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Characterization of topological states via dual multipartite entanglement

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We demonstrate that multipartite entanglement is able to characterize one-dimensional symmetry-protected topological order, which is witnessed by the scaling behavior of the quantum Fisher information of the ground state with respect to the spin operators defined in the dual lattice. We investigate an extended Kitaev chain with a $\mathbf{Z}$ symmetry identified equivalently by winding numbers and paired Majorana zero modes at each end. The topological phases with high winding numbers are detected by the scaling coefficient of the quantum Fisher information density with respect to generators in different dual lattices. Containing richer properties and more complex structures than bipartite entanglement, the dual multipartite entanglement of the topological state has promising applications in robust quantum computation and quantum metrology, and can be generalized to identify topological order in the Kitaev honeycomb model.

Introduction.—In recent years, quantum topological phases [1] in extended systems have become of great significance in modern physics due to its promise for both topological quantum computation [2–7] and condensed matter physics [8, 9]. Topological phase transitions, beyond the Landau symmetry-breaking theory, are described by the change of its topological order or symmetry-protected topological (SPT) order [1]. Topological order [10], e.g. quantum Hall states or spin liquids [11], cannot be described by local order parameters [12, 13] but can be characterized by the long-range entanglement encoded in the states of the systems, such as the topological entanglement entropy [14, 15] and entanglement spectrum [16]. Further enriched by symmetries, SPT phases, corresponding to short-range entangled phases with symmetry-protected edge modes [17–20], are theoretically proposed and experimentally discovered in topological insulators and superconductors [20–26]. These characteristics make topological states robust against local noise, which has emerged as one of the most exciting approaches to realizing topologically protected quantum information processing and fault-tolerant quantum computing [27]. The simplest realization would be the Majorana zero modes (MZMs) at the edges of low-dimensional systems [28–34], e.g., extended Kitaev models [35–38], which have recently been observed in various experimental platforms including nanowire devices [39, 40] and quantum spin liquids [41].

In addition to the fruitful results from bipartite entanglement [14–16], multipartite entanglement [42–45] (witnessed by the quantum Fisher information (QFI) [46–48] with respect to nonlocal operators [49]) displays much richer properties of complex structures of topological states and deserves further investigation. The QFI quantifies useful multipartite entanglement for quantum metrology, which is confirmed by quantum parameter estimation with sub-shot-noise sensitivity [45–48, 50, 51]. Recently, it was shown that the scaling behavior of the QFI with respect to spin operators in the original lattice is sensitive for detecting the topologically nontrivial phases with low winding numbers $\nu = \pm 1, \pm\frac{1}{2}$ [52]. However, we find that topological phases with higher winding numbers cannot be characterized by the QFI with respect to these operators.

In this Letter, we provide a general method to characterize 1D SPT order with higher winding numbers by multipartite entanglement defined in the dual lattice. We focus on an extended Kitaev fermion chain with p-wave superconductivity and a chiral symmetry belonging to the $\mathbf{Z}$-type BDI class [53–55] identified equivalently by high winding numbers and boundary MZMs from the Bogoliubov-de Gennes (BdG) Hamiltonian. Dual multipartite entanglement is signaled by the scaling behavior of the QFI density of the ground state with respect to spin operators by the duality transformation [56–59]. By exploiting the duality of the model, we find that the QFI density in dual lattices, written in terms of string correlation functions (SCFs) [59–61], has a linear scaling behavior versus system size in SPT phases and detects 1D quantum SPT phase transitions. Therefore, together with [52], dual multipartite entanglement can be used to identify SPT order. We also extend our investigation to the Kitaev honeycomb model [62], indicating that our results can be generalized to 2D systems with topological order. Our work reveals the possibility of promising applications of topologically protected multipartite entanglement in robust quantum computation and quantum metrology.

Winding numbers, Majorana zero modes, and topological phase transitions.—We study the extended Kitaev fermion chain with extensive pairing and hopping terms [37],

$$H = \sum_{n=1}^{N_f}\sum_{j=1}^{L}\left(\frac{J_n^+}{2}c_j^\dagger c_{j+n} + \frac{J_n^-}{2}c_j^\dagger c_{j+n}^\dagger + \text{h.c.}\right) - \sum_{j=1}^{L}\mu\left(c_j^\dagger c_j - \frac{1}{2}\right), \tag{1}$$

where L (assumed even) is the total number of sites, N_f denotes the farthest pairing and hopping distance, and the antiperiodic conditions $c_{j+L} = -c_j$ are assumed. The hopping and pairing parameters $J_n^{\pm}$ are all chosen as real numbers, to make the Hamiltonian preserve time-reversal symmetry and belong to the BDI class ($\mathbf{Z}$ type) characterized by a winding number [53, 54]. Through the Jordan-Wigner transformation $c_1 = -\sigma_1^+$, $c_j = -\sigma_j^+ \prod_{i=1}^{j-1} \sigma_i^z$, this spinless fermion model corresponds to the extended Ising model [63–68]

$$H = \sum_{n=1}^{N_f} \sum_{j=1}^{L} \left(\frac{J_n^x}{2} \sigma_j^x \sigma_{j+n}^x + \frac{J_n^y}{2} \sigma_j^y \sigma_{j+n}^y \right) \prod_{l=j+1}^{j+n-1} \sigma_l^z + \sum_{j=1}^{L} \frac{\mu}{2} \sigma_j^z, \tag{2}$$

with $J_{x,y} \equiv (J_n^+ \pm J_n^-)/2$. In the thermodynamic limit $L \gg N_f \geq 1$, the Hamiltonian (1) can be diagonalized by a Fourier-Bogoliubov transformation with energy spectrum $\epsilon_q = \pm \frac{1}{2}\sqrt{y(q)^2 + z(q)^2}$, where $y(q) = \sum_{n=1}^{N_f} J_n^- \sin(nq)$, $z(q) = \sum_{n=1}^{N_f} J_n^+ \cos(nq) - \mu$, with q the wavevector [68].

