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Coulomb-gas electrostatics controls large fluctuations of the KPZ equation

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We establish a large deviation principle for the Kardar-Parisi-Zhang (KPZ) equation, providing precise control over the left tail of the height distribution for narrow wedge initial condition. Our analysis exploits an exact connection between the KPZ one-point distribution and the Airy point process – an infinite particle Coulomb-gas which arises at the spectral edge in random matrix theory. We develop the large deviation principle for the Airy point process and use it to compute, in a straight-forward and assumption-free manner, the KPZ large deviation rate function in terms of an electrostatic problem (whose solution we evaluate). This method also applies to the half-space KPZ equation, showing that its rate function is half of the full-space rate function. In addition to these long-time estimates, we provide rigorous proof of finite-time tail bounds on the KPZ distribution which demonstrate a crossover between exponential decay with exponent 3 (in the shallow left tail) to exponent 5/2 (in the deep left tail). The full-space KPZ rate function agrees with the one computed in Sasorov et al. [J. Stat. Mech, 063203 (2017) [40]] via a WKB approximation analysis of a non-local, non-linear integro-differential equation generalizing Painlevé II which Amir et al. [Comm. Pure Appl. Math. **64**, 466 (2011) [19]] related to the KPZ one-point distribution.

Since its birth in 1986, the Kardar–Parisi–Zhang (KPZ) equation [1] has been applied to describe growth of interfaces [2], transport in one-dimension (1D) and Burgers turbulence [3], directed polymers [4], chemical reaction fronts [5], bacterial growth [6], slow combustion [7], coffee stains [8], conductance fluctuations in Anderson localization [9], polar active fluids [10], Bose Einstein superfluids [11], quantum entanglement growth [12].

Whereas some stochastic models (e.g. exclusion processes [13], random permutations [14], random walks in random media [15]) are directly related (via mappings to 'height functions') to the universality class for the 1D KPZ equation; others – namely random matrix theory (RMT) – rely on hidden connections to KPZ which are only seen from exact solutions to both KPZ and RMT models [16]. In this Letter, we describe such a relationship between the KPZ equation and the Airy point process – an infinite particle Coulomb-gas [17] which arises at the spectral edge in random matrix theory – and exploit variational techniques of electrostatics to precisely quantify the large fluctuations for the KPZ equation.

The 1D KPZ equation describes the stochastic growth of an interface of height $h(t, x)$ at $x \in \mathbb{R}$ and time $t > 0$

$$\partial_t h = \partial_x^2 h + (\partial_x h)^2 + \xi(t, x) , \qquad (1)$$

in convenient units, starting from an initial condition $h(t = 0, x)$. Here $\xi(x, t)$ is a centered Gaussian white noise with $\overline{\xi(t, x)\xi(t', x')} = 2\delta(x - x')\delta(t - t')$ and $\overline{\cdots}$ denotes expectations w.r.t. this noise. Typically, the fluctuations of the height field scale, at large time, like $t^{1/3}$. Recent progress has yielded exact solutions for the probability density function (PDF) of the height at a given space point at arbitrary time when starting from special initial conditions (e.g. droplet, flat, stationary) [18–20]. Focusing here and below on the droplet (a.k.a narrow wedge) initial condition, $h(0, x) = -\frac{|x|}{\delta} - \ln(2\delta)$ for $\delta \ll 1$, the exact formula for the PDF is expressed in terms of a Fredholm determinant. Using this, the scaled and centered height $\mathcal{H}(t)/t^{1/3}$, where $\mathcal{H}(t) = h(t, 0) + \frac{t}{12}$, was shown to converge in law as $t \to +\infty$ to the Tracy-Widom GUE distribution, which also describes the fluctuations of the largest eigenvalue, $\lambda_{\max}$, of a large random matrix from the Gaussian unitary ensemble (GUE).

