Braiding with Borromean Rings in (3+1)-Dimensional Spacetime

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Borromean-Rings Braiding Statistics in (3+1)-dimensional Spacetime

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While winding a particle-like excitation around a loop-like excitation yields the celebrated Aharonov-Bohm phase, we find a distinctive braiding phase in the absence of such mutual winding. In this work, we propose an exotic particle-loop-loop braiding process, dubbed the *Borromean-Rings braiding*. In the process, a particle moves around two unlinked loops, such that its trajectory and the two loops form the Borromean-Rings or more general Brunnian links. As the particle trajectory does not wind with any of the loops, the resulting braiding phase is fundamentally different from the Aharonov-Bohm phase. We derive an explicit expression for the braiding phase in terms of the underlying Milnor's triple linking number. We also propose Topological Quantum Field Theories consisting of an AAB-type topological term which realize the braiding statistics.

Introduction—Braiding statistics is a quantum mechanical phenomenon in which a quantum state acquires a holonomy when winding an excitation adiabatically around other excitations [1–3]. It arises from the ambiguous weightings for distinct homotopy classes of trajectories in the Feynman's path integral which sums over all continuous paths in the configuration space [4, 5]. Not only is quantum statistics an important subject in fundamental physics, it is also a crucial data in characterizing topological order in long-range entangled phases of matter [6–8]. Moreover, braiding statistics has been recently shown to be a powerful diagnostic of Symmetry-Protected Topological (SPT) phases [9–13]. By now, braiding statistics in (2+1)D has been thoroughly studied through the braid group and formulated in the theory of anyons [14–17]. Nevertheless, our understanding of braiding statistics in (3+1)D is still far from mature. The core reason is that the possible loop excitations complicate the configuration space in the path integral. While the simplest particle-particle braiding is always trivial due to the contractibility of particle trajectories around the other particle, the possible braiding statistics is significantly enriched if the spatially extended loop excitations are taken into account. The most well-known example is the particle-loop braiding statistics in which a particle carried along a non-contractible cycle around a loop experiences the Aharonov-Bohm effect [18].

The peculiar braiding phase in the Aharonov-Bohm effect has been understood to be associated with the winding between the particle trajectory and the loop [1–3] [Fig. 1(a)]. In this work, we argue that the statistical interaction between particles and loops can appear in a more general context. We consider the effect of braiding a particle with more than one loop. Importantly, we find that there can be a non-trivial braiding phase even without winding the particle around any loop [Fig. 1(b)]. Particularly, in braiding a particle around two unlinked loops, a braiding phase appears when the particle trajectory and the loops form a Brunnian link, which is formed by three mutually unlinked circles. For example, the simplest Brunnian link is the Borromean-Rings link. While the traditional particle-loop braiding statistics is dictated by the Hopf linking number $\mathfrak{L}$, we show that the *particle-loop-loop braiding* statistics is instead governed by a higher order linking number, called the Milnor's triple linking number $\bar{\mu}$ [19].

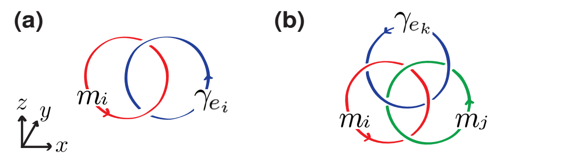

FIG. 1. (Color online) (a) Particle-loop braiding: a particle e_i travels around a loop m_i such that the braiding trajectory γ_{e_i} and m_i form a Hopf link. (b) Borromean-Rings braiding: a particle e_k moves around two unlinked loops m_i, m_j such that m_i, m_j and the trajectory γ_{e_k} form the Borromean rings (or generally the Brunnian link).

