Parity-Controlled 2π Josephson Effect Mediated by Majorana Kramers Pairs

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Parity-controlled 2π Josephson effect mediated by Majorana Kramers pairs

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We study a time-reversal-invariant topological superconductor island hosting spatially separated Majorana Kramers pairs, with weak tunnel couplings to two s-wave superconducting leads. When the topological superconductor island is in the Coulomb blockade regime, we predict that a Josephson current flows between the two leads due to a non-local transfer of Cooper pairs mediated by the Majorana Kramers pairs. Interestingly, we find that the sign of the Josephson current is controlled by the joint parity of all four Majorana bound states on the island. Consequently, this parity-controlled Josephson effect can be used for qubit read-out in Majorana-based quantum computing.

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The past years have shown rapid progress towards the realization of topological superconductors (TSCs) hosting spatially separated Majorana bound states (MBSs) [1–3], which may be useful in building a robust quantum computer. Promising platforms for TSCs to date include hybrid superconductor (SC) - semiconductor nanowire devices under magnetic fields [4–9], chains of magnetic atoms on top of a SC substrate [10–16] as well as vortices in SC-topological insulator heterostructures [17–19]. While all of these setups are designed to search for unpaired MBSs, it was predicted that topological superconductivity also exists in time-reversal-invariant (TRI) systems and gives rise to Kramers doublets of MBSs or Majorana Kramers pairs (MKPs) [20]. In particular, a one-dimensional TRI TSC wire hosts spatially separated MKPs at its two ends. Despite consisting of two MBSs, an isolated MKP is a robust zero-energy degree of freedom protected by time reversal symmetry.

Candidate systems for realizing such TRI topological superconductors comprise nanowires contacted to unconventional SCs [21–24], Josephson π-junctions in proximitized nanowires and topological insulators [25–28] as well as setups of two nanowires or two topological insulator systems coupled via a conventional s-wave SC [29–32]. Additionally, it was pointed out recently that TSCs could appear in systems with an emergent time-reversal symmetry [33–36]. While various schemes were put forward to detect the MKPs in such systems [37–42], novel properties of MKPs and TRI TSCs remain to be explored.

Here, we study the Josephson effect in a mesoscopic TRI TSC island tunnel coupled to two s-wave superconducting leads via two spatially separated MKPs, see Fig. 1(a). When the island is in the Coulomb blockade regime, we show that a finite Josephson current flows due to higher order co-tunnelling processes in which Cooper pairs in the SC leads tunnel in and out of the spatially separated MKPs localized at opposite ends of the island. We find that the sign of the resulting Josephson current is controlled by the joint parity of the two MKPs. For the case of odd joint parity, the two SC leads form a Josephson π-junction, whereas for even joint parity the two SC leads form a Josephson 0-junction. Besides being a robust and easily accessible property of MKPs, we hope that the sign reversal of the Josephson current will prove useful for qubit read-out in Majorana-based quantum computing [43–51].

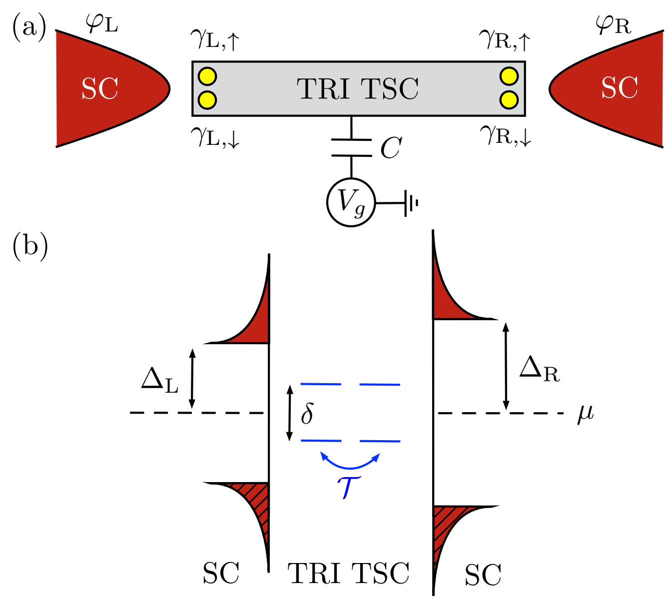

FIG. 1. (Color online) (a) A weak link of two s-wave SC leads $\ell =$ L,R (red) with SC phases φ_ℓ which are coupled to a mesoscopic TRI TSC island (gray). The TRI TSC island hosts a MKP $\gamma_{\ell,s}$ with $s = \uparrow,\downarrow$ (yellow) at each boundary. The island is grounded by a capacitor with capacitance C and thereby attains a finite charging energy which is tunable via an external gate voltage V_g. (b) Schematic energy spectrum of the island and the two SC leads with superconducting gaps Δ_ℓ close to a resonance. The low-energy charge states of the island (blue) are related by time-reversal symmetry $\mathcal{T}$ and are split by an amount δ with $|\delta| \ll \Delta_\ell$ due to a finite detuning away from resonance. The island superconducting gap is assumed to be the largest energy scale.

