Quantum Error Correction with Only Two Extra Qubits

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Noise rates in quantum computing experiments have dropped dramatically, but reliable qubits remain precious. Fault-tolerance schemes with minimal qubit overhead are therefore essential. We introduce fault-tolerant error-correction procedures that use only two extra qubits. The procedures are based on adding "flags" to catch the faults that can lead to correlated errors on the data. They work for various distance-three codes.

In particular, our scheme allows one to test the $[\![5,1,3]\!]$ code, the smallest error-correcting code, using only seven qubits total. Our techniques also apply to the $[\![7,1,3]\!]$ and $[\![15,7,3]\!]$ Hamming codes, thus allowing to protect seven encoded qubits on a device with only 17 physical qubits.

Quantum computers will need protection from noise. A delicately designed circuit can correct errors, and even tolerate faults within itself [1]. However, while there have been experimental tests of quantum codes [2–8], testing *fault-tolerant* error correction remains a major challenge. A main difficulty is the substantial overhead; many physical qubits are needed for each encoded, logical qubit. This means that on small- and medium-scale systems in the near future, it will be difficult to test fault-tolerance theory, and to explore the efficacy of different fault-tolerance schemes. It is important to reduce this overhead.

Limited fault-tolerance schemes have been tested on five-qubit systems [9–12], using the $[\![4,2,2]\!]$ code. This code encodes two qubits into four, and the fifth qubit is used for error detection [13]. Since the code has distance two, it can only detect an error, not correct it.

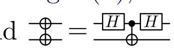
Until recently, the smallest known scheme that can *correct* an error used the $[\![9,1,3]\!]$ Bacon-Shor code, plus a tenth extra qubit. Although smaller, more efficient error-correcting codes are known, such as the $[\![5,1,3]\!]$ perfect code (the smallest distance-three code), fault-tolerance schemes using these codes have still required at least ten qubits. For example, Shor-style syndrome extraction requires $w+1$ [14] or w [15] extra qubits, where w is the largest weight of a stabilizer generator. Steane-style error correction for CSS codes [16, 17] uses at least a full code block of extra qubits.

Yoder and Kim [18] have given fault-tolerance schemes using the $[\![5,1,3]\!]$ code with eight qubits total, and the $[\![7,1,3]\!]$ Steane code with nine qubits total. Extending the latter construction, we introduce fault-tolerance procedures that use only two extra qubits. Table I summarizes some suitable, distance-three codes and compares the qubit overhead of our scheme to other methods. (In particular, in App. A [19] we generalize [18, 20]: the "decoded half cat state" method uses $\max\{3, \lceil w/2 \rceil\}$ extra qubits.)

For example, with the $[\![5,1,3]\!]$ code our scheme uses only seven qubits total, or ten qubits with the $[\![8,3,3]\!]$ code. A particularly promising application is to the $[\![15,7,3]\!]$ Hamming code: 17 physical qubits suffice to protect seven encoded qubits. In [21] we give fault-tolerant procedures for applying arbitrary Clifford operations on these encoded qubits, also using only two extra qubits, and fault-tolerant universal operations with four extra qubits, 19 total. Substantial fault-tolerance tests can therefore be run on a quantum computer with fewer than twenty qubits.

Our procedures here are based on adding "flags" to the syndrome-extraction circuits in order to catch the faults that can lead to correlated errors on the data. Figure 1(c) shows an example. Provided that syndromes are extracted in a careful order, detecting the possible presence of a correlated error is enough to correct it.

Flagged error correction for the $[\![5,1,3]\!]$ *code.* The perfect $[\![5,1,3]\!]$ code [22] has the stabilizer generators, and logical X and Z operators of Fig. 1(a).

The syndrome for the first stabilizer, $XZZXI$, can be extracted by the circuit in Fig. 1(b), where Z indicates a $|0\rangle, |1\rangle$ measurement, and  . However, this circuit is not fault tolerant. For example, if the second gate fails and after it is applied an IZ fault, then this fault will propagate through the subsequent gates to become an $IIZXI$ error on the data. Thus a single fault can create a weight-two error on the data, and even a perfect error-correction procedure will correct this error in the wrong direction, creating the logical error $IIZXZ \sim \overline{X}$.

