Self-Adapted Floquet Dynamics of Ultracold Bosons in a Cavity

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Floquet dynamics of a quantum system subject to periodic modulations of system parameters provide a powerful tool for engineering new quantum matter with exotic properties. While system dynamics are significantly altered, the periodic modulation itself is usually induced externally and independent of Floquet dynamics. Here we propose a new type of Floquet physics for a Bose-Einstein condensate (BEC) subject to a shaken lattice generated inside a cavity, where the shaken lattice and atomic Floquet bands are mutually dependent, resulting in self-adapted Floquet dynamics. In particular, the shaken lattice induces Floquet quasi-energy bands for the BEC, whose back action leads to a self-adapted dynamical normal-superradiant phase transition for the shaken lattice. Such self-adapted Floquet dynamics show two surprising and unique features: i) the normal-superradiant phase transition possesses a hysteresis even without atom interactions; ii) the dynamical atom-cavity steady state could exist at free energy maxima. The atom interactions strongly affect the phase transition of the BEC from zero to finite momenta. Our results provide a powerful platform for exploring self-adapted Floquet physics, which may open an avenue for engineering novel quantum materials.

Introduction.—Floquet physics has been extensively studied in solid state, ultracold atomic, and photonic systems in recent years with significant theoretical and experimental progress [1-14]. In particular, ultracold atoms in periodically driven optical lattices provide a highly controllable and disorder-free platform for studying Floquet physics, yielding many interesting and important phenomena such as coherent ac-induced tunneling and band coupling [15-25], the realization of gauge fields and topological bands [26-37], and the dynamical control of expansion and quantum phase transition of bosonic systems [38-41], etc. These previous studies on Floquet physics assumed that system parameter modulations (e.g., the shaking or moving optical lattices) are determined solely by external driving and do not depend on system dynamics, i.e., no back action of system Floquet states on parameter modulations. Therefore a natural and important question is whether novel Floquet physics can emerge when system Floquet dynamics and parameter modulations are mutually dependent.

In this Letter, we address this important question by studying Floquet dynamics of ultracold boson atoms subject to a shaken optical lattice generated inside an optical cavity. In the past several decades, the interaction between atoms and static cavity fields with atom back actions (no Floquet physics) have been well studied in both theory [42] and experiment [43-53], showcasing rich cavity quantum electrodynamics (QED) physics ranging from few-body problems such as Jaynes-Cummings model [54] to many-body physics such as the Dicke superradiance [55, 56]. However, in these studies, the cavity mode is static without periodic modulations such as shaking or moving.

Here we propose to realize a cavity-mode-induced freely evolving shaken lattice, utilizing transverse pumping and a periodic modulation of the cavity field phase, and study its mutual interaction with a non- or weakly interacting Bose-Einstein condensate (BEC) inside the cavity. While such shaken lattice generates Floquet bands for the BEC, the back action of atom Floquet bands modulates the shaken lattice, leading to a dynamical superradiant phase, where atom Floquet bands and shaken lattice are self-adapted. Such Floquet normal-superradiant phase transition can be dramatically different from non-Floquet one because of the coupling between different Floquet sidebands. In particular, the interplay between intra- and inter-sideband couplings may induce a hysteresis for the Floquet normal-superradiant phase transition of non-interacting atoms, yielding a completely new mechanism different from the well-known interaction-driven hysteresis [57-62]. Surprisingly, the steady state of the atom-cavity system can stabilize at the free energy maximum for dominant inter-sideband coupling because of the non-equilibrium nature of Floquet states. With increasing superradiant field, the Floquet band dispersion gradually evolves from a single minimum to doubly degenerate minima, leading to a second-order phase transition of the BEC. Such transition can be significantly affected by the interaction between atoms through the back action, which changes the critical superradiant field, the Floquet band dispersion and the condensate momenta across the transition.