Despite considerable interest, much less is known about large deviations and tails of the KPZ field or PDF $P(H, t) = \frac{\partial}{\partial H}\mathbb{P}(\mathcal{H}(t) \leq H)$. For general non-equilibrium systems, large deviation rate functions play a role similar to the free energy or entropy in equilibrium systems (see [21] and references therein). Existing large deviation theories fail to apply in the KPZ growth setting. The macroscopic fluctuation theory [22] requires local thermodynamic equilibrium, not realized here. The weak noise theory (see, e.g. [23]) applies, but only at very short times. Understanding the large deviations for the KPZ equation poses an important conceptual challenge.

Quantitative control over the tails of the KPZ equation plays an important role in experimental and numerical works. Precise results can be used e.g. as benchmarks for broadly applicable numerical Monte-Carlo methods such as used in [24]. In experimental work (such as reviewed in [25]), the tail behavior we are probing corresponds to excess growth. While unlikely at a single point, if the growing substrate is sufficiently long, disparate regions (spaced as time$^{2/3}$) will see roughly independent growth. Hence, by standard extreme-value theory, the maximal and minimal height of the entire substrate will

be determined by the one-point tail behaviors. The KPZ equation also models semiconductor film growth [26]. In technological applications, the roughness of these films determines device performance. As many films are grown independently, large deviations dictate failure rates.

In population growth and mass transport models, the KPZ tails play contribute to multi-fractal intermittency [27]. The $H/t^{1/3} \gg 1$ tail is associated with excess mass growth which comes from locally favorable effects; in contrast, the $-H/t^{1/3} \gg 1$ tail is associated with mass die-out which arises from collective effects of wide-spread unfavorable growth regions. Due to this collective effect, the left tail is intrinsically more difficult to analyze at large time. A similar situation arises in RMT for the tails of the PDF of $\lambda_{\max}$: while positive fluctuations arise from the largest eigenvalue $\lambda_{\max}$ simply detaching from the bulk of the spectrum, negative ones requires a reorganisation of the entire Wigner semicircle density of eigenvalues (the pushed Coulomb-gas) [30]. This analogy leads to the prediction [31] that for $t \gg 1$ and large fluctuations $|H| \sim t$ the right tail ($H \gg 0$) scales as $-\ln P(H, t) \sim t$ while the left tail ($H \ll 0$) scales as $-\ln P(H, t) \sim t^2$.

For short times $t \ll 1$ the left tail of the PDF ($H \ll 0$) behaves as $P(H, t) \sim \exp(-\frac{4}{15\pi}|H|^{5/2}/t^{1/2})$, as was shown analytically (via weak noise theory and exact solutions) [32, 33] and numerically [24] (see also [23, 34, 35] for other initial conditions). Extracting this tail in the intermediate or large time limit is much harder. For $t \gg 1$, in the typical scaling region $H \sim t^{1/3}$, the left tail should behave like the Tracy-Widom GUE distribution, i.e. $P(H, t) \sim_{H \to -\infty} \exp(-\frac{1}{12}|H|^3/t)$. Until recently, nothing was known about how far this cubic exponent persists into the very far left tail region $|H| \sim t^\alpha$ with $\alpha > 1/3$, or whether it holds for intermediate times.

Given the similarities between the KPZ and RMT problems, it is natural to try to attack these tail questions using methods inspired by RMT. The left tail behavior for $\lambda_{\max}$ can be accessed by either (i) the Coulomb-gas and associated electrostatic variational problem for the GUE spectrum [29, 30] (see also [36] for other large deviation applications of the Coulomb gas) or (ii) the relationship between gap probabilities and certain classical integrable systems [37] (which, in $N \to \infty$ edge limit, relate to the Painlevé II equation [38]). [19] introduced a non-local, non-linear integro-differential equation which generalizes Painlevé II by including a "Fermi-factor", and showed that its solution relates to the KPZ PDF. Studying this generalized equation via standard 'integrable-integral operator' methods [39] involves infinite-dimensional Riemann-Hilbert problem steepest descent analysis which is beyond current techniques. Employing a certain approximation ansatz, [31] attempted to analyze this equation. While they successfully predicted the scaling form for the large deviation tail $P(H, t) \sim \exp(-t^2 \Phi_-(H/t))$ for $-H \sim t \gg 1$, the approximations were too reductive and [31] predicted $\Phi_-(z) = \frac{1}{12}|z|^3$