Physically, braiding statistics involving particles and loops can be realized in Abelian discrete gauge theories. For example, non-trivial particle-loop braiding statistics can be realized in $\mathbb{Z}_N$ gauge theory, which describes the deconfined phase of (3+1)D type-II superconductor with a charge-N condensate [20, 21]. In such a gapped phase of matter, the excitation spectrum is generated by a particle e and a loop m under fusion, where Ne, Nm are both trivial. Carrying a particle e in a closed path γ_e around a loop m leads to the quantized phase $\frac{2\pi}{N}\mathfrak{L}(m,\gamma_e)$ [Fig. 1(a)]. Recently, more exotic multi-loop braiding statistics and particle statistical transmutation have been demonstrated in discrete gauge theories with larger gauge group $G=\prod_i \mathbb{Z}_{N_i}$ [22–36], which is a system of $\mathbb{Z}_{N_i}$ gauge theories with a collection of flavors $i\in\mathcal{F}$. In these theories, the vacuum expectation of any physical braiding process yields a complex phase factor.

In this work, we introduce the Borromean-Rings (BR) braiding, namely, the particle-loop-loop braiding generating the Brunnian links [Fig. 1(b)], in discrete gauge theories with $G=\prod_i \mathbb{Z}_{N_i}$. Contrary to the particle-loop braiding where the particle trajectory is linked with the loop, the particle trajectory is not linked to any of the two loops in the BR braiding. By following a line of geometric arguments, we derive constraints [Fig. 2] and quantization condition of the BR braiding phase if it exists. Then we obtain an explicit formula for the braiding phase, which is expressed in terms of the Milnor's triple linking number [Eq. (1)]. Also, we construct Topological Quantum Field Theories (TQFTs) with a BF action dressed with an AAB topological term (A and B denote some 1-form and 2-form gauge fields respectively) [Eq. (2)] which support non-trivial BR braiding statistics [Eq. (10)]. The resulting BR braiding phase agrees with the result from geometric arguments. This work is concluded with several remarks and future directions.

Preliminaries—As a warm-up, we discuss general aspects of braiding a particle around loops in Abelian discrete gauge theories. Here, we are primarily interested in the classes of closed paths which could lead to non-trivial braiding statistics [4, 5]. As the particle travels in the complement of loops, the braiding statistics must be trivial if its closed trajectory can be adiabatically shrunk to a point. Equivalently, if the trajectory and the loops are viewed as a link, the braiding statistics is trivial if the trajectory can be unlinked from the loops. Under the deformation, the trajectory can cross with itself since the intersection point corresponds to the particle position at different time instances which can never interact. However, it cannot cross with any of the loops since the Aharonov-Bohm effect can contribute to braiding statistics. Besides, the loops can also undergo adiabatic deformation. Note that while Aharonov-Bohm interaction is possible among loops, there is no Aharonov-Bohm self-interaction [37–40]. In other words, each loop is allowed to cross with itself but not with other loops. Under such *link homotopy*, each link component can cross with itself but not with other link components [19]. Any particle trajectory that cannot be shrunk to a point under link homotopy can in principle lead to a non-trivial braiding phase.

In such formulation, each homotopy class of links is assigned with a braiding phase that depends only on the underlying linking numbers. Hence, while the particle-loop braiding phase is determined by the Hopf linking number $\mathfrak{L}$, the particle-loop-loop braiding phase is governed by the three mutual Hopf linking numbers and the Milnor's triple linking number $\bar{\mu}$ [19]. In this work, we study the particle-loop-loop braiding statistics which cannot be simply explained by the Hopf linking numbers. Nevertheless, the higher order linking number $\bar{\mu}$ is an invariant under link homotopy iff all the three mutual Hopf linking numbers vanish. So when the trajectory and the two loops are mutually unlinked, we expect a well-defined braiding phase determined by $\bar{\mu}$.