System.—As shown in Fig. 1(a), two schemes (A and B) can be used to generate shaking cavity mode as \(2E(x, \varphi) \propto k_0x + \varphi(t)\), with \(\hat{z}\) the polarization direction and \(k_0\) the wavenumber. Scheme A employs two electro-optic modulators (EOMs) [63], while scheme B uses two mirrors synchronously driven by piezoelectric transducers (PZTs) [64] to periodically and slowly [comparing with cavity free spectral range (FSR)] modify the optical phase delay by \(\varphi(t) = \varphi_0 + \omega_0t\). The total optical path length does not change, therefore the cavity resonance frequency is not affected. We consider a quasi-one-dimensional (the dynamics in other directions...
are reduced by a deep harmonic trap) BEC prepared in such a cavity, which is pumped by an external transverse laser. The pumping frequency \( \omega_p \) is close to the cavity resonance frequency \( \omega_c \), both of which are detuned far below the atomic transition frequency \( \omega_a \) [see Fig. 1(b)].

After adiabatically eliminating the excited atomic level, we obtain the Hamiltonian of the atom-cavity system in a rotating frame with \( \hbar = 1 \)

\[
    \mathcal{H} = (\Delta_c - u)e^c + \int dx \Psi^\dagger(x) H_0(t) \Psi(x),
\]

where \( c \) is the annihilation operator of the cavity photon, and \( \Psi(x) \) is the matter wave field of atoms in the ground state, \( \Delta_c = \omega_c - \omega_p \) is the cavity mode detuning, and \( u = \frac{\hbar^2}{\Delta_c} \int dx \Psi^\dagger \Psi \cos^2(k_0 x + \varphi) \) is the detuning induced by the atoms, which is typically small and negligible. The single atom Hamiltonian is

\[
    H_0(t) = -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{ext}}(x) + \hat{V}_c(x,t),
\]

with \( V_{\text{ext}}(x) = v_e \cos^2(k_0 x) \) an static external background lattice potential, which can be realized by additional lasers [65]. It gives rise to a static tight-binding atomic band structure \( \varepsilon_k(q_x) = E_k + t \lambda \cos(q_x) \) with band index \( \lambda \) and Bloch momentum \( q_x \in [-k_0, k_0] \) [Fig. 2(a)].

\[
    \hat{V}_c(x,t) = -\eta \frac{\epsilon^{\delta + \phi}}{\sqrt{N_a}} \cos(k_0 x + \varphi(t)) \text{ is the shaking potential induced by the cavity-assisted ac-Stark shift,}
\]

Utilizing the expansion \( e^{i \omega t} \sum_n \frac{1}{\sqrt{N_a}} n^i J_n(f) e^{i \omega_0 t} \cos[k_0 x + \varphi_0 + f \cos(\omega_0 t)] \), we see \( \hat{V}_c(x,t) \) can change the band structure by inducing a sequence of sidebands (i.e. phonon-dressed bands) and couplings between them (see Fig. 2). We choose \( v_e, \omega_0, \varphi_0 \) and \( f \) such that the static \( s \)-band is near resonance with two-phonon-dressed \( \omega = 2 \omega_0 \) \( p \)-band [Fig. 2(a)], therefore only these two bands need be considered for the calculation of new Floquet band structure \( \tilde{\varepsilon}_\lambda(q_x) \).

The first order expansion \( e^{\pm i \omega_0 t} \) is far-off resonance and can be neglected. The zero-th order expansion corresponds to intra-sideband coupling, thus only terms containing \( \cos(k_0 x) \) are nonzero due to the symmetry of the Wannier functions. Similarly, only terms with \( \sin(k_0 x) \) are nonzero for the second-order expansion \( e^{\pm 2i \omega_0 t} \) that couples \( s \) and \( p \) bands. In total, the cavity-assisted potential can be written as [65]

\[
    \hat{V}_c = -\eta_0 \frac{\epsilon^{\delta + \phi}}{\sqrt{N_a}} \cos(k_0 x) + 4\eta \cos(\omega t) \sin(k_0 x),
\]