which turns out only to hold true for z near 0. Ref. [40] revisited this analysis and employed a WKB approximation along with a 'self-consistency' ansatz for the form of the solution to a Schrödinger equation in which the potential depends upon the solution. Given these assumptions, [40] extracted a formula

$$\Phi_-(z) = \frac{4}{15\pi^6}(1 - \pi^2 z)^{5/2} - \frac{4}{15\pi^6} + \frac{2}{3\pi^4}z - \frac{1}{2\pi^2}z^2 \quad (2)$$

which predicts a crossover between $\Phi_-(z) \simeq_{z \to -\infty} \frac{4}{15\pi}|z|^{5/2}$ and $\simeq_{z \to -0} \frac{1}{12}|z|^3$. This, taken with the short-time estimates, suggests that the $|H|^{5/2}$ tail remains valid at all times (see also [24] and [23, 32, 34]) and that there is a crossover between the $\frac{1}{12}|H|^3/t$ and $\frac{4}{15\pi}|H|^{5/2}/t^{1/2}$ tail when $|H| \approx t$ (once $t \gg 1$).

The purpose of this Letter is to demonstrate how the Coulomb-gas can be utilized in a straight-forward and assumption-free manner to (i) establish, using the large deviations for the Airy point process, an electrostatic variational formula for $\Phi_-(z)$ whose solution (which we derive) agrees with (2), and (ii) demonstrate the first precise tail bounds (13) which are valid for all intermediate and long times and which capture the crossover between the $\frac{1}{12}|H|^3/t$ tail for $|H| \ll t$ and the $\frac{4}{15\pi}|H|^{5/2}/t^{1/2}$ tail for $|H| \gg t$. Our work provides a description of the intermediate and late time left large deviations for the KPZ equation where the connection to RMT and the role of the collective effects is explicit: each fixed value of $z = H/t$ corresponds to an optimal eigenvalue density (see Fig. 1). Finally, we extend our study to the half-line KPZ equation in the critical case, which relates to the Gaussian orthogonal ensemble (GOE), leading (via our RMT approach) to the rate function $\Phi_-^{\text{half-space}}(z) = \frac{1}{2}\Phi_-^{\text{full-space}}(z)$.

Our starting point is a remarkable identity [41, 42], obtained from the exact solution of the droplet initial condition [18, 19] which directly connects KPZ and RMT (as well as fermions in an harmonic well at temperature of order $t^{-1/3}$ [43]): for $\varphi_{t,s}(a) = \log(1 + e^{t^{\frac{1}{3}}(a+s)})$

$$\overline{\exp(-e^{\mathcal{H}(t)+st^{1/3}})} = \mathbb{E}_{\text{Airy}}\left[\exp\left(-\sum_{i=1}^{\infty} \varphi_{t,s}(\mathbf{a}_i)\right)\right]. \quad (3)$$

The l.h.s is an expectation over the KPZ white noise giving access to $P(H, t)$ while the r.h.s is the expectation of a "Fermi factor" over the *Airy point process* (Airy PP) generating the set $\{\mathbf{a}_i\} \in \mathbb{R}$. The Airy PP describes the largest few eigenvalues of a large GUE matrix. It is a 'determinantal' measure on infinite point configurations $\mathbf{a} = (\mathbf{a}_1 > \mathbf{a}_2 > \cdots)$ on $\mathbb{R}$ which means that for all $k \geq 1$, the k-th correlation function $\rho_k(x_1, \ldots, x_k)$ (which equals the probability density for the event that $\{x_i \in \mathbf{a}, \text{ for all } 1 \leq i \leq k\}$) takes the form $\rho_k(x_1, \ldots, x_k) = \det(K(x_i, x_j))_{1 \leq i,j \leq k}$ for some fixed 'correlation kernel' $K : \mathbb{R}^2 \to \mathbb{C}$. The Airy PP correlation kernel is