As a $\mathbf{Z}$ topological invariant [53, 65], the winding number of the closed loop with the vector $\boldsymbol{r}(q) = (0, y(q), z(q))$ in the auxiliary y-z plane around the origin can be written as $\nu = (1/2\pi) \oint (y dz - z dy)/|\boldsymbol{r}|^2$. Substituting $\zeta(q) \equiv \exp(iq)$, for $y(q) \equiv Y(\zeta)$ and $z(q) \equiv Z(\zeta)$, we can define a complex characteristic function $g(\zeta) \equiv Z(\zeta) + iY(\zeta)$ and obtain the winding number by calculating the logarithmic residue of $g(\zeta)$ in accordance with the Cauchy's argument principle [69] $\nu = (1/2\pi i) \oint_{|\zeta|=1} d\zeta\, g'(\zeta)/g(\zeta) = \mathcal{N} - \mathcal{P}$, where in the complex region $|\zeta| < 1$, $\mathcal{N}$ is the number of zeros and $\mathcal{P}$ is the number of poles. Moreover, topological phase transitions are characterized by the change of winding numbers at the critical points that can be calculated by solving $g(\zeta) = 0$ on the contour $|\zeta| = 1$ [68]. Similarly, the topologically nontrivial phases for the model (1) are also identified by the existence of paired boundary MZMs of which the properties are obtained from the solution of the BdG Hamiltonian with open boundary conditions [64, 70, 71]. This can also be transformed to calculating zeros of $g(\zeta)$ in $|\zeta| < 1$, such that the number of MZMs at each end of the open chain, defined as $\mathcal{M}_0$, equals the absolute value of the winding number: $\mathcal{M}_0 = |\mathcal{N} - \mathcal{P}| = |\nu|$. Therefore, these two approaches [(i) by winding numbers from the geometric topology in the 2D auxiliary space, and (ii) by MZMs from BdG equations to characterize topological phases] in the extended Kitaev chain in Eq. (1) are equivalent [68] (see, e.g., Fig. 1).

Multipartite entanglement and QFI density.—Multipartite entanglement [43, 44] plays a key role in quantum physics and quantum metrology, and moreover, it is central to understanding quantum many-body systems. QFI, similar as quantum spin squeezing [72, 73], is a significant quantity in both large-scale multipartite entanglement detection and high-precision quantum metrology [45–48, 50, 51]. Given a generator $\mathcal{O}$ and a mixed state $\rho = \sum_i p_i |i\rangle\langle i|$, with $\langle i|j\rangle = \delta_{ij}$, the QFI of a state $\rho(t) = \exp(-it\mathcal{O})\rho\exp(it\mathcal{O})$ with respect to a parameter t is [46] $F_Q[\mathcal{O}, \rho] = \sum_{p_i + p_j \neq 0} \frac{2(p_i - p_j)^2}{p_i + p_j} |\langle i|\mathcal{O}|j\rangle|^2$. For a pure state $|\psi\rangle$, the QFI can be simplified as $F_Q[\mathcal{O}, |\psi\rangle] = 4(\Delta_\psi \mathcal{O})^2$, where the variance of the generator is $(\Delta_\psi \mathcal{O})^2 \equiv \langle \mathcal{O}^2 \rangle_\psi - \langle \mathcal{O} \rangle_\psi^2$. The QFI relates to dynamic susceptibilities [74] that are routinely measured in laboratory experiments. Furthermore, the scaling of the QFI with respect to nonlocal operators [49] would be sensitive to topological quantum phase transitions [52]. For critical systems with L sites, we consider a QFI density with form $f_Q = F_Q/L$, and the violation of the inequality $f_Q \leq \kappa$ signals $(\kappa+1)$-partite entanglement $(1 \leq \kappa \leq L-1)$ [42].

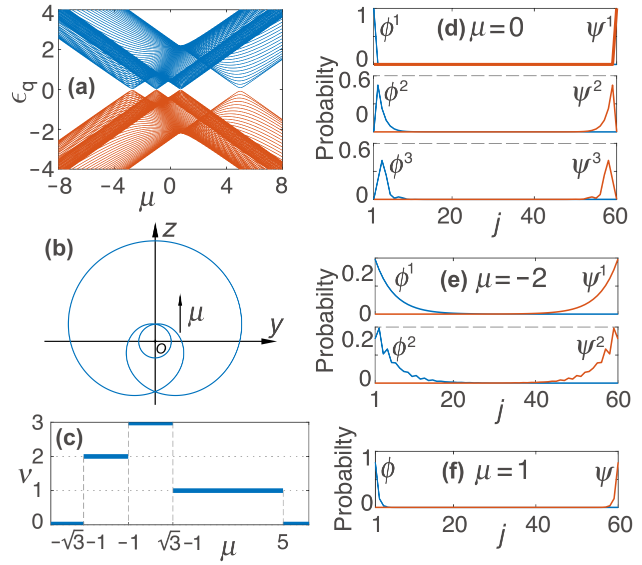

FIG. 1. (color online) (a) Energy spectrum for $L = 200$ sites, (b) trajectory of the winding vector $\boldsymbol{r}(q) = (0, y(q), z(q))$, (c-e) probability distributions [blue (red) curve is for left (right) modes] of MZMs for $L = 60$ sites given different values of a chemical potential μ for the extended Kitaev chain with $N_f = 3$ and $J_1^{\pm} = 1$, $J_2^{\pm} = 2$, $J_3^{\pm} = 2$. (c) The phase diagram characterized by the winding number. (d) For $\mu = 0$, the winding number $\nu = 3$ and we have three pairs of non-degenerate MZMs exponentially localized at the domain wall. (e) For $\mu = -2$, $\nu = 2$ and there are two pairs of MZMs. (f) When $\mu = 1$, $\nu = 1$ which leads to one pair of MZMs.