where \( \eta_0 = \eta J_0(f) \cos(\varphi_0), \eta = \frac{J_0(f)}{2 J_0(1)} \tan(\varphi_0) \) (tunable by \( f \) and \( \varphi_0 \)) is the ratio between inter- and intra-sideband coupling strengths. Note that the spatial period of \( \hat{V}_c(x,t) \) is twice of \( V_{\text{ext}}(x) \), therefore the Brillouin zone (BZ) reduces by half to \( q_x \in [-k_0/2, k_0/2] \) through the band folding. Each band \( \lambda \) is split into two bands \( \lambda_0, \lambda \pi \) \( \lambda = s, p \) as shown in Fig. 2(b) and the lattice potential \( \hat{V}_c \) can only couple 0 and \( \pi \) bands due to the momentum transfer. \( \varphi_0 \) characterizes the relative phase between background lattice and the shaking center of cavity field: for \( \varphi_0 = 0 \) (\( \pi \)), the shaking potential is symmetric (asymmetric) at each site of the background lattice, therefore can only induce intra-sideband (inter-sideband) couplings. Both couplings coexist for \( \varphi_0 \neq j \pi/2 \) (\( j \) is an integer) [65].

**Method.**—Under mean-field approximation, the cavity field satisfies the Langevin equation \( \mathcal{L}(\hat{c}(t)) = \langle [c, \mathcal{H}] \rangle - i\gamma/(2c(t)) \) due to the weak leakage of the high-Q cavity, yielding

\[
    i\dot{\alpha} = (\Delta_c - u - i\gamma/2) \alpha - \eta_0 \Theta(t),
\]

with \( \alpha(t) = \langle c(t) \rangle/\sqrt{N_a} \) and \( \gamma \) the cavity loss rate. Here \( \Theta(t) = N_a^{-1} \int dx \langle \Psi^\dagger \Psi \rangle \cos(k_0 x) + 4\eta \sin(k_0 x) \cos(\omega t) \rangle \) is the atomic density order. The frequency \( \omega \) is chosen to be much larger than \( \eta_0 \Theta, \Delta_c - u, \gamma/2 \) (see section experimental consideration), so that high-order oscillation
where density $F$ find that solving the self-consistent equation Eq. (4) is the cavity field $\Theta$ can be calculated [65]. $\Theta$ in turn drives Floquet states for the BEC, from which the atom den-

\vspace{0.5cm}

\section{Results}

We focus on non-interacting BECs and discuss the interaction effects later. The numerically calculated phase diagrams and corresponding self-adapted Floquet bands with $\omega \gtrsim E_{sp} + |t_s| + |t_p|$ are shown in Fig. 3. For a small $\eta_t (\lesssim 1)$, $\hat{V}_c$ is dominated by the time-independent term that couples $s_0$ and $s_p$ bands, which would lower the energy of $s_0$ band [see the inset in Fig. 3(a)], indicating $\langle \hat{V}_c \rangle = -\eta_0 \phi \Theta_0 < 0$. According to Eq. (4), a non-trivial steady state solution exists only for blue effective detuning $\Delta_c > 0$ [Figs. 3(a) (b)]. The phase transition requires a stronger $\eta_0$ as $\eta_t$ increases, indicating that the transition becomes harder due to the competition between inter- and intra-sideband couplings. As $\Delta_c$ decreases, the critical value of $\eta_0$ required for superradiance first decreases then increases and tends to infinity at $|\Delta_c| = 0$. This is because $V_c$, which drives the atomic density order $\Theta_0$, is proportional to $\phi$ and approaches zero as $|\Delta_c| \rightarrow 0$ [see Eq. (4)]. We find that the solution is located at the minima of $F(\phi)$, which can be expanded as $F(\phi) = a_2 \phi^2 + a_4 \phi^4 + \cdots$. $F(\phi)$ exhibits a continuous transition from a single minimum at $\phi = 0$ to double minima at $\phi \neq 0$ [see the inset in Fig. 3(b)], where $a_2$ and $a_4$ are both positive before the transition, and $a_2$ changes sign when the phase transition (second order) occurs.