Borromean-Rings Braiding—Here we introduce the BR braiding in the context of Abelian discrete gauge theories. Pick any $i, j, k \in \mathcal{F}$, the BR braiding is a particle-loop-loop braiding in which a $\mathbb{Z}_{N_k}$ particle e_k is carried around mutually unlinked $\mathbb{Z}_{N_i}$ loop m_i and $\mathbb{Z}_{N_j}$ loop m_j such that the closed path and the two loops form the Borromean-Rings, or generally the Brunnian link [Fig. 1(b)]. Let $\mathsf{B}_{i,k}$ and $\mathsf{B}_{j,k}$ be the quantum operators of braiding e_k around m_i and m_j respectively. Given the two loops, any braiding process can be written as a sequential operation in $\mathsf{B}_{i,k}$ and $\mathsf{B}_{j,k}$ as well as their inverses, in which m_i and m_j together with the braiding trajectory γ_{e_k} can be viewed homotopically as a link L. For the BR braiding, since the braiding trajectory is not linked

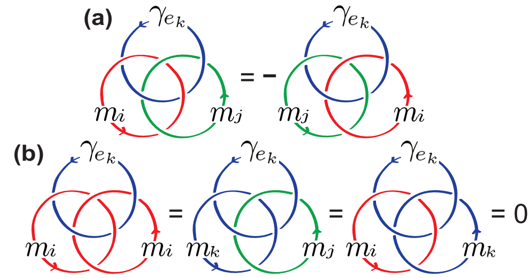

FIG. 2. (Color online) Constraints on the BR braiding phase $\Theta(L)$. (a) $\Theta(L)$ changes sign if m_i and m_j are exchanged. (b) $\Theta(L)$ vanishes if any two of three components belong to the same gauge group.

with any of the two loops, the sum of exponents is zero for both $\mathsf{B}_{i,k}$ and $\mathsf{B}_{j,k}$. For example, the braiding process giving Borromean-Rings link is written as $\mathsf{B}_{j,k}^{-1}\mathsf{B}_{i,k}^{-1}\mathsf{B}_{j,k}\mathsf{B}_{i,k}$ [41]. Since the exponent sum of each of them is zero, if any of the two constituent braidings $\mathsf{B}_{i,k}$ and $\mathsf{B}_{j,k}$ gives only an Abelian phase, the BR braiding statistics must be trivial. Hence, non-trivial BR braiding statistics implies that m_i, m_j and e_k support non-Abelian braiding statistics, despite the Abelianess of the gauge group G. We denote the overall BR braiding phase as $\Theta(L)$. Below, we are going to extract several constraints on the BR braiding phase geometrically.

First of all, since the BR braiding generates the Brunnian links, the geometric properties of the braiding phase are dictated by the Milnor's triple linking number $\bar{\mu}$. Let $c_{ij,k}$ be the braiding phase for the simplest BR braiding of particle e_k around m_i and m_j forming Borromean-Rings with $\bar{\mu} = 1$. Consider carrying the particle e_k along its original path γ_{e_k} by w times. It generates a Brunnian link L with $\bar{\mu} = w$. Generally, any Brunnian link with $\bar{\mu} = w$ can be generated this way up to link homotopy. Since the BR braiding is operated repeatedly, the braiding phase accumulates over w times. Therefore the BR braiding phase should take the linear form $\Theta(L) = c_{ij,k}\bar{\mu}(m_i, m_j, \gamma_{e_k})$, where the three entries for $\bar{\mu}$ are respectively the first, second and third component of L. Physically, $c_{ij,k}$ encodes the braiding data while $\bar{\mu}$ gives the geometric properties of the BR braiding process.

Next, we demonstrate the anti-symmetry of the BR braiding statistics. Consider braiding a particle e_k around m_i and m_j in a way generating the Borromean-Rings [Fig. 2(a)]. Viewing the process from the opposite side, the particle e_k travels around loops m_j and m_i with the orientation of the three components are flipped. Since reversing the orientation of any component of L causes $\bar{\mu}$ to change its sign, flipping the orientation of the three components gives a minus sign to the braiding phase. Hence braiding around m_i and m_j is minus the braiding around m_j and m_i, so $c_{ji,k} = -c_{ij,k}$. Such sign change can also be understood as coming from braiding e_k around m_i and m_j but with flipped labels m_i and m_j in L, which changes the sign of $\bar{\mu}$. Generally, for arbitrary Brunnian link L, if the labels m_i and m_j are interchanged, $\Theta(L) \to c_{ij,k}\bar{\mu}(m_j, m_i, \gamma_{e_k}) = -\Theta(L)$. Thus interchanging the labels of m_i and m_j flips the sign of $\Theta(L)$.