To detect a topological phase of an extended Kitaev chain with a winding number $\nu = \pm 1$, the generators in terms of spin operators in the x, y directions through the Jordan-Wigner transformation are chosen as [52] $\mathcal{O}_{\nu=\pm 1} = \sum_{j=1}^{L} \sigma_j^{x,y}/2$, and staggered operators as $\mathcal{O}_{\nu=\pm 1}^{(\mathrm{st})} = \sum_{j=1}^{L} (-)^j \sigma_j^{x,y}/2$. Then, the QFI density for the ground state $|\mathcal{G}\rangle$ becomes $f_Q[\mathcal{O}_{\nu=\pm 1}, |\mathcal{G}\rangle] = 1 + \sum_{r=1}^{L-1} C_{\nu=\pm 1}(r)$ and $f_Q[\mathcal{O}_{\nu=\pm 1}^{(\mathrm{st})}, |\mathcal{G}\rangle] = 1 + \sum_{r=1}^{L-1} (-)^r C_{\nu=\pm 1}(r)$, where the spin-spin correlation functions are $C_{\nu=\pm 1}(r) \equiv \langle \sigma_j^{x,y} \sigma_{j+r}^{x,y} \rangle_{\mathcal{G}}$, with $\langle \cdots \rangle_{\mathcal{G}}$ the average of the ground state $|\mathcal{G}\rangle$. A topological phase with a low winding number can be characterized by power-law diverging finite-size scaling of the QFI density, $f_Q \propto L$, as discussed in [52].

Characterization of topological phases by multipartite entanglement in the dual lattice.—Duality in physics provides different but equivalent mathematical descriptions of a system and provides an overall understanding of the same physical phenomena from different angles [58]. For example, an Ising

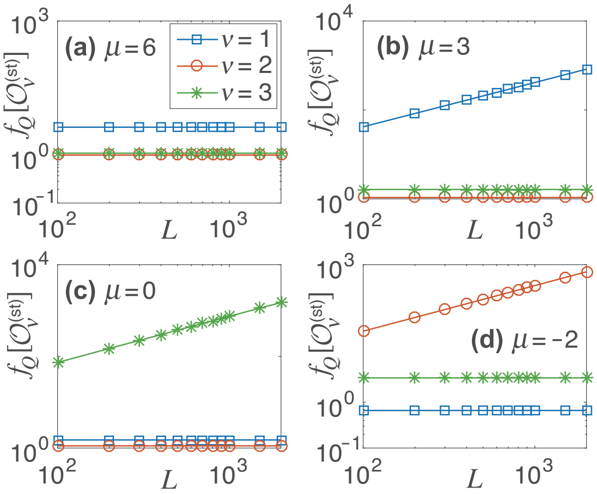

FIG. 2. (color online) Dual QFI density $f_Q[\mathcal{O}_\nu^{(\mathrm{st})}, |\mathcal{G}\rangle]$ of the ground state $|\mathcal{G}\rangle$ versus L for the extended Kitaev chain with $N_f = 3$ and nonzero parameters ($J_1^{\pm} = 1$, $J_2^{\pm} = 2$, $J_3^{\pm} = 2$) in different topological phases. (a) For $\mu = 6$, the winding number $\nu = 0$. (b) For $\mu = 3$, $\nu = 1$, and the fitting nontrivial scaling topological index $\lambda_1^{(\mathrm{st})} = 0.9965$. (c) For $\mu = 0$, $\nu = 3$, and $\lambda_3^{(\mathrm{st})} = 1.0047$. (d) For $\mu = -2$, $\nu = 2$, and $\lambda_2^{(\mathrm{st})} = 0.9957$.

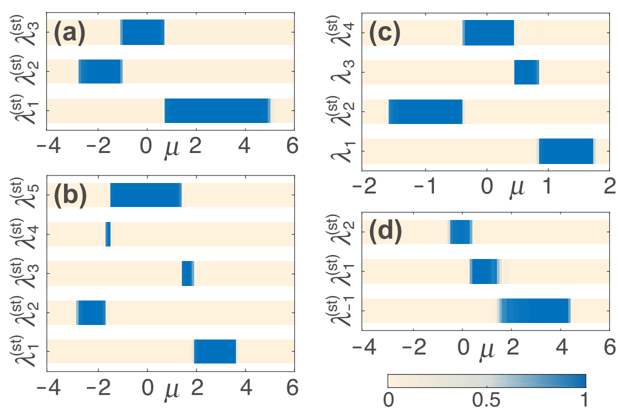

FIG. 3. (color online) Scaling topological index λ_ν and $\lambda_\nu^{(\mathrm{st})}$ of the dual QFI density $f_Q[\mathcal{O}_\nu, |\mathcal{G}\rangle]$ and $f_Q[\mathcal{O}_\nu^{(\mathrm{st})}, |\mathcal{G}\rangle]$, respectively, versus system size L up to 1200. The extended Kitaev chain in Eq. (1) has the following nonzero parameters: (a) $J_1^{\pm} = 1$, $J_2^{\pm} = 2$, $J_3^{\pm} = 2$ ($N_f = 3$); (b) $J_1^{\pm} = 0.1$, $J_2^{\pm} = 0.21$, $J_3^{\pm} = 0.44$, $J_4^{\pm} = 0.9$, $J_5^{\pm} = 2$ ($N_f = 5$); (c) $J_1^{\pm} = 0.1$, $J_2^{\pm} = 0.21$, $J_3^{\pm} = -0.74$, $J_4^{\pm} = 0.9$ ($N_f = 4$); and (d) $J_2^{\pm} = 2.4$, $J_3^{\pm} = \pm 2$ ($N_f = 3$).