For a large $\eta_t (\gg 1)$, $\hat{V}_c$ is dominated by the inter-band coupling between $s_0$ band and $p_\pi$ band that has a lower energy than $s_0$ band, therefore atoms stays at the high-energy excited band and increasing the cavity field would rise the band energy [the inset in Fig. 3(c)] of the BEC, leading to $V_c = -\eta_0 \phi \Theta_0 > 0$. As a result, the non-trivial steady state solution exists only for red effective detuning $\Delta_c < 0$ [Figs. 3(c) (d)]. Surprisingly, the steady state solution is found at the maxima of $F(\phi)$ which exhibits a transition from a single maximum at $\phi = 0$ to double maxima at $\phi \neq 0$ [the inset in Fig. 3(d)] that minimizes the non-equilibrium phase transition of the dynamical steady states which may not minimize the energy. Without Floquet sidebands, such superradiance at energy maximum for $\Delta_c < 0$ would not exist because atoms generally prefer staying in the lowest static band which can only couple with higher static bands.

Moreover, depending on the value of $\eta_t$, the transition can be either continuous (second order) or discontinuous with a hysteresis loop (first order) [Fig. 4(b)]. Such hysteresis originates from the interplay between intrasideband and inter-sideband couplings, which induces a second-order coupling (similar to a two-photon Raman process) between $s_0$ and $p_\pi$ bands mediated by $s_\pi$ and $p_\pi$ bands (see Fig. 2(b)). Notice that the $p_\pi$ band is just below the $s_0$ band, therefore this second-order coupling rises the $s_0$ band by increasing $a_4$. As a result, $a_4$ may change sign (from negative to positive) prior to $a_2$ changes sign (from negative to positive) when the second-order coupling is strong enough, leading to a multi-stability behavior where $F(\phi)$ exhibits three maxima simultaneously.

Hysteresis phenomena are related to strong nonlinearities [78], which are usually induced by strong atom-atom interactions [57, 60]. For example, the strong Ising interaction in the Dicke model can lead to a hysteresis loop.
of the superradiant phase transition [57–59]. However, the hysteresis effect in our system does not need atom-atom interaction at all, and has a completely different mechanism originating from the coupling between Floquet sidebands induced by the shaken cavity mode. Our study offers an excellent example and a realistic system for observing hysteresis effects without atom interactions.

As the superradiant field increases, the $s_0p_0$ band coupling would induce a transition of the Floquet band $\varepsilon_{s_0}(q_x)$ from a single minimum at $q_x = 0$ to doubly degenerate minima at $q_x \neq 0$ [see inset in Fig. 4(a)]. In our system, the $s_0p_0$ band coupling is a Raman-like coupling mediated by $s_\pi$ and $p_\pi$, therefore such transition should be observed when the inter- and intra-sideband couplings coexist. We consider atom-atom interaction (tunable through Feshbach resonance [79]) that is weak and repulsive, therefore the interaction energy is always minimized when the system stays in a single momentum state. We assume that $\varepsilon_{s_0}(q_x)$ is minimized at $\pm q_{m}^n$. For negligibly weak interaction, the condensate momentum $q_c^x$ would locate at one of the band minima (either $+q_{m}^n$ or $-q_{m}^n$) due to spontaneous symmetry breaking [80]. Such a transition in the BEC would also lead to a second order transition [stars in Figs. 4(a) (c)] of the shaking cavity field $\phi_s$ due to the back action.