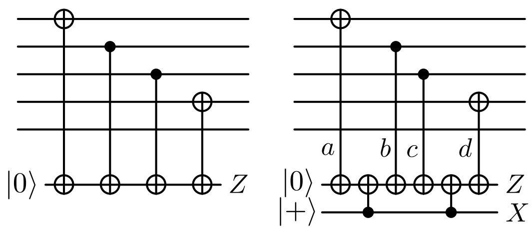
To fix this problem, we instead extract the $XZZXI$ syndrome using the circuit in Fig. 1(c). With no faults, this circuit behaves exactly the same as that of Fig. 1(b), and the X-basis ($|+\rangle, |-\rangle$) measurement will always give $|+\rangle$. The purpose of the extra $|+\rangle$ qubit, which we term a "flag," is to detect gate faults that can propagate to correlated errors, weight two or higher, on the data. Failures after gates a and d cannot create correlated errors. The failures after gates b and c that can create correlated errors are listed in Fig. 1(d). (Y failures on the second qubit have the same effect on the data as Z failures.) These failures will all be detected, causing a $|-\rangle$ measurement outcome. Moreover, observe that the seven distinct data errors have distinct, nontrivial syndromes.

The complete error-correction procedure is given by:

1. Use the circuit of Fig. 1(c) to measure $XZZXI$.
   (a) If the flag qubit is measured as $|-\rangle$, then use the un-

	X	Z	Z	X	I
	I	X	Z	Z	X
	X	I	X	Z	Z
	Z	X	I	X	Z
$\overline{X} =$	X	X	X	X	X
$\overline{Z} =$	Z	Z	Z	Z	Z

(a)



(b) (c)

ⓑ failure	Data error
IZ	$IIZXI$
XZ	$IXZXI$
YZ	$IYZXI$
ZZ	$IZZXI$

ⓒ failure	Data error
IZ	$IIIXI$
XZ	$IIXXI$
YZ	$IIYXI$
ZZ	$IIZXI$

(d)

FIG. 1. Flagged syndrome extraction for the $[\![5,1,3]\!]$ code. (a) Stabilizers and logical operators. (b) Circuit to extract the syndrome of the $XZZXI$ stabilizer into an extra qubit. This is not fault tolerant because a fault on the extra qubit can spread to a weight-two error on the data. (c) This circuit also extracts the $XZZXI$ syndrome, and if a single fault spreads to a data error of weight ≥ 2 then the X measurement will return $|-\rangle$, flagging the failure. (d) The nontrivial data errors that can result from a single gate failure in (c) that triggers the flag; these errors are distinguishable by their syndromes and so can be corrected.

flagged circuits analogous to Fig. 1(b) to extract all four syndromes. Finish by applying the corresponding correction from among $IIIII$, $IIZXI$, $IXZXI$, $IYZXI$, $IZZXI$, $IIIXI$, $IIXXI$, $IIYXI$.

(b) Otherwise, if the syndrome is -1, i.e., the syndrome qubit is measured as $|1\rangle$, then use unflagged circuits to extract *all four* syndromes. Finish by applying the corresponding correction of weight ≤ 1.

2. (If the flag was not raised and the syndrome was 1, then) Similarly measure $IXZZX$. If the flag is raised, then use unflagged circuits to extract the four syndromes, and finish by applying the correction from among $IIIII$, $IIIIX$, $IXXII$, $IIIXX$, $XIIIY$, $IXIII$, $IIIZX$, $IIIYX$. If the syndrome is -1, then use unflagged circuits to extract the four syndromes, and finish by applying the correction of weight ≤ 1.
3. Similarly measure $XIXZZ$, and correct if the flag is raised or the syndrome is nontrivial.
4. Similarly measure $ZXIXZ$, and correct if the flag is raised or the syndrome is nontrivial.

We now argue that this procedure is fault tolerant according to the extended rectangle formalism for distance-three codes [23].