$K_{\mathbf{Ai}}(x,y) = \int_0^\infty \mathbf{Ai}(x+r)\mathbf{Ai}(y+r)\mathrm{d}r$. In particular the mean density is $\rho(a) = \rho_1(a) = K_{\mathbf{Ai}}(a,a) \simeq_{a\to-\infty} \pi^{-1}\sqrt{|a|}$. This agrees with the square-root behavior of the Wigner semi-circle at the edge. Remarkably, the $|H|^{5/2}$ tail emerges quite simply from this $\sqrt{|a|}$ density as we show from the first term in the cumulant expansion of the r.h.s. of (3), see (6). After observing this, we describe the Airy PP large deviation principle (LDP) derived via Coulomb-gas, and use it to compute the full crossover rate function $\Phi_-(z)$. Finally, we provide the bounds (13) which describes intermediate time behavior of the tail.

Cumulant expansion. As $st^{\frac{1}{3}} \to \infty$ the l.h.s. of (3) approaches $\mathbb{P}(\mathcal{H}(t) \leq -st^{\frac{1}{3}}) = \mathbb{P}(\mathcal{H}(t) \leq zt)$ with $z = -st^{-2/3}$. The r.h.s. of (3) is evaluated via cumulants as

$$\log\big(\text{r.h.s.}(3)\big) = \sum_{n=1}^\infty \frac{\kappa_n}{n!} \qquad (4)$$

where κ_n is the n-th cumulant of the Airy PP whose general form is known [44], e.g. for $n = 1, 2$

$$\kappa_1 = -\mathrm{Tr}(\varphi_{t,s}K_{\mathbf{Ai}}) = -\int_{-\infty}^{+\infty} \mathrm{d}a\,\varphi_{t,s}(a)\rho(a). \quad (5)$$

and $\kappa_2 = \mathrm{Tr}(\varphi_{t,s}^2 K_{\mathbf{Ai}}) - \mathrm{Tr}(\varphi_{t,s}K_{\mathbf{Ai}}\varphi_{t,s}K_{\mathbf{Ai}})$, where $(\psi K)(x,y) = \psi(x)K(x,y)$, $\mathrm{Tr}K = \int_{\mathbb{R}} \mathrm{d}a\,K(a,a)$. In the limit $z \to -\infty$, it is sufficient to keep only the first cumulant (the $n = 1$ term) in (4), which, using the above asymptotics $\rho(a) \simeq_{a\to-\infty} \pi^{-1}\sqrt{|a|}$, is estimated as (we use the notation $(\cdot)_+ = \max(\cdot, 0)$ below)

$$\kappa_1 \simeq -t^{1/3} \int_{-\infty}^{+\infty} \mathrm{d}a\,(a+s)_+\rho(a)$$
$$\simeq -t^{1/3}\frac{4}{15\pi}s^{5/2} = -t^2\frac{4}{15\pi}|z|^{5/2}. \qquad (6)$$

This simple argument gives the leading behavior as $z \to -\infty$ of the left large deviation rate function, $\Phi_-(z) \simeq \frac{4}{15\pi}|z|^{5/2}$, hence the desired $|H|^{5/2}$ tail. Explicit calculation (see [45]) of the next higher cumulants

$$\kappa_2 \simeq t^{2/3}\frac{s^2}{\pi^2} = t^2\frac{z^2}{\pi^2}\,,\; \kappa_3 \simeq -t\frac{4|s|^{3/2}}{\pi^3} = -t^2\frac{4|z|^{3/2}}{\pi^3} \quad (7)$$

shows their subdominance both (i) for $-z \gg 1$ with $t \gg 1$ and $z = H/t$ fixed and (ii) t fixed and large $s = -H/t^{1/3}$ and reproduces the large $|z|$ expansion of (2).