chain with an external field h has a self-dual symmetry, mapping between the ordered and disordered phases, expressed as $H_{\text{Ising}} = \sum_j (\sigma_j^x \sigma_{j+1}^x + h\sigma_j^z) = h\sum_j (s_j^x s_{j+1}^x + h^{-1} s_j^z)$, with the duality transformation $s_j^x = \prod_{k\le j} \sigma_k^z$, $s_j^z = \sigma_j^x \sigma_{j+1}^x$, and $s_j^y = -i s_j^z s_j^x$ [75]. Here both σ and s satisfy the same algebra. Furthermore, the *nonlocal* SCF [59–61], characterizing SPT order by the $\mathbf{Z}_2 \times \mathbf{Z}_2$ symmetry in the cluster Ising model [1] with Hamiltonian $H_{\text{cluster}} = \sum_j (\sigma_{j-1}^x \sigma_j^z \sigma_{j+1}^x + h\sigma_j^z)$, can be written as a *local* correlator $(-)^r \langle s_j^y s_{j+r}^y \rangle_{\mathcal{G}}$ in the dual lattice of the Ising model [60, 61]. Through the Jordan-Wigner transformation (also regarded as a duality transformation using a bond-algebraic approach [76]), the self-duality properties of a spin-$\frac{1}{2}$ model can help to study topological phases and multipartite entanglement in the extended Kitaev chain (1).

To detect a SPT phase with a positive integer winding number $\nu = n \ge 2$, we consider the duality transformation of an extended Ising model $H = \sum_j (\sigma_j^x \sigma_{j+n-1}^x \prod_{l=1}^{n-2} \sigma_{j+l}^z + h\sigma_j^z) = h\sum_j (\hat{s}_j^x \hat{s}_{j+n-1}^x \prod_{l=1}^{n-2} \hat{s}_{j+l}^z + h^{-1}\hat{s}_j^z)$, corresponding to an extended Kitaev chain with $\nu = n-1$. We can define the dual operator $\tau_j^{(\nu=n)} \equiv \hat{s}_j^y = -i\hat{s}_j^z \hat{s}_j^x$. For a negative winding number, $\nu = -n$, we consider another extended Ising model by transforming $x \to y$: $H = \sum_j (\sigma_j^y \sigma_{j+n-1}^y \prod_{l=1}^{n-2} \sigma_{j+l}^z + h\sigma_j^z) = h\sum_j (\tilde{s}_j^y \tilde{s}_{j+n-1}^y \prod_{l=1}^{n-2} \tilde{s}_{j+l}^z + h^{-1}\tilde{s}_j^z)$ and obtain the dual spin operator $\tau_j^{(\nu=-n)} \equiv \tilde{s}_j^x = i\tilde{s}_j^z \tilde{s}_j^y$. The expressions of the dual spin operators $\tau_j^{(\nu)}$ differ according to the parity of the winding number ν [77]. Explicitly with $p \ge 1$, we have [68] for even winding numbers,

$$\tau_j^{(\nu\pm 2p)} = -\left(\prod_{k=1}^{j-1} \sigma_k^z\right)\left(\prod_{l=1}^{p} \sigma_{j+2l-2}^{y,x} \sigma_{j+2l-1}^{x,y}\right), \quad (3)$$

and for odd winding numbers,

$$\tau_j^{(\nu=\pm(2p+1))} = \sigma_j^{x,y}\left(\prod_{l=1}^{p} \sigma_{j+2l-1}^{y,x} \sigma_{j+2l}^{x,y}\right). \quad (4)$$

The SCF [60, 61] equals the spin correlation function from site j to $(j+r)$ in the dual lattice:

$$C_\nu(r) \equiv \langle \tau_j^{(\nu)} \tau_{j+r}^{(\nu)} \rangle_{\mathcal{G}} = \left\langle \prod_{l=j}^{j+r-1} \left(\sigma_l^\alpha \sigma_{l+|\nu|}^\alpha \prod_{k=1}^{|\nu|-1} \sigma_{l+k}^z\right) \right\rangle_{\mathcal{G}}, \quad (5)$$

where $\alpha = x$ (or y) for a positive (or negative) ν. It is clearer to write the SCF, in terms of Majorana fermion operators $a_j = c_j^\dagger + c_j$ and $b_j = i(c_j^\dagger - c_j)$, as

$$C_\nu(r) = \left\langle \prod_{l=j}^{j+r} (-ib_l a_{l+\nu}) \right\rangle_{\mathcal{G}} = \left\langle \prod_{l=j}^{j+r} (1 - 2d_{l,\nu}^\dagger d_{l,\nu}) \right\rangle_{\mathcal{G}}, \quad (6)$$

where we define $d_{l,\nu} = (b_l + ia_{l+\nu})/2$ and $d_{l,\nu}^\dagger = (b_l - ia_{l+\nu})/2$ as Dirac fermion operators [71]. Therefore, the SCF can also be regarded as the ground-state average of $\mathbf{Z}$ type Majorana parity [77], and in particular, $\Delta_\nu \equiv \lim_{r\to\infty} C_\nu(r)$ and $\Delta_\nu^{(\mathrm{st})} \equiv \lim_{r\to\infty} (-)^r C_\nu(r)$ are the string order parameters [59, 61], capturing hidden SPT order.