- If there are no faults, then it appropriately corrects the data to the codespace.
- If the data lies in the codespace and there is at most one faulty gate in the error-correction procedure, then:
  - If all syndromes and flags are trivial, then the data can have at most a weight-one error. (No correction is applied.)
  - If a flag is raised or a syndrome is nontrivial, then the subsequent unflagged syndrome extractions are perfect, and suffice to correct either a possibly correlated error (if the flag is raised) or a weight ≤ 1 error (if no flag is raised but the syndrome is nontrivial).

Let us point out that when a syndrome extracted by a flagged circuit is nontrivial, then even if the flag is not raised we still extract all four syndromes (using unflagged circuits) before applying a correction. This is because a fault on the data could have been introduced in the middle of syndrome extraction. For example, if we extract the first syndrome as $+1$, but a Z_1 error is then added to the data, the remaining syndromes will be $+1, -1, +1$. The correction for $(+,+,-,+)$ is Z_3, but were we to apply this correction the data would end up with error $ZIZII$. This moral is that nontrivial syndromes cannot be trusted unless they are repeated.

Flagged error correction for Hamming codes. The Hamming codes are a family of $[\![2^r-1, 2^r-1-2r, 3]\!]$ quantum error-correcting codes, for $r \geq 3$ [24, 25]. They are self-dual perfect CSS codes.

As an example, consider the $r = 4$, $[\![15,7,3]\!]$ code, whose four X and four Z stabilizers are given by the parity checks of Fig. 2(a).

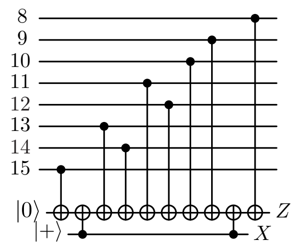
For a subset S, let $Z_S = \prod_{j \in S} Z_j$. The circuit of Fig. 2(b) extracts the first Z syndrome, for $Z_{\{8,\ldots,15\}}$. As

TABLE I. Extra qubits required for fault-tolerant syndrome-extraction methods, including Shor's cat state method and the Stephens-Yoder-Kim decoded half cat state method. Our flagged error-correction procedure needs only two extra qubits, as observed for the $[\![7,1,3]\!]$ code by [18]. Codes marked ◇ are Hamming codes.

Code qubits		Extra qubits required for		
Physical	Logical	Shor cat state	Decoded half cat	Flagged
5	1	5	3	2
7	1 ◇	5	3	2 [18]
9	1	1	–	–
8	3	7	3	2
10	4	9	4	2
11	5	9	4	2
15	7 ◇	9	4	2
31	21 ◇	17	8	2
2^r-1	2^r-1-2r ◇	$2^{r-1}+1$	2^{r-2}	2

0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
1	0	1	0	1	0	1	0	1	0	1	0	1	0	1

(a)



(b)

FIG. 2. Flagged syndrome extraction for Hamming codes. (a) $[\![15,7,3]\!]$ code parity checks. (b) Flagged circuit to extract the $Z_{\{8,\ldots,15\}}$ syndrome fault tolerantly.

in Fig. 1(b), any single gate fault that leads to a data error of weight ≥ 2 will also make the X measurement output $|-\rangle$. Moreover, if that measurement gives $|-\rangle$ with a single fault, the error's possible Z components are

$$\mathbf{1}, Z_8, Z_{\{8,9\}}, Z_{\{8,9,10\}}, Z_{\{8,9,10,12\}}, Z_{\{8,9,10,11,12\}}, \\ Z_{\{8,9,10,11,12,14\}}, Z_{\{8,9,10,11,12,13,14\}}$$

These errors are distinguishable by their X syndromes.

Similar circuits work for the other stabilizers, thus giving a two-qubit fault-tolerant error-correction procedure for the $[\![15,7,3]\!]$ code.

Similar schemes work for all Hamming codes:

Claim. *Syndromes for the $[\![2^r-1, 2^r-1-2r, 3]\!]$ Hamming code can be fault-tolerantly extracted with two qubits.*

Proof. Consider the stabilizer $Z_{\{2^{r-1},\ldots,2^r-1\}}$. As in Fig. 2(b), we give a permutation of the last 2^{r-1} qubits so that, when the flag is triggered, the possible Z errors have distinct syndromes.