Model. We consider a mesoscopic TRI TSC island connected to the ground by a capacitor and weakly coupled to two s-wave SC leads, see Fig. 1(a). The two SC leads $\ell =$ L,R are described by the BCS (Bardeen-Cooper-

Schrieffer) Hamiltonian,

$$H_0 = \sum_{\ell=\mathrm{L,R}} \sum_{\mathbf{k}} \Psi^\dagger_{\ell,\mathbf{k}} \left(\xi_{\mathbf{k}}\eta_z + \Delta_\ell \eta_x e^{i\varphi_\ell \eta_z}\right) \Psi_{\ell,\mathbf{k}}. \qquad (1)$$

Here, $\Psi_{\ell,\mathbf{k}} = (c_{\ell,\mathbf{k}\uparrow}, c^\dagger_{\ell,-\mathbf{k}\downarrow})^T$ is a Nambu spinor with $c_{\ell,\mathbf{k}s}$ the electron annihilation operator, where $\mathbf{k}$ denotes single-particle states with normal state dispersion $\xi_{\mathbf{k}}$ and Kramers index $s =\uparrow,\downarrow$. By definition, s-wave pairing occurs between Kramers pairs $(\mathbf{k}, s)$ and $(-\mathbf{k}, -s)$, resulting in the superconducting gap Δ_ℓ. The SC phase difference $\varphi_\mathrm{L} - \varphi_\mathrm{R}$ is controlled by a flux-threaded SC loop far away from the junction so that the effect of the magnetic field in the junction region is zero. The Pauli matrices acting in Nambu space are $\eta_{x,y,z}$. We assume the magnitudes of the superconducting gaps are identical, $\Delta \equiv \Delta_\mathrm{L} = \Delta_\mathrm{R}$.

The TRI TSC island hosts a MKP $\gamma_{\ell,s}$ at each boundary. The two members of a MKP are related by time-reversal symmetry,

$$\mathcal{T}\gamma_{\ell,\uparrow}\mathcal{T}^{-1} = \gamma_{\ell,\downarrow},\ \mathcal{T}\gamma_{\ell,\downarrow}\mathcal{T}^{-1} = -\gamma_{\ell,\uparrow}. \qquad (2)$$

We assume that the island is much wider than the MBS localization lengths, so that the wavefunction overlap of MKP on opposite boundaries is negligible. Since MBSs are zero-energy degrees of freedom hosting unpaired electrons without energy cost, the TRI TSC island can accommodate even and odd numbers of electrons on equal ground. For a TRI TSC island of mesoscopic size, there is also a finite charging energy given by

$$U_C(n) = (ne - Q_0)^2/2C. \qquad (3)$$

Here, Q_0 is a gate charge that is continuously tunable via a gate voltage V_g across a capacitor with capacitance C.

Finally, we introduce the tunnel coupling between the TRI TSC island and the s-wave SC leads. We assume temperature is sufficiently small compared to the charging energy $U \equiv e^2/2C$ and the superconducting gaps of both the SC leads and the island, so that no quasiparticle states are occupied with a notable probability and the Josephson current is predominantly carried by the ground state of the junction. We assume that the SC gap in the island is sufficiently large so that virtual transitions via quasiparticle states in the island are negligible.

Single-particle tunneling between MKPs in the island and the SC leads is then described by the Hamiltonian

$$H_T = \sum_{\ell=\mathrm{L,R}} \sum_{\mathbf{k},s',s} \lambda_{\ell ss'} c^\dagger_{\ell,\mathbf{k}s'} \gamma_{\ell,s} e^{-i\phi/2} + \mathrm{H.c.} \qquad (4)$$

Here, the tunneling amplitudes at the junction between the island and the lead $\lambda_{\ell ss'}$ are allowed to take the most general form, *i.e.*, complex and spin-dependent. Time reversal symmetry implies $\lambda^*_{\ell ss'} = (s_y)_{st}\lambda_{\ell tt'}(s_y)_{t's'}$ with $s_{x,y,z}$ denoting the Pauli matrices in spin-space. We note that one can always choose a proper spin basis transformation so that the tunneling amplitude becomes real and spin-independent, *i.e.*, $\lambda_{\ell ss'} = \lambda_\ell \delta_{ss'}$. Without loss of generality, this choice will be adopted below [52].