Coulomb-gas and large deviation rate function. Using (3), $\Phi_-(z)$ can be computed as (write $\mathbb{E}$ for $\mathbb{E}_{\mathrm{Airy}}$)

$$\Phi_-(z) = \lim_{t\to\infty}\frac{1}{t^2}\log\mathbb{E}\Big[\exp\Big(-\sum_{i=1}^\infty \varphi_{t,-zt^{2/3}}(\mathbf{a}_i)\Big)\Big].$$

For large t, we have $\varphi_{t,-zt^{2/3}}(t^{2/3}a) \approx t(a-z)_+$. Let $\mu_t(a)\mathrm{d}a = t^{-1}\sum_{i\geq 1}\delta_{-t^{-2/3}\mathbf{a}_i}(a)\mathrm{d}a$ denote the *scaled,*

space-reversed Airy PP empirical measure. Then we have

$$\Phi_-(z) = \lim_{t\to\infty}\frac{1}{t^2}\log\mathbb{E}\Big[\exp\Big(-t^2\int_{\mathbb{R}}\mathrm{d}a\,\mu_t(a)(-z-a)_+\Big)\Big]. \tag{8}$$

Like the GUE, the Airy PP should enjoy an LDP so that for a suitable class of functions μ, $\mathbb{P}(\mu_t \approx \mu) \approx \exp\big(-t^2 I_{\mathrm{Airy}}(\mu)\big)$. To our knowledge, this rate function is not in the literature, and we describe it below and in [28]. Given this, the r.h.s. of (8) can be evaluated via a variational problem, $\Phi_-(z) = \min_\mu \Sigma(\mu)$, with cost function

$$\Sigma(\mu) = \int_{\mathbb{R}} \mathrm{d}a\,\mu(a)(-z-a)_+ + I_{\mathrm{Airy}}(\mu). \tag{9}$$

To derive the LDP for the Airy PP we will appeal to the fact that the Airy PP arises as an edge limit of the GUE. The GUE spectrum is a 1D Coulomb-gas with logarithmic interaction which immediately leads to an electrostatic variational formulation for the GUE LDP [17, 29] (with the Wigner semi-circle representing the minimizer of this electrostatic energy). Our approach is to rewrite the GUE LDP in such a manner that it admits an edge scaling limit to yield the Airy PP LDP.

Recall from [29] that the empirical measure $\Lambda_N(\lambda)\mathrm{d}\lambda = \frac{1}{N}\sum_{i=1}^N \delta_{\boldsymbol{\lambda}_i}(\lambda)\mathrm{d}\lambda$ associated to the eigenvalues $\{\boldsymbol{\lambda}_1,\ldots,\boldsymbol{\lambda}_N\}$ of the GUE (normalized to have typical support $[-2,2]$ – see [28] for a precise definition) enjoys an LDP so that, for a generic density Λ with unit mass, $\mathbb{P}(\Lambda_N \approx \Lambda) \approx \exp\big(-N^2 I_2(\Lambda)\big)$. The rate function $I_\beta(\Lambda)$ is the difference of the electrostatic energy of a Coulomb-gas of charge β (with $\beta = 2$ for GUE, and $\beta = 1$ for GOE) with density Λ, as compared to that of the Wigner semi-circle density $\Lambda_{\mathrm{sc}}(\lambda) = \frac{1}{2\pi}\sqrt{4-\lambda^2}\mathbf{1}_{\{|\lambda|<2\}}$. I_β can be rewritten (see [28] for details) as

$$I_\beta(\Lambda) = \frac{\beta}{2}J(\Lambda) + \frac{\beta}{2}\int_{\mathbb{R}}\mathrm{d}\lambda\,V(\lambda)\Lambda(\lambda), \tag{10}$$