Next, we show that the BR braiding phase vanishes if any two objects involved are from the same gauge group. Consider the decomposition $\mathsf{B}_{j,k}^{-1}\mathsf{B}_{i,k}^{-1}\mathsf{B}_{j,k}\mathsf{B}_{i,k} = e^{ic_{ij,k}}$ for the BR braiding generating the Borromean-Rings link [Fig. 2(b)]. If $i = j$, the product of operators reduces to identity and hence $c_{ii,k} = 0$. If $i = k$, since $\mathsf{B}_{i,k}$ is guaranteed to give an Aharonov-Bohm phase $\frac{2\pi}{N_k}$ under the $\mathbb{Z}_{N_k}$ gauge group, the product of operators again reduces to identity and hence $c_{kj,k} = 0$. Similarly, we also have $c_{ik,k} = 0$. Consequently, $\Theta(L)$ vanishes if any two of the indices in L are identical. In other words, non-trivial BR braiding appears only for distinct indices i, j, k. In particular, non-trivial BR braiding implies non-Abelian particle-loop braidings $\mathsf{B}_{i,k}, \mathsf{B}_{j,k}$ for *distinct* flavors, rendering each of them to be gauge non-invariant.

We now derive the quantization rule of the BR braiding phase. Consider the BR braiding forming the Borromean-Rings with braiding phase $c_{ij,k}$. Imagine scaling up the phase by N_k by carrying e_k along γ_{e_k} repeatedly for N_k times. The whole process is equivalent to carrying $N_k e_k$ once along γ_{e_k}. Since $N_k e_k$ is a trivial particle, and braiding is compatible with fusion, we have $N_k c_{ij,k} = 0 \bmod 2\pi$. Now imagine scaling up the phase by N_i by winding m_i along its locus for N_i times instead. Again, since $N_i m_i$ is trivial and braiding is compatible with fusion, we have $N_i c_{ij,k} = 0 \bmod 2\pi$. Similarly, $N_j c_{ij,k} = 0 \bmod 2\pi$. Combining all the three conditions, we have $c_{ij,k} = \frac{2\pi k_{ij,k}}{N_{ijk}}$, where $k_{ij,k}$ is an integer and N_{ijk} denotes the greatest common divisor of N_i, N_j and N_k. Finally, we get the formula for the BR braiding phase

$$\Theta(L) = \frac{2\pi k_{ij,k}}{N_{ijk}} \bar{\mu}(m_i, m_j, \gamma_{e_k}) \,, \tag{1}$$

where all properties of the coefficient $c_{ij,k}$ propagate to $k_{ij,k}$. That is, $k_{ji,k} = -k_{ij,k}$ and $k_{ij,k}$ vanishes if any of the two indices are the same. Since $\Theta(L)$ is defined up to 2π, the parameter $k_{ij,k} \in \mathbb{Z}_{N_{ijk}}$. We conclude one of our main results: if the BR braiding statistics exists in discrete gauge theories, the braiding phase must take the form as Eq. (1). Below, we construct explicitly field-theoretic models which support non-trivial BR braiding statistics.