Let $p(x)$ be a degree-$(r-1)$ primitive polynomial over GF(2). For $j = 0, \ldots, 2^{r-1}-2$, let $q_j(x) = x^j \mod p$; these are distinct polynomials of degree $\leq r-2$. Furthermore, the remainders of their cumulative sums, $\sum_{i=0}^{j} x^i$, are also distinct. (Otherwise, if $0 \equiv x^{j+1} + \ldots + x^k = x^{j+1}(x^{k-j}+1)/(x+1)$, then considering the lowest-order terms give $x^{k-j}+1 \equiv 0$, contradicting that p is primitive.)

The desired permutation is $2^{r-1}, q_0(2) + 2^{r-1}, q_1(2) + 2^{r-1}, \ldots, q_{2^{r-1}-2}(2) + 2^{r-1}$. To see this, identify both qubit indices and syndromes with polynomial coefficients, and then combine the above two properties. □

Other distance-three codes. Figure 3 presents $[\![8,3,3]\!]$, $[\![10,4,3]\!]$ and $[\![11,5,3]\!]$ codes [25–27].

The property required for our flagged procedure to work is: for measuring an operator $Z_1 Z_2 Z_3 \ldots Z_w$ (up to qubit permutations and local Clifford unitaries), the different errors $P_j Z_{j+1} \ldots Z_w$, for $P \in \{I, X, Y, Z\}$ and $j \in \{2, \ldots, w-1\}$, should have distinct and nontrivial syndromes. (For a CSS code, for which X and Z errors can be corrected separately, it is enough that the syndromes for different errors be different for $P \in \{I, Z\}$.)

X	X	X	X	X	X	X	X
Z	Z	Z	Z	Z	Z	Z	Z
I	I	Z	Y	X	Z	Y	X
I	Z	X	I	X	Y	Z	Y
I	X	I	Z	Z	X	Y	Y

(a)

X	X	X	X	X	X	X	X	X	X
Z	Z	Z	Z	Z	Z	Z	Z	Z	Z
X	Z	Z	X	I	Z	Y	Y	Z	I
I	X	Z	Z	X	I	Z	Y	Y	Z
X	I	X	Z	Z	Z	I	Z	Y	Y
Z	X	I	X	Z	Y	Z	I	Z	Y

(b)

I	I	I	Z	Z	Y	Y	X	X	X	X
I	I	I	X	X	Z	Z	Y	Y	Y	Y
I	X	X	Z	Z	I	I	Y	Y	Y	Y
I	Z	Z	Z	Z	Z	Z	Z	Z	Z	Z
Z	X	I	I	Z	Z	X	I	Y	X	Z
Y	I	Z	I	Y	Z	Y	Y	Z	X	I

(c)

FIG. 3. (a) Stabilizers for an $[\![8,3,3]\!]$ code. (b) Stabilizers for a $[\![10,4,3]\!]$ code, based on the $[\![5,1,3]\!]$ code [26]. (c) Stabilizers for an $[\![11,5,3]\!]$ code.

This property indeed holds for some qubit permutation of each of the stabilizer generators for the $[\![11,5,3]\!]$ code.

For the $[\![10,4,3]\!]$ code, there are permutations that work for stabilizers $X^{\otimes 10}$ and $Z^{\otimes 10}$, which can *detect* any single-qubit error. For example, the order $1,2,3,4,6,7,5,8,9,10$ works for both. (With $1,2,\ldots,10$, the errors $Z_9 X_{10}$ and $Y_6 X_{\{7,8,9,10\}}$ have the same syndrome.) If an error is detected or the flag is triggered, the stabilizers can all be measured without flags to diagnose the problem.

For the $[\![8,3,3]\!]$ code, there are no valid permutations for $X^{\otimes 8}$ and $Z^{\otimes 8}$. Instead, to detect single-qubit errors we can measure with flags $XXYZIYZI$, $ZZIXYIXY$ and $IIZYXZYX$.

