Parity-Controlled $2\pi$ Josephson Effect Mediated by Majorana Kramers Pairs

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Parity-controlled $2\pi$ Josephson effect mediated by Majorana Kramers pairs

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We study a time-reversal-invariant topological superconductor island hosting spatially separated Majorana Kramers pairs, with weak tunnel couplings to two $s$-wave superconducting leads. When the topological superconductor island is in the Coulomb blockade regime, we predict that a Josephson current flows between the two leads due to a non-local transfer of Cooper pairs mediated by the Majorana Kramers pairs. Interestingly, we find that the sign of the Josephson current is controlled by the joint parity of all four Majorana bound states on the island. Consequently, this parity-controlled Josephson effect can be used for qubit read-out in Majorana-based quantum computing.

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The past years have shown rapid progress towards the realization of topological superconductors (TSCs) hosting spatially separated Majorana bound states (MBSs) [1–3], which may be useful in building a robust quantum computer. Promising platforms for TSCs to date include hybrid superconductor (SC) - semiconductor nanowire devices under magnetic fields [4–9], chains of magnetic atoms on top of a SC substrate [10–16] as well as vortices in SC-topological insulator heterostructures [17–19]. While all of these setups are designed to search for unpaired MBSs, it was predicted that topological superconductivity also exists in time-reversal-invariant (TRI) systems and gives rise to Kramers doublets of MBSs or Majorana Kramers pairs (MKPs) [20]. In particular, a one-dimensional TRI TSC wire hosts spatially separated MKPs at its two ends. Despite consisting of two MBSs, an isolated MKP is a robust zero-energy degree of freedom protected by time reversal symmetry.

Candidate systems for realizing such TRI topological superconductors comprise nanowires contacted to unconventional SCs [21–24], Josephson $\pi$-junctions in proximitized nanowires and topological insulators [25–28] as well as setups of two nanowires or two topological insulator systems coupled via a conventional $s$-wave SC [29–32]. Additionally, it was pointed out recently that TSCs could appear in systems with an emergent time-reversal symmetry [33–36]. While various schemes were put forward to detect the MKPs in such systems [37–42], novel properties of MKPs and TRI TSCs remain to be explored.

Here, we study the Josephson effect in a mesoscopic TRI TSC island tunnel coupled to two $s$-wave superconducting leads via two spatially separated MKPs, see Fig. 1(a). When the island is in the Coulomb blockade regime, we show that a finite Josephson current flows due to higher order co-tunnelling processes in which Cooper pairs in the SC leads tunnel in and out of the spatially separated MKPs localized at opposite ends of the island. We find that the sign of the resulting Josephson current is controlled by the joint parity of the two MKPs. For the case of odd joint parity, the two SC leads form a Josephson $\pi$-junction, whereas for even joint parity the two SC leads form a Josephson 0-junction. Besides being a robust and easily accessible property of MKPs, we hope that the sign reversal of the Josephson current will prove useful for qubit read-out in Majorana-based quantum computing [43–51].

Model. We consider a mesoscopic TRI TSC island connected to the ground by a capacitor and weakly coupled to two $s$-wave SC leads, see Fig. 1(a). The two SC leads $\ell = L,R$ are described by the BCS (Bardeen-Cooper-
Schrieffer) Hamiltonian,
\[ H_0 = \sum_{\ell=L,R} \sum_{k} \Psi_{\ell,k}^\dagger (\xi_k \eta_k + \Delta_{\ell} \eta_k e^{i\phi_{\ell}} \eta_k) \Psi_{\ell,k}. \]  

Here, \( \Psi_{\ell,k} = (c_{\ell,k^s}, c_{\ell,k^s}^\dagger)^T \) is a Nambu spinor with \( c_{\ell,k^s} \) the electron annihilation operator, where \( k \) denotes single-particle states with normal state dispersion \( \xi_k \) and Kramers index \( s = \uparrow, \downarrow \). By definition, spin-wave pairing occurs between Kramers pairs \((k, s)\) and \(-(k, -s)\), resulting in the superconducting gap \( \Delta_\ell \). The SC leads are then described by the Hamiltonian via quasiparticle states in the island are negligible. The gap in the island is sufficiently large so that virtual transitions across a capacitor with capacitance \( C \) increases/decreases the total charge of the TRI TSC island by one charge unit, \( [n, e^{\pm i\phi/2}] = \pm e^{\pm i\phi/2} \) while the MBS operators \( \gamma_{\ell,s} \) change the electron number parity in the TRI TSC island \([53]\). We assume that the SC leads through the TRI TSC island. For states with \( n \) electrons on the island, but the intermediate steps of 