TQFTs with AAB Topological Term— It is believed that low energy physics of long-range entangled phases of matter is captured by some TQFTs [42]. For example, the topological features of discrete gauge theories with $G = \prod_i \mathbb{Z}_{N_i}$ are known to be described by the BF theories with action $S_{\mathrm{BF}} = \int \sum_i \frac{N_i}{2\pi} B^i dA^i$, where the 1-form A^i and 2-form B^i are compact U(1) gauge fields describing the loop and particle degrees of freedom respectively [43, 44]. The $\mathbb{Z}_{N_i}$ fusion structure of particles and loops is encoded in the cyclic Wilson integrals of A^i and B^i. Moreover, the Aharonov-Bohm effect is captured by the effective action $S_{\mathrm{Hopf}} = \sum_i \frac{2\pi}{N_i} I_{\mathrm{Hopf}}[\Sigma^i, J^i]$, where the 3-form J^i and 2-form Σ^i are respectively the particle and the loop sources describing the braiding process, and $I_{\mathrm{Hopf}}[\Sigma^i, J^i] = \int \Sigma^i d^{-1} J^i$ counts the Hopf linking $\mathfrak{L}(m_i, \gamma_{e_i})$. On top of the BF theories, exotic braiding statistics can be realized by introducing an extra topological term [22–36, 45–49]. Below, we introduce the AAB term and show that the resulting theories support the BR braiding statistics [Eq. (10)].

We begin by exhausting the possible AAB terms for physical theories. Consider adding an $A^i A^j B^k$ term with some real coefficient $c_{ij,k}$ upon the BF theories [Eq. (2)]. We are going to show that $c_{ij,k}$ here satisfies the same set of constraints as that in the previous discussion. First, notice that not all possible terms are independent, more specifically, interchanging A^i and A^j gives the same term but with a minus sign, hence $c_{ji,k} = -c_{ij,k}$. Second, some choices of indices are improper. We see $i \neq j$, otherwise $A^i A^j B^k$ vanishes. For any flavor i, since either A^i or B^i is reserved as the Lagrange multiplier which enforces the $\mathbb{Z}_{N_i}$ fusion structure, A^i and B^i of the same flavor cannot simultaneously appear on top of the BF theories, so $i, j \neq k$. In other words, $c_{ij,k} = 0$ if any two indices are the same. So G requires at least three $\mathbb{Z}_{N_i}$ group components for a legitimate $A^i A^j B^k$ term. Lastly, we show that $c_{ij,k}$ is quantized due to large gauge invariance. To this end, we pick some distinct $i, j, k \in \mathcal{F}$ for the $A^i A^j B^k$ term and focus on the three cyclic group components involved. Consider

$$S = S_{\mathrm{BF}} + S_{\mathrm{AAB}} \,, \quad S_{\mathrm{AAB}} = \int \frac{n c_{ij,k}}{(2\pi)^3} A^i A^j B^k \,, \tag{2}$$

where $n = N_i N_j N_k$. Let $a, b = i, j$, the action S is invariant up to a surface term under the gauge transformation

$$\begin{aligned} &A^a \to A^a + d\alpha^a \,,\; B^a \to B^a + d\beta^a + \mathcal{X}^a \,, \\ &B^k \to B^k + d\beta^k \,,\; A^k \to A^k + d\alpha^k + \mathcal{X}^k \,, \end{aligned} \tag{3}$$

where, to compensate the gauge change of the S_{AAB} term, the Lagrange multipliers B^a and A^k transform with extra twists

$$\begin{aligned} \mathcal{X}^a &= -\textstyle\sum_b \frac{n c_{ab,k}}{(2\pi)^2 N_a} (\alpha^b B^k - A^b \beta^k + \alpha^b d\beta^k) \,, \\ \mathcal{X}^k &= -\textstyle\sum_{ab} \frac{n c_{ab,k}}{(2\pi)^2 N_k} (\alpha^a A^b + \frac{1}{2} \alpha^a d\alpha^b) \,. \end{aligned} \tag{4}$$