Therefore, for all of these codes, two extra qubits are enough to fault-tolerantly extract the syndromes and apply error correction [28].

Flagged error detection for $[\![n, n-2, 2]\!]$ codes. The idea of flagging faults that can spread badly is also useful for error-detecting codes.

For even n, the $[\![n, n-2, 2]\!]$ error-detecting code has stabilizers $X^{\otimes n}$ and $Z^{\otimes n}$, and logical operators $\overline{X}_j = X_1 X_{j+1}$, $\overline{Z}_j = Z_{j+1} Z_n$ for $j = 1, \ldots, n-2$.

Observe that extracting a syndrome with a single extra qubit is not fault tolerant because, for example, a Z fault at either location indicated with a ★ in Fig. 4(a) results in an undetectable logical error. With a flag qubit, the circuit is fault tolerant; any single fault is either detectable or creates no data error.

This approach works for any n. One way of interpreting it is that the extra qubit is encoded into the two-qubit Z-error detecting code, with stabilizers XX and logical operators $\overline{X} = XI$ and $\overline{Z} = ZZ$. This code detects the single Z faults that can propagate back to the data.

Simulations and conclusion. In order to get a sense for the practicality of the two-qubit error-correction schemes,

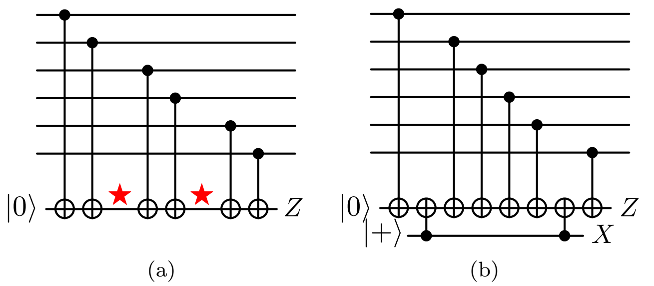


FIG. 4. (a) Circuit to extract the $Z^{\otimes n}$ syndrome. (b) Adding a flag makes it fault tolerant.

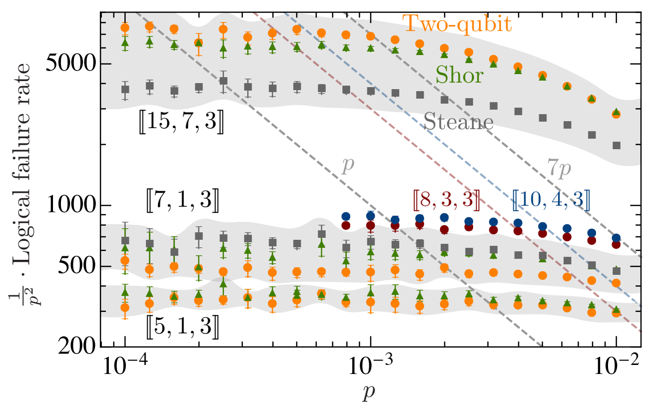


FIG. 5. Comparison to previous error-correction schemes. Two-qubit error correction (•) slightly outperforms Shor-style correction (▲) for the [[5, 1, 3]] code, and both Shor- and Steane-style (■) error correction for the [[7, 1, 3]] code. For the [[15, 7, 3]] code, Steane's method performs best. Logical error rates are plotted divided by p^2 to reveal leading-order coefficients, and $p, 3p, 4p$ and $7p$, i.e., error rates without encoding, are plotted to help judge pseudo-thresholds [29]. Two-qubit error correction for the [[8, 3, 3]] (•) and [[10, 4, 3]] (•) codes can achieve higher pseudo-thresholds.

we simulate error correction using a standard depolarizing noise error model on the one- and two-qubit operations [30]. (The source code is available [19].) Figure 5 shows the results from simulating at least 10^6 consecutive error-correction rounds for each value of the CNOT failure rate p. Observe that for the [[15, 7, 3]] code, Steane-style error correction, which extracts all stabilizer syndromes at once, performs better than either the Shor-style or two-qubit procedures. For the [[5, 1, 3]] and [[7, 1, 3]] codes, the different error-correction methods are all very close. By carefully using their stabilizer generators in the flagged schemes, multi-qubit block codes like the [[8, 3, 3]] and [[10, 4, 3]] codes give higher pseudo-thresholds.