Due to the superconducting gap \( \Delta \) in the SC leads, single charge transfer across the TRI TSC island is suppressed at low energy. Cooper pair transport occurring separately between each SC lead and the island is also forbidden, as these processes alter the charge of the island by \( 2e \) and thereby leak out of the low-energy Hilbert space. Hence, upon fourth order in the tunneling amplitudes \( \lambda_\ell \), only two types of co-tunneling processes give rise to coherent Josephson coupling between the two SC leads. These processes transfer charge \( 2e \) between the two SC leads through the TRI TSC island. For states with \( n_0 \) electrons on the island, the transfer of a \( 2e \)-charged Cooper pair across the junction entails four steps of subsequently adding and removing electrons on the island, see Fig. 2(a). This is also the case for states with \( n_0 + 1 \) electrons on the island, but the intermediate steps of adding and removing charges are reversed, see Fig. 2(b).

The amplitude of these processes at and near resonance are derived in the limit of weak tunnel coupling, \( \Gamma_\ell \equiv \pi \nu_\ell |\lambda_\ell|^2 \ll \Delta \) with \( \nu_\ell \) the normal-state density of states per spin of the \( \ell \)-SC at the Fermi energy \([52]\). The resulting effective Hamiltonian acting on the reduced Hilbert space consisting of the BCS ground states of the SC leads and the charge states \( n_0 \) and \( n_0 + 1 \) of the mesoscopic TRI TSC island reads,

\[ H_{\text{eff}} = \frac{\delta}{2} \tau_z \left( \gamma_{R,\uparrow} \gamma_{L,\uparrow} \gamma_{R,\downarrow} \gamma_{L,\downarrow} \right) \left( J_0 + J_1 \delta \right) \cos(\varphi_L - \varphi_R) \] 

where \( \tau_z = \pm 1 \) denotes the charge state \( n_0 \) and \( n_0 + 1 \) in the island, respectively. Here, the first term describes the energy splitting \( \delta \) of the two charge states due to detuning the gate charge \( Q_0 \) away from the resonant point \( Q_0/e = n_0 + 1/2 \). Moreover, \( J_0 \) is the Josephson coupling
The Josephson current between the SC leads is given by \( I = \frac{2e}{\hbar} \left( J_0 \sigma_z + \frac{J_1 \delta}{\Delta} \right) \sin(\varphi_L - \varphi_R), \) 

(9)
provide a way of measuring the quasiparticle poisoning rate.

(3) Eq. (9) shows that on resonance ($\delta = 0$), $J_1 = 0$, i.e., the magnitude of critical current in even and odd parity branches is identical. Away from resonance ($\delta \neq 0$), $J_1 \neq 0$. Hence this symmetry is lifted and the critical current mediated by the TSC island in even or odd configurations differs in magnitude. When the even parity state is higher (lower) in energy $\delta > 0$ ($< 0$), the corresponding critical current is larger (smaller) in magnitude, see Eq. (9) and Fig. 2(c).

(4) The Josephson coupling for the limit of large charging energy ($U \gg \pi \nu e |\lambda|^2$) is qualitatively different from the zero charging energy case. In the latter case the dominant contributions are of second-order in the tunnel amplitudes leading to a Josephson coupling $\propto \sin \varphi_L$ [37]. In the intermediate charging energy regime ($U \sim \pi \nu e |\lambda|^2$) both sinusoidal and cosinusoidal contributions are present yielding an interaction-dependent anomalous phase shift in the current-phase relation that interpolates between the zero and large charging energy limit.

**Josephson current near a Coulomb valley.** In this section, we show that the proposed parity-controlled Josephson effect is more general and also arises near a Coulomb valley when $Q_0/e$ is close to an integer value, $2N + 1$ or $2N$, so that the ground states of the island consist of either an odd number of electrons, $n_0 = 2N + 1$, or an even number of electrons, $n_0 = 2N$.