After integrating out the Lagrange multipliers, the action S reduces to S_{AAB}, where A^a and B^k are enforced to be closed with cyclic Wilson integrals $\oint A^a \in \frac{2\pi}{N_a}\mathbb{Z}_{N_a}$ and $\oint B^k \in \frac{2\pi}{N_k}\mathbb{Z}_{N_k}$ over any closed manifolds. Under large gauge transformation, the gauge change of the action S_{AAB} consists of terms which take values in integral multiple of $N_i c_{ij,k}$, $N_j c_{ij,k}$ and $N_k c_{ij,k}$ (SM Part 2.1.3 [50]). The large gauge invariance of the resulting S_{AAB} term, which implies that $N_i c_{ij,k}$, $N_j c_{ij,k}$ and $N_k c_{ij,k}$ vanish mod 2π, leads to the desired coefficient quantization $c_{ij,k} = \frac{2\pi k_{ij,k}}{N_{ijk}}$ for integral $k_{ij,k}$.

Next, we discuss the constraints on the braiding. Consider a generic braiding described by some closed world lines for particles and closed world sheets for loops. The corresponding conserved particle sources J^i, J^j and J^k, and loop sources Σ^i, Σ^j and Σ^k can be incorporated into S via the source term

$$S_{\mathrm{s}} = -\int \sum_a \left(J^a A^a + \Sigma^a \mathcal{B}^a\right) + \Sigma^k B^k + J^k \mathcal{A}^k, \tag{5}$$

where $a = i, j$. The sources Σ^a and J^k are respectively coupled to the modified Lagrange multipliers $\mathcal{B}^a$ and $\mathcal{A}^k$ defined as

$$\begin{aligned}\mathcal{B}^a &= B^a - \textstyle\sum_b \frac{n c_{ab,k}}{2(2\pi)^2 N_a}(A^b d^{-1} B^k - d^{-1}A^b\, B^k)\,, \\ \mathcal{A}^k &= A^k - \textstyle\sum_{ab} \frac{n c_{ab,k}}{2(2\pi)^2 N_k} A^a d^{-1} A^b\,,\end{aligned} \tag{6}$$

which transform like ordinary gauge fields.Under gauge transformation, A^a and B^k change by a pure gauge, so $J^a A^a$ and $\Sigma^k B^k$ must be gauge invariant. However, $\mathcal{B}^a$ and $\mathcal{A}^k$ change by a total derivative of non-local terms, which is not strictly a pure gauge, so $\Sigma^a \mathcal{B}^a$ and $J^k \mathcal{A}^k$ may not be gauge invariant for arbitrary braiding. Remarkably, S_{s} is gauge invariant iff

$$I_{\mathrm{Hopf}}[\Sigma^a, J^k] = 0\,, \quad I_{\mathrm{Hopf}}[\Sigma^a, \hat{J}^b] = 0 \;\; (a \neq b)\,, \tag{7}$$

for any $\hat{J}^b$ describing current on the world sheet of m_b, for $a, b{=}i, j$ (SM Part 2.1.4 [50]). The first constraint means that the particle-loop braiding of e_k and m_a alone is not gauge invariant for $a = i, j$. The physical meaning of the second constraint can be understood by considering different choices of $\hat{J}^b$. Take $\hat{J}^b$ as the current of any point on m_b, it means that no point on the loop m_b can braid around the loop m_a for $a \neq b$. Take $\hat{J}^b$ as a time slice of the world sheet of m_b, which corresponds to the locus of m_b at a fixed time, it means that there is no linking between the loops m_a and m_b for $a \neq b$. In particular, since crossing between two loops always changes their linking number, loop crossing of m_a and m_b is not gauge invariant for $a \neq b$. If the loops are static, then the constraints in Eq. (7) simply mean that the loops m_i and m_j and the particle trajectory γ_{e_k} must be mutually unlinked [Fig. 3].

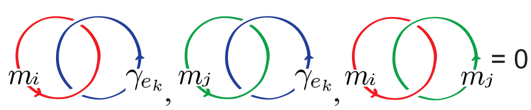

FIG. 3. (Color online) Illustration of the braiding constraints in Eq. (7). If the loops m_i and m_j are static, then m_i, m_j and γ_{e_k} are mutually unlinked circles for gauge invariant braiding process.