with a Coulomb interaction term $J(\Lambda) = -\int_{\mathbb{R}^2}\log|\lambda_1 - \lambda_2|\prod_{i=1}^2 \mathrm{d}\lambda_i(\Lambda(\lambda_i) - \Lambda_{\mathrm{sc}}(\lambda_i))$ (note that $\Lambda - \Lambda_{\mathrm{sc}}$ is a signed density with integral over $\mathbb{R}$ equal to 0) and potential term $V(\lambda) = \int_0^{|\lambda|} \mathrm{d}\lambda'\,((\lambda'^2 - 4)_+)^{1/2}$. The (space-reversed) Airy PP arises as a scaling limit of the GUE spectrum near its lower edge $\lambda = -2$. To deduce the Airy PP LDP from that of the GUE, we introduce the scaling $\lambda = -2 + t^{2/3}N^{-2/3}a$. As $N \to \infty$, $N\mathrm{d}\lambda\,\Lambda_N(\lambda) \simeq t\mathrm{d}a\,\mu_t(a)$, which when inserted into (10) gives $N^2 I_\beta(\Lambda) \simeq t^2 I_{\mathrm{Airy}}(\mu)$, with

$$I_{\mathrm{Airy}}(\mu) = J_{\mathrm{Airy}}(\mu) + U(\mu).$$

Here $J_{\mathrm{Airy}}(\mu) = -\int\log|a_1 - a_2|\prod_{i=1}^2 \mathrm{d}a_i(\mu(a_i) - \mu_{\mathrm{Airy}}(a_i))$ is defined for densities μ satisfying mass-conservation $\int \mathrm{d}a\,(\mu(a) - \mu_{\mathrm{Airy}}(a)) = 0$, where $\mu_{\mathrm{Airy}}(a) = \frac{1}{\pi}\sqrt{a}\mathbf{1}_{\{a>0\}}$, and $U(\mu) = \frac{4}{3}\int_{-\infty}^0 \mathrm{d}a\,|a|^{\frac{3}{2}}\mu(a)$.

Instead of searching directly for the minimum of Σ in (9), we first consider a simpler cost function

$$\Sigma_J(\mu) = \int_{\mathbb{R}} da\,(-z-a)_+ \mu(a) + J_{\text{Airy}}(\mu)$$

that drops the term $U(\mu)$. The minimizer μ_* of Σ_J is the unique measure (see [28] for details) such that

$$(-z-a)_+ - 2\int_{\mathbb{R}} da'\log|a-a'|(\mu_*(a') - \mu_{\text{Airy}}(a')) \geq c \tag{11}$$

for some constant c with strict equality on the support of μ_*. Differentiating the l.h.s. of (11) in a yields

$$-\mathbf{1}_{\{a<-z\}} - 2\int_{\mathbb{R}} da'\frac{\mu_*(a') - \mu_{\text{Airy}}(a')}{a-a'}. \tag{12}$$

Consider a generic interval $[u,\infty)$ and let

$$\mu_{*,u}(a) = \Big(\frac{1}{\pi}\sqrt{a-u} + \frac{1}{2\pi^2}\log\Big|\frac{\sqrt{a-u}+\sqrt{v}}{\sqrt{a-u}-\sqrt{v}}\Big|$$
$$+ \frac{1}{\pi}\Big(\frac{u}{2}-\frac{\sqrt{v}}{\pi}\Big)\frac{1}{\sqrt{a-u}}\Big)\mathbf{1}_{\{a>u\}},$$

where $v = -z-u$. Ref. [28] verifies that substituting this density $\mu_{*,u}(a)$ for $\mu_*(a)$ implies that (12)$=0$ on $[u,\infty)$. Furthermore, [28] shows that $u = u_0 = \frac{2}{\pi^2}(\sqrt{1-\pi^2 z}-1)$ is the unique choice of u for which one also has (12)≥ 0 on $(-\infty,u_0)$ and $=0$ on $[u_0,\infty)$. This means that $\mu_*(a) = \mu_{*,u_0}(a)$ satisfies (11) and hence is the unique minimizer of Σ_J. Evaluating yields (see Fig 1)