The generators of the dual QFI density are defined in the dual lattice as $\mathcal{O}_\nu = \sum_{j=1}^{M} \tau_j^{(\nu)}$, and $\mathcal{O}_\nu^{(\mathrm{st})} = \sum_{j=1}^{M} (-)^j \tau_j^{(\nu)}$, with $M \equiv L - |\nu| + 1$, where the choice of dual generators depends on the sign of the direct interaction between the Majorana fermions at chain ends [68, 78]. The operator $\mathcal{O}_\nu$ applies for the positive interaction, and the staggered operator $\mathcal{O}_\nu^{(\mathrm{st})}$ is for the negative one. Then, we obtain the dual QFI density of the ground state for $L \gg N_f \ge 1$ as $f_Q[\mathcal{O}_\nu, |\mathcal{G}\rangle] \simeq$

$1+\sum_{r=1}^{M-1}C_\nu(r)$, and $f_Q[\mathcal{O}_\nu^{(\mathrm{st})},|\mathcal{G}\rangle] \simeq 1+\sum_{r=1}^{M-1}(-)^r C_\nu(r)$, where we have used $(\tau_j^{(\nu)})^2 = \mathbb{I}$, with $\mathbb{I}$ the identity. Using Wick's theorem, the dual QFI density can be expressed in terms of fermion correlators and may be measured in many-body systems using experimentally mature techniques, such as Bragg spectroscopy [79, 80] or neutron scattering [81].

The SCF has a similar scaling behavior in the topologically nontrivial phase with a higher winding number as the spin correlator used in [52] (see, e.g., [68]). Thus, we find that the dual QFI density as a function of L also follows an asymptotic power law scaling in the thermodynamic limit as $f_Q[\mathcal{O}_\nu,|\mathcal{G}\rangle] = 1+\gamma_\nu L^{\lambda_\nu}$, and $f_Q[\mathcal{O}_\nu^{(\mathrm{st})},|\mathcal{G}\rangle] = 1+\gamma_\nu^{(\mathrm{st})}L^{\lambda_\nu^{(\mathrm{st})}}$, where the scaling coefficients γ and λ depend on the choice of the dual generators and the parameters of the Hamiltonian (1). For a topological phase with a definite winding number ν, we could find that λ_ν or $\lambda_\nu^{(\mathrm{st})}$ is equal to 1 ($F_Q \propto L^2$), and the scaling coefficients λ_ω and $\lambda_\omega^{(\mathrm{st})}$ for other integer winding numbers, $\omega \neq \nu$, are approximately zero (see, e.g., Fig. 2). Thus, the scaling topological index λ_ν or $\lambda_\nu^{(\mathrm{st})}$, relating directly to the SCF, characterizes the features of the topological phase with a winding number ν of the extended Kitaev model. In Fig. 3, we consider four different types of extended Kitaev chain models and plot the fitting scaling coefficients λ_ν or $\lambda_\nu^{(\mathrm{st})}$ of the QFI density versus system size L up to 1200, and also versus the chemical potential μ, which clearly show the topological phase diagrams. Therefore, we conclude that by choosing the generators in different dual lattices, the scaling behavior of the QFI density, a witness of multipartite entanglement, can detect 1D SPT phase transitions. In the topologically nontrivial phase with integer winding number, the quadratic growth of the QFI can also be broadly applicable to practical quantum metrology [45–48, 50, 51]. The scaling coefficients of the QFI density in phases with half-integer winding numbers or on the critical boundary between two topological phases would be complicated [52] and deserve further investigations, of which more simulations and discussions are given in [68].

Dual multipartite entanglement in the Kitaev honeycomb model.—The Kitaev honeycomb model [62], on a hexagonal lattice with topological order at zero temperature, has been widely investigated using a variety of quantum-information methods [82–85]. The Hamiltonian is $H_{\mathrm{hc}} = -\sum_{\alpha=x,y,z}J_\alpha\sum_{\langle ij\rangle_\alpha}\sigma_i^\alpha\sigma_j^\alpha$, where $\langle ij\rangle_\alpha$ denotes the nearest neighbor bonds in the α-direction. We consider positive bonds, $J_{x,y,z} > 0$, and focus on the $J_x+J_y+J_z = 1$ parametric plane. The phase diagram is shown in Fig. 4(a).

Here, through the two-leg spin ladder [57] of the Kitaev honeycomb model, we find that the quantum phase with hidden topological order can also be characterized by dual multipartite entanglement. As shown in Fig. 4(b), we relabel all the sites along a special path and rewrite the Hamiltonian with third-nearest-neighbor couplings [57]: $H_{21} = -\sum_{j=1}^{L}(J_x\sigma_{2j-1}^x\sigma_{2j}^x + J_y\sigma_{2j}^y\sigma_{2j+3}^y + J_z\sigma_{2j}^z\sigma_{2j+1}^z)$. With the duality transformation $\check{s}_j^x = \prod_{k=1}^{j}\sigma_k^x$, $\check{s}_j^z = \sigma_j^z\sigma_{j+1}^z$, and

