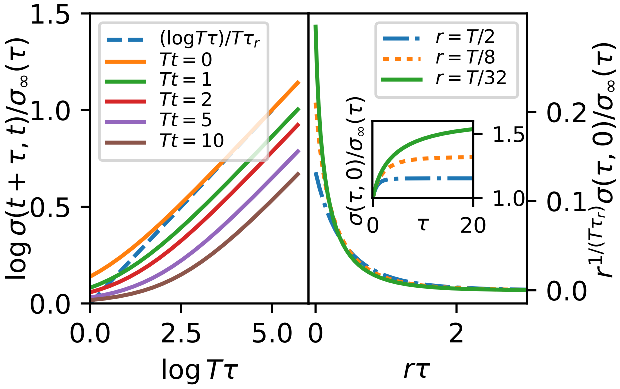

FIG. 3. Aging in the limit $\alpha = 0$ for the hard quench. Left: Log-Log plot of the ratio $\sigma(t+\tau,t)$ and $\lim_{t\to\infty}\sigma(t+\tau,t)$ against $T\tau$. The conductivity is enhanced at early times because of the absence of superfluid fluctuations. In the case of $t = 0$ the ratio converges to $(T\tau)^\gamma$, where $\gamma = 1/T\tau_r$. Right: Scaling plot of the $\sigma(\tau,0)$ at $\alpha = 0$ for different detunings r. Inset: unscaled $\sigma(\tau,0)$. Plots shown for $T\tau_r = 5$.

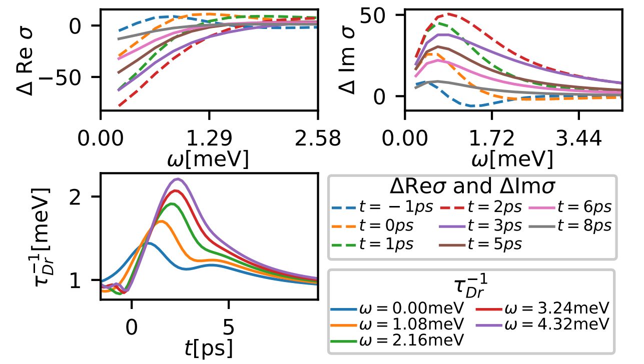

FIG. 4. Optical conductivity (in $e^2/\hbar$) for the soft quench with the profile for the detuning $r(t)$ shown in right panel of Fig. 1. Top panel: Re,Im $[\sigma(\omega,t) - \sigma(\omega, t=\infty)] = \Delta\sigma$ for several values of t and $T = 100$K. Early times $t \leq 2$ps are shown with dashed lines, later times $t > 2$ps in full lines. Lower panel: Drude parameter as a function of time for several frequencies.

simplify to

$$A(t) = 1 - (Tt)^{-1},\ B(t,t') = (Tt - Tt')^{-2} - (Tt)^{-2},$$
$$C(t,t') = (Tt - Tt')^{-1} - T(t+t')(Tt)^{-2}. \quad (7)$$

The conductivity $\sigma(t+\tau,t)$ for these parameters is shown in Fig. 2. The slow relaxation leads to "aging" phenomena, where the conductivity approaches its equilibrium value not on the microscopic time scale T^{-1}, but much slower, as r^{-1}. In particular at $r = 0$ the approach is via scale free power laws. There are two aging effects, one arising from the dissipative term $A(t)$ which affects the short time conductivity, and the other due to the Cooper channel $C(t)$ which affects the long time conductivity. The effect of the latter is shown in Fig. 2 (left, inset), where for $t \to \infty$ (orange line), the conductivity decays with a long tail ($\sim \tau^{-1}$). However the tail is absent for $t = 0$ (blue line) as there are no fluctuations at $t = 0$.

The short time aging is qualitatively different. It arises because $A(t)$ converges to its equilibrium value as $1/t$, causing the conductivity at short times after the quench to be larger than its equilibrium value. In particular, at short times, setting $\alpha = 0$ in Eq. (4) gives,

$$\sigma(t+\tau,t) \to_{Tt\gg 1} e^{-\tau/\tau_r};\ \sigma(\tau,0) = \tau^{1/T\tau_r} e^{-\tau/\tau_r}. \quad (8)$$

This behavior is shown in Fig. 3. The strong power law amplification, with material dependent exponent, $(T\tau_r)^{-1}$, is a consequence of the scale free nature of the $r = 0$ critical point, and thus provides an example of critical aging. Since the $t = 0$ conductivity is enhanced at short times by the lack of inelastic scattering, but is suppressed due to the lack of fluctuation conductivity at long times, the aging is curiously non-monotonic in time as seen in Fig. 2 (left panel). In Fourier space, Fig. 2 (right panel), aging manifests as a nearly logarithmic approach to the long time behavior of Reσ at low frequencies. Overall the dominant effect is a spreading of the initially very sharp peak in the initial state (due to weak scattering) to the wider line-shape in the inset. The conductivity is not Drude-like: as a consequence of the τ^{-1} tail of $\sigma(t+\tau,t)$, Re$\sigma(\omega, t\to\infty)$ diverges as $\log\omega$, as $\omega \to 0$.