Now, we show that these theories support non-trivial BR braiding statistics. With the source term S_{s}, the Lagrange multipliers B^a and A^k enforce that $\Sigma^a{=}\frac{N_a}{2\pi}dA^a$ and $J^k{=}\frac{N_k}{2\pi}dB^k$, for $a = i, j$. Consequently, $S + S_{\mathrm{s}}$ leads to the effective action

$$S_{\mathrm{eff}} = S_{\mathrm{Hopf}} + S_{\mathrm{BR}}\,, \quad S_{\mathrm{BR}} {=} \frac{2\pi k_{ij,k}}{N_{ijk}} I_{\mathrm{BR}}[\Sigma^i, \Sigma^j, J^k]\,. \tag{8}$$

As in the BF theories, S_{Hopf} accounts for the Aharonov-Bohm effect for particle-loop braiding within the same flavor. Here, the AAB term induces an extra effect described by S_{BR} with

$$\begin{aligned} I_{\mathrm{BR}}[\Sigma^i, \Sigma^j, J^k] = \textstyle\int & d^{-1}\Sigma^i d^{-1}\Sigma^j d^{-1} J^k \\ & -\tfrac{1}{2}\Sigma^i (d^{-1}\Sigma^j d^{-2} J^k - d^{-1} J^k d^{-2}\Sigma^j) \\ & -\tfrac{1}{2}\Sigma^j (d^{-1} J^k d^{-2}\Sigma^i - d^{-1}\Sigma^i d^{-2} J^k) \\ & -\tfrac{1}{2} J^k (d^{-1}\Sigma^i d^{-2}\Sigma^j - d^{-1}\Sigma^j d^{-2}\Sigma^i)\,. \end{aligned} \tag{9}$$

Analogous to I_{Hopf} that counts the Hopf linking $\mathfrak{L}(m_i, \gamma_{e_i})$, the integral I_{BR} also admits a geometric interpretation (SM Part 2.2 [50]). Consider the gauge invariant particle-loop-loop braiding, where e_k travels around two static loops m_i and m_j with mutually unlinked m_i,m_j and γ_{e_k}. I_{BR} counts the Milnor's triple linking number $\bar{\mu}(m_i, m_j, \gamma_{e_k})$. Hence,

$$S_{\mathrm{BR}} = \frac{2\pi k_{ij,k}}{N_{ijk}} \bar{\mu}(m_i, m_j, \gamma_{e_k})\,. \tag{10}$$

In other words, the BR braiding process produces a BR braiding phase $\Theta(L) = S_{\mathrm{BR}}$. The field-theoretic result here further justifies the main result (1) obtained independently by geometric arguments. By noting that the braiding phase S_{BR} is defined mod 2π, we see $k_{ij,k} \in \mathbb{Z}_{N_{ijk}}$ can be used to classify discrete gauge theories with BR braiding statistics.

Conclusions—In this work, we introduced the BR braiding statistics from both geometric arguments and field-theoretic approach. Same quantum phenomenon is expected to appear also in discretized spacetime [51, 52]. The BR braiding statistics reveals exotic phases with non-abelian topological order under Abelian gauge group G. Due to the duality correspondence between topological order and SPT order [9, 22], the proposed BR braiding statistics immediately implies a new class of SPT order with global symmetry G (SM Part 3 [50]). In principle, the BR braiding phase can be observed by interferometry experiments similar to the measurements of Aharonov-Bohm effect, though the experimental design could be challenging. Nevertheless, it is expected to show up numerically as Berry phase in lattice Hamiltonian with higher form gauge symmetry [51, 52]. Lastly, it will be amusing to study entanglement properties of Eq. (2) [53] and explore the fermionic analog of BR braiding statistics.

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