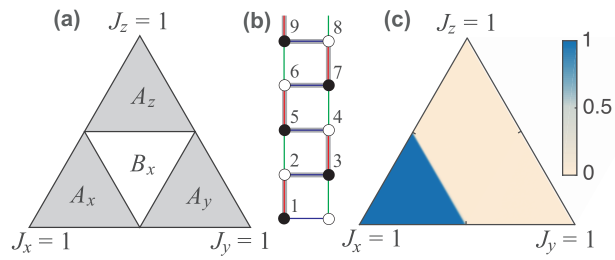

FIG. 4. (color online) (a) The phase diagram of the Kitaev honeycomb model on the $J_x+J_y+J_z = 1$ plane. In the region $J_x \leq J_y+J_z$, $J_y \leq J_z+J_x$, and $J_z \leq J_x+J_y$, there is a gapless phase B with non-Abelian excitations, and in other regions, there are three gapped phases $A_{x,y,z}$ with Abelian anyon excitations. (b) A single-chain representation of the two-leg spin ladder of the Kitaev model. (c) The scaling topological index $\lambda_x^{(\mathrm{st})}$ of the dual QFI density $f_Q[\mathcal{O}_x^{(\mathrm{st})},|\mathcal{G}\rangle]$ for different values of $J_{x,y,z}$ versus system size $2L$ up to 400.

$\check{s}_j^y = -i\check{s}_j^z\check{s}_j^x$, we obtain an anisotropic XY spin chain with a transverse field in the dual space

$$H_{21} = -\sum_{j=1}^{L}(J_x\check{s}_{2j}^x\check{s}_{2j+2}^x + J_yW_j\check{s}_{2j}^y\check{s}_{2j+2}^y + J_z\check{s}_{2j}^z), \quad (7)$$

where $W_j \equiv \check{s}_{2j-1}^x\check{s}_{2j+1}^z\check{s}_{2j+3}^x$ is the plaquette operator in the dual lattice (a good quantum number [57]) and has $W_j = -1$ (π-flux phase [86]) for the ground state. Then, with respect to the dual generator $\mathcal{O}_x^{(\mathrm{st})} = \sum_{j=1}^{L}(-)^j\check{s}_{2j}^x$, the QFI density is $f_Q[\mathcal{O}_x^{(\mathrm{st})},|\mathcal{G}\rangle] \equiv 1+\sum_{r=1}^{L-1}(-)^r C_x(r) \simeq 1+\gamma_x^{(\mathrm{st})}L^{\lambda_x^{(\mathrm{st})}}$, where the staggered SCF is $(-)^r C_x(r) \equiv (-)^r\langle\check{s}_{2j}^x\check{s}_{2j+2r}^x\rangle_{\mathcal{G}} = (-)^r\langle\prod_{k=1}^{2r}\sigma_{2j+k}^x\rangle_{\mathcal{G}}$. The dual QFI density is linear versus L in the gapped phase A_x ($J_x \geq J_y+J_z$) and constant in other regions [see Fig. 4(c)]. The other two gapped phases A_y and A_z [see Fig. 4(a)] can be obtained by the substitutions $J_x \to J_{y,z} \to J_{z,y} \to J_x$, respectively. Moreover, when considering the equivalent brick-wall lattice [57] of the Kitaev honeycomb model, these results can also be extended to the general 2D lattice by transforming the second index of site to the momentum space [57, 68].

Conclusions.—Recent work [52] shows that 1D SPT order with winding numbers $\nu = \pm 1$ can be characterized by a super-extensive QFI with respect to the spin operator, $F_Q \propto L^2$. By introducing the above duality, we have shown that 1D SPT order with higher winding numbers can be characterized by the scaling behavior of multipartite entanglement with respect to the spin generators in the dual lattice. By choosing the generators in different dual lattices, the scaling coefficients λ_ν and $\lambda_\nu^{(\mathrm{st})}$ of the dual QFI density, as a witness of multipartite entanglement [43, 44], effectively identify different nontrivial topological phases with high winding numbers. Moreover, further investigations on the Kitaev honeycomb model have shown that our results for detecting 1D SPT order could be well generalized to characterize topological order in 2D systems (e.g., the toric code model [62] and fractional quantum Hall states [87]) and SPT order in non-Hermitian systems [88]. This work paves the way to characterizing topological phases using multipartite entanglement of the ground state, and also the detection of topologically protected multipartite entanglement, with promising applications in both quantum computation and quantum metrology.

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[1] B. Zeng, X. Chen, D. L. Zhou, and X. G. Wen, "Quantum information meets quantum matter–From quantum entanglement to topological phase in many-body systems," arXiv:1508.02595 (2015).

[2] R. S. K. Mong, D. J. Clarke, J. Alicea, N. H. Lindner, P. Fendley, C. Nayak, Y. Oreg, A. Stern, E. Berg, K. Shtengel, and M. P. A. Fisher, "Universal topological quantum computation from a superconductor-Abelian quantum Hall heterostructure," Phys. Rev. X **4**, 011036 (2014).

[3] C. V. Kraus, P. Zoller, and M. A. Baranov, "Braiding of atomic Majorana fermions in wire networks and implementation of the Deutsch-Jozsa algorithm," Phys. Rev. Lett. **111**, 203001 (2013).

[4] J. D. Sau, R. M. Lutchyn, S. Tewari, and S. Das Sarma, "Generic new platform for topological quantum computation using semiconductor heterostructures," Phys. Rev. Lett. **104**, 040502 (2010).

[5] J. Alicea, Y. Oreg, G. Refael, F. von Oppen, and M. P. A. Fisher, "Non-Abelian statistics and topological quantum information processing in 1D wire networks," Nat. Phys. **7**, 412–417 (2011).

[6] J. Alicea, "New directions in the pursuit of Majorana fermions in solid state systems," Rep. Prog. Phys. **75**, 076501 (2012).