Under this condition, Cooper pair transport occurs via virtually excited states of order $U$ on the island. Up to fourth order in the couplings $\lambda_t$, three types of cotunneling processes contribute to the Josephson coupling: The first type of process involves subsequently adding and removing a unit of charge on the island. For the second type of process, the first two intermediate steps involve adding/removing a charge on the island, while in the final two intermediate steps this order of adding/removing a charge is reversed. In the third type of process, a Cooper pair from one lead is added/removed on the island in the first two intermediate steps, which alters the island charge by $2e$. Subsequently, the Cooper pair is again removed/added from/to the other lead in the final two intermediate steps so that the island returns to its ground state. Importantly, the processes of the second and third type involve intermediate charge states $n_0 = 1$, $n_0 \pm 2$, which are energetically unfavourable in the close-to-resonance case, but in the Coulomb valley case, should be included.

The amplitudes of the processes can be calculated in the limit of weak tunnel couplings, $\Gamma_t \ll \Delta, U$, using fourth-order perturbation theory. The resulting effective Hamiltonian acting on the BCS ground states of the SC leads and the charge ground states on the island reads,

$$H_{\text{eff}}' = -(\gamma_R \gamma_L \gamma_R^+ \gamma_L^+ J') \cos(\varphi_L - \varphi_R).$$

Here, we have introduced the coupling constant $J' = J_0 + J_1' + J_2'$ with,

$$J_0' = \frac{32 \Gamma_L \Gamma_R}{\pi^2 \Delta} \int_1^\infty \frac{dx \ dy}{f(x) f(y) [f(x) + f(y)] g(x) g(y)}$$

$$J_1' = \frac{32 \Gamma_L \Gamma_R}{\pi^2 \Delta} \int_1^\infty \frac{dx \ dy}{f(x) f(y) [f(x) + f(y)] g(x)^2}$$

and $g(x) = \sqrt{1 + x^2 + U' / \Delta}$. The effective Hamiltonian given in Eq. (10) is the second main result of our work.

Crucially, we observe that the direct coupling of the effective Hamiltonian to the joint parity $\gamma_R \gamma_L \gamma_R^+ \gamma_L^+$ is preserved near a Coulomb valley. For the simplest case when the joint parity is fixed by the total island charge mod 2,

$$\gamma_L \gamma_R \gamma_L \gamma_R^+ = (-1)^{n_0}$$

the resulting Josephson current is given by,

$$I' = (-1)^{n_0} (2e / h) J' \sin(\varphi_L - \varphi_R).$$

We want to emphasize three features of this result:

(1) Unlike in the close-to-resonance case, the Josephson current consists of only a single branch for either an even parity ground state, $n_0 = 2N$, or an odd parity ground state, $n_0 = 2N + 1$. However, the sign of the critical current $I_0 = (-1)^{n_0} (2e / h) J'$ remains to be a direct measure of the joint parity $\gamma_R \gamma_L \gamma_R^+ \gamma_L^+$ through the gauge constraint given in Eq. (12).

(2) In comparison to the close-to-resonance case, the sign of the supercurrent is expected to be more stable against quasiparticle poisoning events due to the large charging energy.

(3) At the Coulomb valleys, the magnitude of the critical current is identical for both even and odd configurations. This behavior is in contrast with weak links of two SC leads coupled via a quantum dot, where odd and even charge states of the quantum dot create Josephson 0- and $\pi$-junctions, respectively, but with critical current generally of different magnitude [57].

Before closing, we point out that under rather general conditions no Josephson current is observed when the TRI TSC island is replaced by a time-reversal-breaking TSC island in symmetry class D [58]. This is because after a proper spin basis transformation, a non-degenerate MBS in the TSC island couple only to a single spin species [59] and not to both spin species as MKPs do in the case for a TRI TSC island. Finally, a parity-controlled Josephson effect can also appear for a trivially species [59] and not to both spin species as MKPs do in the case for a TRI TSC island. Finally, a parity-controlled Josephson effect can also appear for a trivially

**Conclusions.** We have shown that in a weak link of two $s$-wave SCs coupled via a TRI TSC island, a Josephson current can flow due to Cooper pairs tunneling in
and out of spatially separated MKPs. We have demonstrated that the sign of the resulting Josephson current is fixed by the joint parity of the four MBSs on the island. This parity-controlled Josephson effect can hence be used as a read-out mechanism for the joint parity, a key requirement in Majorana-based quantum computing [43–51].

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[52] In the Supplemental Material, we provide more details on the derivation of tunneling Hamiltonian coupling the SC leads to the TRI TSC island as well as on the derivation of the effective Hamiltonians describing the parity-controlled Josephson effect.