$$\mu_*(a) = \Big(\frac{1}{\pi}\sqrt{a-u_0} + \frac{1}{2\pi^2}\log\Big|\frac{\sqrt{a-u_0}+\frac{\pi}{2}u_0}{\sqrt{a-u_0}-\frac{\pi}{2}u_0}\Big|\Big)\mathbf{1}_{\{a>u_0\}}.$$

The associated minimum of Σ_J is

$$\min_\mu \Sigma_J(\mu) = \frac{4}{15\pi^6}(1-\pi^2 z)^{\frac{5}{2}} - \frac{4}{15\pi^6} + \frac{2}{3\pi^4}z - \frac{1}{2\pi^2}z^2,$$

which coincides precisely with $\Phi_-(z)$ in (2).

Returning to Σ from (9), we note that $U(\mu) \geq 0$ implies $(\min\Sigma) \geq (\min\Sigma_J)$. Since $\mu_*(a)$ vanishes for $a < 0$ (since $u_0 > 0$), we have $U(\mu_*) = 0$ and hence $\Sigma(\mu_*) = \Sigma_J(\mu)$. Thus, the minimizer and minimum for Σ_J in fact also applies to Σ. Since $\Phi_-(z) = \min_\mu \Sigma(\mu)$, this confirms the formula in (2) and the calculation of [40].

Tail bounds for intermediate times. While the KPZ LDP holds for $t \to \infty$, the crossover behavior between exponents 3 and 5/2 remains valid at all intermediate times. Precisely: For any $\varepsilon, \delta \in]0,\frac{1}{3}[$ and $t_0 > 0$ then there exists constants $S = S(\varepsilon,\delta,t_0)$, $K_1 = K_1(\varepsilon,\delta,t_0) > 0$ and $K_2 = K_2(t_0) > 0$ such that for all $s \geq S$ and $t \geq t_0$,

$$\mathbb{P}(H \leq -st^{\frac{1}{3}}) \leq e^{-\frac{4(1-\varepsilon)}{15\pi}t^{\frac{1}{3}}s^{\frac{5}{2}}} + e^{-K_1 s^{3-\delta} - \varepsilon t^{\frac{1}{3}}s} + e^{-\frac{1-\varepsilon}{12}s^3}$$

$$\mathbb{P}(H \leq -st^{\frac{1}{3}}) \geq e^{-\frac{4(1+\varepsilon)}{15\pi}t^{\frac{1}{3}}s^{5/2}} + e^{-K_2 s^3} \tag{13}$$
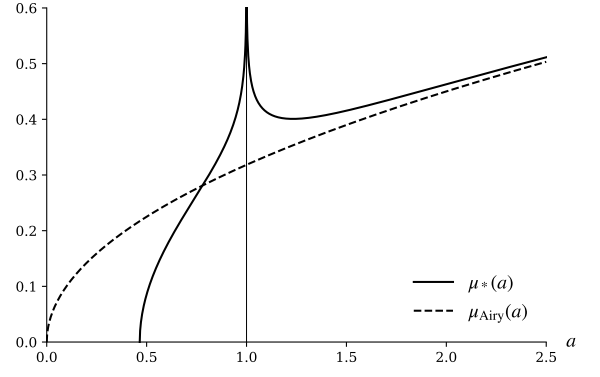

FIG. 1. Optimal density $\mu_*(a)$ at $z = -1$ compared to $\mu_{\text{Airy}}(a)$. The density μ_* has a log singularity at $a = -z$ (c.f. [46] for another Coulomb gas problem with similar behavior).