[7] A. Y. Kitaev, "Fault-tolerant quantum computation by anyons," Ann. Phys. **303**, 2–30 (2003).

[8] Y. Xia, D. Qian, D. Hsieh, L. Wray, A. Pal, H. Lin, A. Bansil, D. Grauer, Y. S. Hor, R. J. Cava, and M. Z. Hasan, "Observation of a large-gap topological-insulator class with a single Dirac cone on the surface," Nat. Phys. **5**, 398–402 (2009).

[9] S. M. Albrecht, A. P. Higginbotham, M. Madsen, F. Kuemmeth, T. S. Jespersen, J. Nygard, P. Krogstrup, and C. M. Marcus, "Exponential protection of zero modes in Majorana islands," Nature **531**, 206–209 (2016).

[10] X. G. Wen and Q. Niu, "Ground-state degeneracy of the fractional quantum Hall states in the presence of a random potential and on high-genus Riemann surfaces," Phys. Rev. B **41**, 9377–9396 (1990).

[11] X. G. Wen, *Quantum field theory of many-body systems: from the origin of sound to an origin of light and electrons* (Oxford University Press on Demand, 2004).

[12] L. Amico, R. Fazio, A. Osterloh, and V. Vedral, "Entanglement in many-body systems," Rev. Mod. Phys. **80**, 517–576 (2008).

[13] J. Eisert, M. Cramer, and M. B. Plenio, "Colloquium: Area laws for the entanglement entropy," Rev. Mod. Phys. **82**, 277–306 (2010).

[14] A. Kitaev and J. Preskill, "Topological entanglement entropy," Phys. Rev. Lett. **96**, 110404 (2006).

[15] M. Levin and X. G. Wen, "Detecting topological order in a ground state wave function," Phys. Rev. Lett. **96**, 110405 (2006).

[16] H. Li and F. D. M. Haldane, "Entanglement spectrum as a generalization of entanglement entropy: Identification of topological order in non-Abelian fractional quantum Hall effect states," Phys. Rev. Lett. **101**, 010504 (2008).

[17] Z. C. Gu and X. G. Wen, "Tensor-entanglement-filtering renormalization approach and symmetry-protected topological order," Phys. Rev. B **80**, 155131 (2009).

[18] X. Chen, Z. C. Gu, and X. G. Wen, "Classification of gapped symmetric phases in one-dimensional spin systems," Phys. Rev. B **83**, 035107 (2011).

[19] T. Scaffidi, D. E. Parker, and R. Vasseur, "Gapless symmetry-protected topological order," Phys. Rev. X **7**, 041048 (2017).

[20] K. Y. Bliokh, D. Smirnova, and F. Nori, "Quantum spin Hall effect of light," Science **348**, 1448–1451 (2015).

[21] M. Konig, S. Wiedmann, C. Brune, A. Roth, H. Buhmann, L. W. Molenkamp, X. L. Qi, and S. C. Zhang, "Quantum spin Hall insulator state in HgTe quantum wells," Science **318**, 766–770 (2007).

[22] L. Fu and C. L. Kane, "Superconducting proximity effect and Majorana fermions at the surface of a topological insulator," Phys. Rev. Lett. **100**, 096407 (2008).

[23] D. Hsieh, D. Qian, L. Wray, Y. Xia, Y. S. Hor, R. J. Cava, and M. Z. Hasan, "A topological Dirac insulator in a quantum spin Hall phase," Nature **452**, 970–975 (2008).

[24] M. Z. Hasan and C. L. Kane, "Colloquium: Topological insulators," Rev. Mod. Phys. **82**, 3045–3067 (2010).

[25] X. L. Qi and S. C. Zhang, "Topological insulators and superconductors," Rev. Mod. Phys. **83**, 1057–1110 (2011).

[26] M. Sato and Y. Ando, "Topological superconductors: a review," Rep. Prog. Phys. **80**, 076501 (2017).

[27] C. Nayak, S. H. Simon, A. Stern, M. Freedman, and S. Das Sarma, "Non-Abelian anyons and topological quantum computation," Rev. Mod. Phys. **80**, 1083–1159 (2008).

[28] S. D. Sarma, M. Freedman, and C. Nayak, "Majorana zero modes and topological quantum computation," npj Quantum Inf. **1**, 15001 (2015).

[29] S. R. Elliott and M. Franz, "Colloquium: Majorana fermions in nuclear, particle, and solid-state physics," Rev. Mod. Phys. **87**, 137–163 (2015).

[30] J. Q. You, X. F. Shi, X. D. Hu, and F. Nori, "Quantum emulation of a spin system with topologically pro-

tected ground states using superconducting quantum circuits," Phys. Rev. B **81**, 014505 (2010).

[31] J. Q. You, Z. D. Wang, W. X. Zhang, and F. Nori, "Encoding a qubit with Majorana modes in superconducting circuits," Sci. Rep. **4**, 5535 (2014).

[32] R. S. Akzyanov, A. L. Rakhmanov, A. V. Rozhkov, and F. Nori, "Majorana fermions at the edge of superconducting islands," Phys. Rev. B **92**, 075432 (2015).

[33] R. S. Akzyanov, A. L. Rakhmanov, A. V. Rozhkov, and F. Nori, "Tunable Majorana fermion from Landau quantization in 2D topological superconductors," Phys. Rev. B **94**, 125428 (2016).

[34] P. Zhang and F. Nori, "Majorana bound states in a disordered quantum dot chain," New J. Phys. **18**, 043033 (2016).

[35] D. Vodola, L. Lepori, E. Ercolessi, A. V. Gorshkov, and G. Pupillo, "Kitaev chains with long-range pairing," Phys. Rev. Lett. **113**, 156402 (2014).