The operator $e^{\pm i\phi/2}$ in Eq. (4) increases/decreases the total charge of the TRI TSC island by one charge unit, $[n, e^{\pm i\phi/2}] = \pm e^{\pm i\phi/2}$ while the MBS operators $\gamma_{\ell,s}$ change the electron number parity in the TRI TSC island [53]. We remark that the MBSs at one end of the island do not couple to the SC lead at the opposite end, as the MBS localization length is assumed to be much shorter than the island length. To summarize, the full Hamiltonian of our setup is $H = H_0 + U_C(n) + H_T$.

Josephson current near a resonance. We now show that a Josephson current occurs due to Cooper pair tunnelling between the TRI TSC island and the two SC leads enabled by the two MKPs. We first focus on the near-resonant case, $|\delta| \ll \Delta$ with $\delta \equiv U_C(n_0) - U_C(n_0+1)$. This allows us to truncate the Hilbert space of the island retaining only the states with n_0 and n_0+1 units of charge, see Fig. 1(b). All remaining charge states are separated from this low-energy subspace by a large charging energy, $U \gg \Delta \gg \pi\nu_\ell|\lambda_\ell|^2$, and hence have negligible contribution to the Josephson current.

Due to the superconducting gap Δ in the SC leads, single charge transfer across the TRI TSC island is suppressed at low energy. Cooper pair transport occurring separately between each SC lead and the island is also forbidden, as these processes alter the charge of the island by $2e$ and thereby leak out of the low-energy Hilbert space. Hence, up to fourth order in the tunneling amplitudes λ_ℓ, only two types of co-tunnelling processes give rise to coherent Josephson coupling between the two SC leads. These processes transfer charge $2e$ between the two SC leads through the TRI TSC island. For states with n_0 electrons on the island, the transfer of a $2e$-charged Cooper pair across the junction entails four steps of subsequently adding and removing electrons on the island, see Fig. 2(a). This is also the case for states with n_0+1 electrons on the island, but the intermediate steps of adding and removing charges are reversed, see Fig. 2(b).

The amplitude of these processes at and near resonance are derived in the limit of weak tunnel coupling, $\Gamma_\ell \equiv \pi\nu_\ell|\lambda_\ell|^2 \ll \Delta$ with ν_ℓ the normal-state density of states per spin of the ℓ-SC at the Fermi energy [52]. The resulting effective Hamiltonian acting on the reduced Hilbert space consisting of the BCS ground states of the SC leads and the charge states n_0 and n_0+1 of the mesoscopic TRI TSC island reads,

$$H_\mathrm{eff} = \frac{\delta}{2}\tau_z - (\gamma_{\mathrm{R},\uparrow}\gamma_{\mathrm{L},\uparrow}\gamma_{\mathrm{R},\downarrow}\gamma_{\mathrm{L},\downarrow}) \left(J_0 + \frac{J_1\delta}{\Delta}\tau_z\right)\cos(\varphi_\mathrm{L} - \varphi_\mathrm{R}) \qquad (5)$$

where $\tau_z = \pm 1$ denotes the charge state n_0 and n_0+1 in the island, respectively. Here, the first term describes the energy splitting δ of the two charge states due to detuning the gate charge Q_0 away from the resonant point $Q_0/e = n_0 + 1/2$. Moreover, J_0 is the Josephson coupling