We now consider the effect of finite detuning r from the critical point. The effect of the detuning is to suppress the aging behavior. This may be seen analytically in the $\alpha = 0$ limit, where we find that for $rt \gg 1$, $\sigma(\tau,0) \to (T/r)^{1/T\tau_r}\exp(-\tau/\tau_r)$, and the power law amplification saturates to a constant.

Similar to an equilibrium phase transition, the fact that the dynamics is controlled by the single diverging scale r means that the functions $B(t,t')$ and $C(t,t')$ can be expressed as a function of the quantity rt,rt', and the function $A(t)$ may be written as $A(t) = c + \frac{r}{T}g(rt)$, for some order one constant c. As a result of this scaling behavior, for $\alpha \ll 1$, the ratio of short and long time conductivity is "universal",

$$\sigma(\tau,0)/\lim_{t\to\infty}\sigma(t+\tau,t) = r^{-1/(T\tau_r)}F_s(r\tau), \quad (9)$$

for some scaling function $F_s(x)$. This scaling is demonstrated in Fig. 3, right panel.

Smooth quench protocols

The hard quench may not accurately describe the behavior of pumped K_3C_{60} films. In these experiments, the effective fermion interactions may vary smoothly on the scale T^{-1}, as energy is slowly transferred from the laser to the active phonon mode, and then to the environment.

Thus we consider a "soft" quench where r smoothly decreases to zero and then smoothly increases back to a large value, sketched in the right panel of Fig. 1 for the particular choice $r(t>0)/T = 1-(t/t_*)e\exp(-t/t_*)$ where $Tt_* = 30$. We consider the regime where $t_* \gg \tau_r \gg T^{-1}$. The time dependent optical conductivity is shown in Fig. 4 for $T = 100$K for which $t_* = 2.5$ps. The primary effect seen in the top left panel is a strong dip in $\mathrm{Re}\Delta\sigma$ around $t \sim t_*$ for low frequencies $\omega \ll T$, but a peak at $t \sim t_*$ for higher frequencies $\omega \sim T$. This can be understood as a spreading of the peak of $\mathrm{Re}\sigma$, as in the hard quench. This behavior is accompanied by a peak in $\mathrm{Im}\Delta\sigma$ (upper right panel) at low frequencies at $t \sim t_*$.

Fig. 4 lower panel shows $\tau_{\mathrm{Dr}}(t,\omega)$. This parameter varies strongly with frequency during the quench. This highly non-Drude behavior may be understood as coming from the two different effects that increasing fluctuations have on the conductivity. The inelastic scattering effect - the A term - leads to increasing scattering as a function of time, at high frequencies. The memory effect - the B and C terms - leads to decreasing scattering as a function of time, at low frequencies. As these are sensitive in different ways to the trajectory of $r(t)$, the total effect is non-monotonic in time. Further the peak in $\tau_{\mathrm{Dr}}(t,\omega)$ happens at different times for different frequencies. These non-monotonic behaviors and the dispersion in frequency are a consequence of slow charged modes, and therefore a necessary consequence of proximity to a second order superfluid transition.

Conclusions

We have discussed the effect of non-equilibrium superfluid fluctuations on the conductivity of fermions in a lattice potential, with applications to cold atoms and pump-probe experiments. Two primary effects were discussed, a characteristic aging of the conductivity at short times, and a power law tail of the conductivity at long times. The variation of these two effects with detuning from the critical point, and corresponding signals in frequency space were discussed. A key feature of crossing a superfluid critical point is a non-monotonic frequency dependence of the optical conductivity.

Our results generalize to quenches below T_c provided the transient state is too short-lived for long range order to develop. However future experiments which manage to create transient states of a longer duration require extending the study of optical conductivity to include coarsening, followed by vortex dynamics, and eventually the role of strong phase fluctuations. This work provides a theoretical treatment missing in the current literature on how to analyze transport in transient superfluids, and may be generalized to include disorder [22], and explore hydrodynamic features in other transport properties.

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