As the atom-atom interaction increases (still weak enough such that the superradiant phase transition is not affected and the long-time behaviors (heating, atom loss, etc.) are not significant [22, 25, 81]), the phase transition of the BECs from $q_c^x = 0$ to $q_c^x \neq 0$ may be shifted be-

cause $q_c^x \neq \pm q_{m}^n$. $q_c^x$ should be determined by minimizing $\varepsilon_{s_0}(q_x) + \varepsilon_{\text{int}}(q_x)$, with $\varepsilon_{\text{int}}(q_x)$ the momentum-dependent interaction energy,

$$
\varepsilon_{\text{int}}(q_x) = \frac{1}{T} \int dt \int dq_n q^2 |n_{s_0,q_n}(x,t)|^2.
$$

Here $g$ is the interaction constant and $n_{s_0,q_n}(x,t) = \langle \Psi^\dagger \Psi \rangle$ is the atom density. In our system, band mixing enhances spatial modulation of the density, and the interaction energy $\varepsilon_{\text{int}}(q_x)$ is minimized at $q_x = 0$ where the mixing is the smallest. As a result, $q_c^x$ is smaller than $q_{m}^n$, and bosons may be condensed into the local maximum of the single-particle band at $q_x = 0$ [see Figs. 4(b) (d)]. The transition from $q_c^x = 0$ to $q_c^x \neq 0$ for $g > 0$ requires a stronger super-radiant field than the transition for $g = 0$ [see Figs. 4(a) (c)], and it also leads to a second order transition of $\phi_s$ which in turn leads to a transition in $q_c^x$.

**Experimental consideration.**—We consider a high-finesse (low-loss) cavity with $\gamma = 2E_R$ (with $E_R = h^2k_0^2/2m$ the recoil energy). Generally, atoms with a small mass are preferred to obtain a large $E_R$, thus a large $\gamma$, which makes the cavity easy to realize. For example, $^7$Li ($^{23}$Na) atoms has a recoil energy $E_R \sim 40$kHz (10kHz), corresponding to $\gamma = 80$kHz (20kHz), which can be realized with current technique [65, 82]. The shaking frequency $\omega_0$ is about several ten kHz for $^{23}$Na and several hundred kHz for $^7$Li (both are much smaller than the free spectral range $\sim$GHz), and such phase modulation can be implemented by PZT-driving mirrors or low-loss EOMs [65]. The system studied here only involves a change of introducing PZT-driving mirrors or inserting two EOMs into the setups used in the ETH and Hamburg laboratories [45, 49, 52, 53, 82], and thus should be feasible with current technology. Moreover, our model can also be implemented by combining a shaking external lattice and a periodic driving force (e.g., using a periodically modulated magnetic field gradient) [65].

**Discussion.**—We proposed a new type of Floquet physics where the parameter modulations are not only related to external driving, but also mutually coupled with system dynamics. As an example, we studied such Floquet dynamics of BECs in shaking optical lattices, which lead to interesting new phenomena including self-adapted shaking fields, Floquet bands and hysteresis effects. Such self-adapted Floquet physics may also arise in various other systems such as Rydberg-atom or molecule micromaser [83], ion-trap cavity [84] and circuit quantum electrodynamics (circuit QED) [85], etc. For example, in a circuit QED system, the electromagnetic field on the waveguide resonator can be periodically modulated by attaching superconducting quantum interference devices (SQUIDs) to the ends of the resonator, with the SQUIDs driving by external magnetic fluxes [86]. Such a modulated waveguide resonator can couple with su-
perconducting qubits (artificial atoms) and these qubits may also strongly couple with each other, where interesting self-adapted Floquet dynamics may emerge. Such self-adapted Floquet circuit QED may find important applications in quantum information processing and will be addressed in future works. Within the shaking lattice cavity system, interesting physics may also arise by considering strong atom-atom interaction (Bose-Hubbard model [52, 53]) or strong atom-single-photon coupling (limit cycles and chaos [87]), and superradiance of fermion gases (topological bands [88, 89]). In this spirit, our proposal opens up new possibilities for studying self-adapted Floquet physics in various systems, which may pave a way for engineering new exotic quantum matter.

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See Supplemental Material for more details about the cavity-assisted potential, the Floquet bands and energy density, the stability analysis and the experimental consideration, which includes Refs. [66–77].