For $t^{2/3} \gg s \gg 1$, the second and third terms in the first line of (13) dwarf the first term and represent cubic decay (in the exponential) in s. In particular, as t gets large, only the third term survives and we recover (up to an ε correction) the predicted $\frac{1}{12}s^3$ decay. On the other hand, for $s \gg t^{2/3}$ the first term in the second line of (13) dwarfs the others and recovers the predicted $\frac{4}{15\pi}s^{5/2}$ decay for all t. The second line of (13) contains corresponding lower bounds – though notice that for t large and $t^{2/3} \gg s \gg 1$, our bounds do not recover the $\frac{1}{12}$ constant for the lower bound on the cubic decay. This result recovers the large and small z behavior of the $t \to \infty$ rate function $\Phi_-(z)$. Prior to (13), the only finite time bounds were in [47] which provided a Gaussian upper-bound on the decay (hence, the wrong exponent). Moreover, those bounds are not adapted to large t center and scaling—becoming ineffective as t grows.

Eqs. (13) follow from two considerations. The typical locations of the $\mathbf{a}_i$ are governed by $\rho(a)$. Plugging these typical values into (3) yields the 5/2 exponential term. However, the $\mathbf{a}_i$ are random and may deviate from their typical locations. For instance $\mathbf{a}_1 \leq -s$ with probability $\approx \exp(-\frac{1}{12}s^3)$. Such deviations lead to the cubic exponential terms. In order to provide matching upper and lower tail bounds, we precisely control the LDP for the counting function of the Airy PP in large intervals. This can be done via asymptotics of the Ablowitz-Segur solution to Painlevé II [48, 49] which relates to the exponential moment generating function for this counting process, as well as by using of the relation of the AAP to the stochastic Airy operator [50]. The main ideas and steps of this derivation are provided in [28] (and further technical details and complete rigorous proofs are in [51]).

Extensions and Summary. The approach developed in this Letter is applicable to certain variants of the KPZ equation which enjoy identities similar to (3) – namely half-space KPZ [52], the stochastic six vertex model and ASEP [42, 53]. Briefly we consider the half-space KPZ

equation, i.e. (1) restricted to $x \in \mathbb{R}^+$, with Neumann b.c. $\partial_x h(t,x)|_{x=0} = A$, for the value $A = -1/2$ corresponding to the so-called critical case. In that case and for droplet initial condition [52] proved that

$$\overline{\exp(-\frac{1}{4}e^{\mathcal{H}(t)+st^{1/3}})} = \mathbb{E}_{\text{GOE}}\Big[\prod_{i=1}^{+\infty}\frac{1}{\sqrt{1+e^{t^{1/3}(\mathbf{a}_i+s)}}}\Big]$$

where the r.h.s. expectation is over the $\beta = 1$ version of the Airy PP (which describes the top few eigenvalues at the spectral edge for the GOE instead of GUE – see also [28]). Employing the Airy PP Coulomb-gas approach from this Letter, we find that due to the square-root in the r.h.s. above (which introduces a factor of $1/2$ in exponential form), and the value of $\beta = 1$ (instead of $\beta = 2$), the half-space KPZ rate function $\Phi_-^{\text{half-space}}(z) = \frac{1}{2}\Phi_-(z)$ where $\Phi_-(z)$ is the full-space function in (2).

In conclusion, by relating the distribution of the height for the KPZ equation to an expectation over the Airy point process, we are able to employ the Coulomb-gas formalism and associated electrostatic problem large deviation principle (first for the GUE and, through a limit transition which we present, for the Airy point process) to identify the KPZ rate function. Solving the variational problem produces the formula in (2). This argument brings the role of random matrix theory in the study of KPZ to the forefront and provides a straight-forward and assumption-free derivation of the KPZ rate function. Additionally, a similar approach should be applicable to other exactly solvable KPZ class models such as ASEP or the stochastic six vertex model which connect to discrete Coulomb-gases. This approach also permits us to derive results valid for all intermediate times and opens the way to systematically calculate higher order corrections between the long time and finite time PDF, as is useful in experiments and numerics.

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