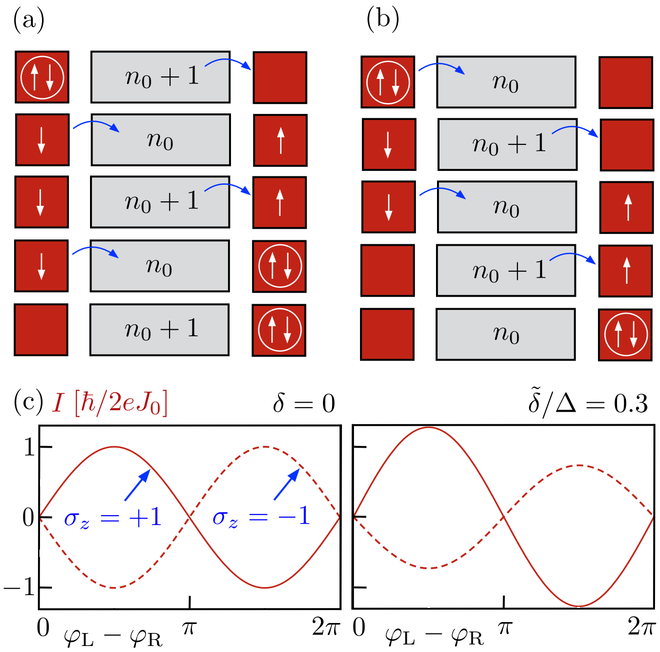

FIG. 2. (Color online) (a) Typical process for the Cooper pair transport between the SC leads via the TRI TSC island which in this case initially carries $n_0 + 1$ units of charge. (b) Same as (a) but here the island initially carries n_0 units of charge. Compared to (a), the intermediate steps of adding/removing a charge from the TRI TSC island are reversed. (c) Josephson current I (in units of $\hbar/2eJ_0$) versus the SC phase difference $\varphi_L - \varphi_R$ for $\delta = 0$ (left panel) and $\tilde{\delta}/\Delta = 0.3$ (right panel). If the joint parity of the MBSs is even, $\sigma_z = +1$, the weak link forms a Josephson 0-junction. Otherwise, it forms a Josephson π-junction. At resonance, when $\delta = 0$, the magnitude of the critical current is identical for both the even and odd parity branches (left panel). This symmetry is lifted away from resonance when $\delta \neq 0$ (right panel).

at resonance, while J_1 is the lowest-order correction for a small detuning δ/Δ away from resonance. Their expressions are given by

$$J_0 = \frac{16\Gamma_L\Gamma_R}{\pi^2\Delta}\int_1^\infty \frac{dx\, dy}{[f(x)+f(y)]\,[f(x)f(y)]^2}$$
$$J_1 = \frac{16\Gamma_L\Gamma_R}{\pi^2\Delta}\left(\frac{3}{2}-\sqrt{2}\right), \tag{6}$$

where $f(x) \equiv \sqrt{1+x^2}$.

The effective Hamiltonian given in Eq. (5) is the first main result. Notably, it directly relates the Josephson current to the joint fermion parity of the four MBSs on the island, $\gamma_{R,\uparrow}\gamma_{L,\uparrow}\gamma_{R,\downarrow}\gamma_{L,\downarrow}$. Depending on the fermion parity being even or odd, $\gamma_{R,\uparrow}\gamma_{L,\uparrow}\gamma_{R,\downarrow}\gamma_{L,\downarrow} = \pm 1$, the Josephson current between the SC leads is given by

$$I = \pm\frac{2e}{\hbar}\left(J_0 + \frac{J_1\delta}{\Delta}\tau_z\right)\sin(\varphi_L - \varphi_R). \tag{7}$$

Importantly, Eq. (5) also applies to the general case of a TRI TSC island hosting any number of spatially separated MKPs. When two of these MKPs couple to separate SC leads, they mediate a Josephson current given by (7) where $\pm$ denotes their joint fermion parity being even or odd, respectively.

For the simplest case of a TSC island with only two MKPs, in the absence of quasiparticle poisoning, the joint parity is given by the total island charge mod 2 [53, 54], $\gamma_{L,\uparrow}\gamma_{R,\uparrow}\gamma_{L,\downarrow}\gamma_{R,\downarrow} = (-1)^n$. After truncating the Hilbert space of the island to two charge states $n = n_0, n_0+1$, the joint parity becomes

$$\gamma_{L,\uparrow}\gamma_{R,\uparrow}\gamma_{L,\downarrow}\gamma_{R,\downarrow} = (-1)^{n_0}\tau_z. \tag{8}$$

The Josephson current given in Eq. (7) then simplifies to

$$I = \frac{2e}{\hbar}\left(J_0\sigma_z + \frac{J_1\tilde{\delta}}{\Delta}\right)\sin(\varphi_L - \varphi_R), \tag{9}$$

where $\sigma_z \equiv (-1)^{n_0}\tau_z = \pm 1$ denotes the even- (odd-) parity member of the two nearly-degenerate charge states n_0 and n_0+1 in the TRI TSC island, and $\tilde{\delta} \equiv (-1)^{n_0}\delta$ is the energy difference of the $\sigma_z = +1$ and $\sigma_z = -1$ state. Four aspects are noteworthy:

(1) The Josephson current between the SC leads is mediated solely by the MKPs localized at opposite boundaries of the TRI TSC island with its magnitude being determined by the coupling strengths at the two junctions only. This is remarkable because the MKPs have essentially zero wavefunction overlap and so no *direct* coupling exists between the MKPs in Eq. (4). We remark that the proposed Josephson effect may require longer measurement times as it emerges only at fourth order in the tunnelling amplitudes.