Note that these simulations do not introduce memory errors on resting qubits, nor add errors for moving qubits into position to apply the gates. See [19] for simulations with rest errors; as one may expect, the advantage of the two-qubit schemes over Steane's scheme could reverse when the memory error increases.

We have focused on extracting syndromes using two extra qubits in order to minimize the qubit overhead. With just one more qubit, however, fault-tolerant syndrome extraction becomes considerably simpler. Consider for example the circuit in Fig. 6(a) for extracting a $ZZZZ$ syndrome. Every CNOT gate into the syndrome qubit has its own flag, to catch Z faults that can lead to correlated Z errors on the data. The flags allow for closely localizing any fault, thereby easing error recovery. For example, if there is a single fault and only the second flag above is triggered, then the Z error on the data can be $IIII$, $IZZZ$ or $IIZZ$. The regions covered by the flags overlap so that no Z fault is missed. This technique is most effective if qubit initialization and measurement is fast. Less extreme versions of the technique, in which some gates share flags, can also be used. In [21] we use variants of this technique to apply operations fault tolerantly to data in a block code, with little qubit overhead.

For a large code with many stabilizers, and especially with rest errors, it can be inefficient to extract the syndromes one at a time. The circuit in Fig. 6(b) extracts two of the [[7, 1, 3]] code's syndromes at once, using a shared flag. Figure 6(c) uses a shared flag to extract all three Z syndromes. A single fault can lead to at most a weight-one X error and, if the flag is not triggered, a weight-one Z error. If the flag is triggered, the gates are arranged so that the different Z errors are distinguishable. The circuit uses four qubits and 15 CNOT gates, versus seven qubits and 25 CNOT gates for Steane-style extraction with the decoding trick [15]; however the syndrome 001 can occur with errors $\mathbf{1}$, Z_1, Z_3 or Z_5 and so, unlike in Steane's scheme, must be verified before applying a correction.

The design space for fault-tolerant error correction thus expands considerably with more allowed qubits. A natural problem is to extend the flag technique to medium-size codes of higher distance [31].

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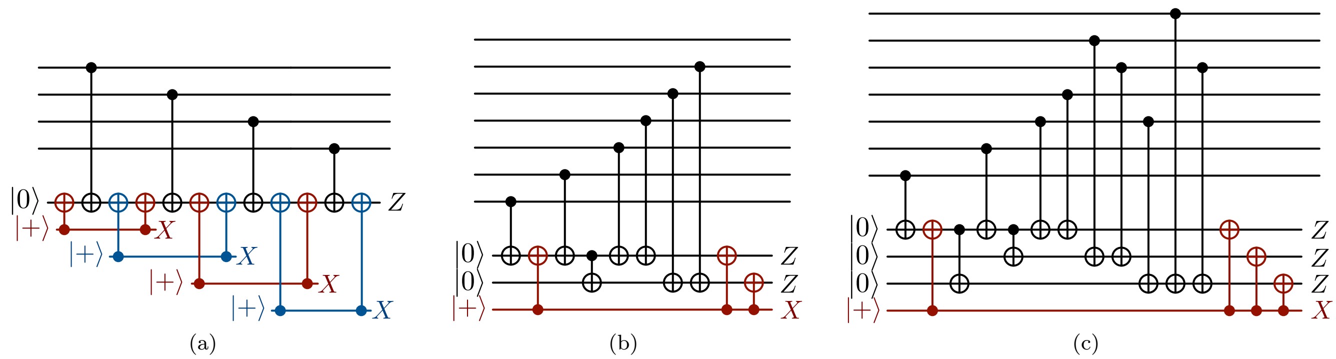

FIG. 6. (a) Overlapping flags can closely localize Z faults. (b, c) Extracting multiple syndromes with a shared flag.

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