(2) For a given phase difference $\varphi_L - \varphi_R$, the sign of the Josephson current in Eq.(7) depends on the joint fermion parity of the four MBSs. For a given fermion parity state, the Josephson current is 2π periodic with respect to the phase. We hence refer to Eq. (7) as the *parity-controlled 2π Josephson effect*. When the joint parity of the four MBSs is even, $\gamma_{L,\uparrow}\gamma_{R,\uparrow}\gamma_{L,\downarrow}\gamma_{R,\downarrow} = +1$, the critical current is positive, $I_c > 0$, and the superconducting weak link forms a Josephson 0-junction. When the joint parity of the four MBSs is odd, $\gamma_{L,\uparrow}\gamma_{R,\uparrow}\gamma_{L,\downarrow}\gamma_{R,\downarrow} = -1$, the sign of the critical current is negative, $I_c < 0$, and the weak link forms a Josephson π-junction, see also Fig. 2(c).

The parity-controlled Josephson effect has two immediate applications: First, for sufficiently long parity lifetimes, the sign of the critical current permits a direct measurement of the joint parity of *four MBSs* in the island, an essential element for Majorana-based quantum computing [43–51]. In contrast, for time-reversal-breaking TSCs with unpaired MBSs, the sign of the Josephson current in the 4π-periodic Josephson effect only permits measuring the parity of *two MBSs* in the weak limit [55, 56]. Second, the switching times between positive and negative critical currents through the island provide a way of measuring the quasiparticle poisoning rate.

(3) Eq. (9) shows that on resonance ($\delta = 0$), $J_1 = 0$, *i.e.*, the magnitude of critical current in even and odd parity branches is identical. Away from resonance ($\delta \neq 0$), $J_1 \neq 0$. Hence this symmetry is lifted and the critical current mediated by the TSC island in even or odd configurations differs in magnitude. When the even parity state is higher (lower) in energy $\tilde{\delta} > 0$ (< 0), the corresponding critical current is larger (smaller) in magnitude, see Eq. (9) and Fig. 2(c).

(4) The Josephson coupling for the limit of large charging energy ($U \gg \pi\nu_\ell|\lambda_\ell|^2$) is qualitatively different from the zero charging energy case. In the latter case the dominant contributions are of second-order in the tunnel amplitudes leading to a Josephson coupling $\propto \sin\varphi_\ell$ [37]. In the intermediate charging energy regime ($U \sim \pi\nu_\ell|\lambda_\ell|^2$) both sinusoidal and cosinusoidal contributions are present yielding an interaction-dependent anomalous phase shift in the current-phase relation that interpolates between the zero and large charging energy limit.

Josephson current near a Coulomb valley. In this section, we show that the proposed parity-controlled Josephson effect is more general and also arises near a Coulomb valley when Q_0/e is close to an integer value, $2N+1$ or $2N$, so that the ground states of the island consist of either an odd number of electrons, $n_0 = 2N+1$, or an even number of electrons, $n_0 = 2N$.

Under this condition, Cooper pair transport occurs via virtually excited states of order U on the island. Up to fourth order in the couplings λ_ℓ, three types of cotunnelling processes contribute to the Josephson coupling: The first type of process involves subsequently adding and removing a unit of charge on the island. For the second type of process, the first two intermediate steps involve adding/removing a charge on the island, while in the final two intermediate steps this order of adding/removing a charge is reversed. In the third type of process, a Cooper pair from one lead is added/removed on the island in the first two intermediate steps, which alters the island charge by $2e$. Subsequently, the Cooper pair is again removed/added from/to the other lead in the final two intermediate steps so that the island returns to its ground state. Importantly, the processes of the second and third type involve intermediate charge states $n_0 - 1, n_0 \pm 2$, which are energetically unfavourable in the close-to-resonance case, but in the Coulomb valley case, should be included.

The amplitudes of the processes can be calculated in the limit of weak tunnel couplings, $\Gamma_\ell \ll \Delta, U$, using fourth-order perturbation theory. The resulting effective Hamiltonian acting on the BCS ground states of the SC leads and the charge ground states on the island reads,

$$H'_{\rm eff} = -(\gamma_{{\rm R},\uparrow}\gamma_{{\rm L},\uparrow}\gamma_{{\rm R},\downarrow}\gamma_{{\rm L},\downarrow}) J' \cos(\varphi_{\rm L} - \varphi_{\rm R}). \tag{10}$$

Here, we have introduced the coupling constant $J' = J'_0 + J'_1 + J'_2$ with,

$$\begin{aligned} J'_0 &= \frac{32\Gamma_{\rm L}\Gamma_{\rm R}}{\pi^2\Delta}\int_1^\infty \frac{{\rm d}x\ {\rm d}y}{f(x)f(y)\left[f(x)+f(y)\right]g(x)g(y)} \\ J'_1 &= \frac{32\Gamma_{\rm L}\Gamma_{\rm R}}{\pi^2\Delta}\int_1^\infty \frac{{\rm d}x\ {\rm d}y}{f(x)f(y)\left[f(x)+f(y)\right]g(x)^2} \\ J'_2 &= \frac{8\Gamma_{\rm L}\Gamma_{\rm R}}{\pi^2 U}\left[\int_1^\infty \frac{{\rm d}x}{f(x)g(x)}\right]^2, \end{aligned} \tag{11}$$

and $g(x) \equiv \sqrt{1+x^2} + U/\Delta$. The effective Hamiltonian given in Eq. (10) is the second main result of our work.

Crucially, we observe that the direct coupling of the effective Hamiltonian to the joint parity $\gamma_{{\rm R},\uparrow}\gamma_{{\rm L},\uparrow}\gamma_{{\rm R},\downarrow}\gamma_{{\rm L},\downarrow}$ is preserved near a Coulomb valley. For the simplest case when the joint parity is fixed by the total island charge mod 2,

$$\gamma_{{\rm L},\uparrow}\gamma_{{\rm R},\uparrow}\gamma_{{\rm L},\downarrow}\gamma_{{\rm R},\downarrow} = (-1)^{n_0}. \tag{12}$$

the resulting Josephson current is given by,

$$I' = (-1)^{n_0}(2e/\hbar)J' \sin(\varphi_{\rm L} - \varphi_{\rm R}). \tag{13}$$

We want to emphasize three features of this result:

(1) Unlike in the close-to-resonance case, the Josephson current consists of only a single branch for either an even parity ground state, $n_0 = 2N$, or an odd parity ground state, $n_0 = 2N+1$. However, the sign of the critical current $I'_c \equiv (-1)^{n_0}(2e/\hbar)J'$ remains to be a direct measure of the joint parity $\gamma_{{\rm R},\uparrow}\gamma_{{\rm L},\uparrow}\gamma_{{\rm R},\downarrow}\gamma_{{\rm L},\downarrow}$ through the gauge constraint given in Eq. (12).

(2) In comparison to the close-to-resonance case, the sign of the supercurrent is expected to be more stable against quasiparticle poisoning events due to the large charging energy.

(3) At the Coulomb valleys, the magnitude of the critical current is identical for both even and odd configurations. This behavior is in contrast with weak links of two SC leads coupled via a quantum dot, where odd and even charge states of the quantum dot create Josephson 0- and π-junctions, respectively, but with critical current generally of different magnitude [57].

Before closing, we point out that under rather general conditions no Josephson current is observed when the TRI TSC island is replaced by a time-reversal-breaking TSC island in symmetry class D [58]. This is because after a proper spin basis transformation, a non-degenerate MBS in the TSC island couple only to a single spin species [59] and not to both spin species as MKPs do in the case for a TRI TSC island. Finally, a parity-controlled Josephson effect can also appear for a trivially superconducting wire with a low-lying Andreev bound state described by two hybridized MBSs at each end. In this case, the parity-controlled Josephson effect enables the read-out of non-topological Andreev qubits.

Conclusions. We have shown that in a weak link of two s-wave SCs coupled via a TRI TSC island, a Josephson current can flow due to Cooper pairs tunneling in and out of spatially separated MKPs. We have demonstrated that the sign of the resulting Josephson current is fixed by the joint parity of the four MBSs on the island. This *parity-controlled Josephson effect* can hence be used as a read-out mechanism for the joint parity, a key requirement in Majorana-based quantum computing [43–51].

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