[36] D. Vodola, L. Lepori, E. Ercolessi, and G. Pupillo, "Long-range Ising and Kitaev models: phases, correlations and edge modes," New J. Phys. **18**, 015001 (2015).

[37] A. Alecce and L. Dell'Anna, "Extended Kitaev chain with longer-range hopping and pairing," Phys. Rev. B **95**, 195160 (2017).

[38] L. Lepori and L. Dell'Anna, "Long-range topological insulators and weakened bulk-boundary correspondence," New J. Phys. **19**, 103030 (2017).

[39] V. Mourik, K. Zuo, S. M. Frolov, S. R. Plissard, E.P.A.M. Bakkers, and L. P. Kouwenhoven, "Signatures of Majorana fermions in hybrid superconductor-semiconductor nanowire devices," Science **336**, 1003–1007 (2012).

[40] S. Nadj-Perge, I. K. Drozdov, J. Li, H. Chen, S. Jeon, J. Seo, A. H. MacDonald, B. A. Bernevig, and A. Yazdani, "Observation of Majorana fermions in ferromagnetic atomic chains on a superconductor," Science **346**, 602–607 (2014).

[41] S.-H. Do, S.-Y. Park, J. Yoshitake, J. Nasu, Y. Motome, Y. S. Kwon, D. T. Adroja, D. J. Voneshen, K. Kim, T.-H. Jang, J.-H. Park, K.-Y. Choi, and S. Ji, "Majorana fermions in the Kitaev quantum spin system α-$RuCl_3$," Nat. Phys. **13**, 1079–1084 (2017).

[42] L. Pezzè and A. Smerzi, "Entanglement, nonlinear dynamics, and the Heisenberg limit," Phys. Rev. Lett. **102**, 100401 (2009).

[43] P. Hyllus, W. Laskowski, R. Krischek, C. Schwemmer, W. Wieczorek, H. Weinfurter, L. Pezzè, and A. Smerzi, "Fisher information and multiparticle entanglement," Phys. Rev. A **85**, 022321 (2012).

[44] G. Tóth, "Multipartite entanglement and high-precision metrology," Phys. Rev. A **85**, 022322 (2012).

[45] H. Strobel, W. Muessel, D. Linnemann, T. Zibold, D. B. Hume, L. Pezzè, A. Smerzi, and M. K. Oberthaler, "Fisher information and entanglement of non-Gaussian spin states," Science **345**, 424–427 (2014).

[46] S. L. Braunstein and C. M. Caves, "Statistical distance and the geometry of quantum states," Phys. Rev. Lett. **72**, 3439–3443 (1994).

[47] V. Giovannetti, S. Lloyd, and L. Maccone, "Quantum metrology," Phys. Rev. Lett. **96**, 010401 (2006).

[48] V. Giovannetti, S. Lloyd, and L. Maccone, "Advances in quantum metrology," Nat. Photon. **5**, 222–229 (2011).

[49] L. Pezzè, Y. Li, W. D. Li, and A. Smerzi, "Witnessing entanglement without entanglement witness operators," PNAS **113**, 11459–11464 (2016).

[50] C. Gross, T. Zibold, E. Nicklas, J. Esteve, and M. K. Oberthaler, "Nonlinear atom interferometer surpasses classical precision limit," Nature **464**, 1165–1169 (2010).

[51] B. Lucke, M. Scherer, J. Kruse, L. Pezzè, F. Deuretzbacher, P. Hyllus, O. Topic, J. Peise, W. Ertmer, J. Arlt, L. Santos, A. Smerzi, and C. Klempt, "Twin matter waves for interferometry beyond the classical limit," Science **334**, 773–776 (2011).

[52] L. Pezzè, M. Gabbrielli, L. Lepori, and A. Smerzi, "Multipartite entanglement in topological quantum phases," Phys. Rev. Lett. **119**, 250401 (2017).

[53] C. K. Chiu, J. C. Y. Teo, A. P. Schnyder, and S. Ryu, "Classification of topological quantum matter with symmetries," Rev. Mod. Phys. **88**, 035005 (2016).

[54] L. H. Li, C. Yang, and S. Chen, "Topological invariants for phase transition points of one-dimensional Z_2 topological systems," Eur. Phys. J. B **89**, 195 (2016).

[55] S. Tewari and J. D. Sau, "Topological invariants for spin-orbit coupled superconductor nanowires," Phys. Rev. Lett. **109**, 150408 (2012).

[56] E. Fradkin and L. Susskind, "Order and disorder in gauge systems and magnets," Phys. Rev. D **17**, 2637–2658 (1978).

[57] X. Y. Feng, G. M. Zhang, and T. Xiang, "Topological characterization of quantum phase transitions in a spin-1/2 model," Phys. Rev. Lett. **98**, 087204 (2007).

[58] Y. Q. Qin, Y. Y. He, Y. Z. You, Z. Y. Lu, A. Sen, A. W. Sandvik, C. K. Xu, and Z. Y. Meng, "Duality between the deconfined quantum-critical point and the bosonic topological transition," Phys. Rev. X **7**, 031052 (2017).

[59] P. Smacchia, L. Amico, P. Facchi, R. Fazio, G. Florio, S. Pascazio, and V. Vedral, "Statistical mechanics of the cluster Ising model," Phys. Rev. A **84**, 022304 (2011).

[60] L. C. Venuti and M. Roncaglia, "Analytic relations between localizable entanglement and string correlations in spin systems," Phys. Rev. Lett. **94**, 207207 (2005).

[61] J. Cui, L. Amico, H. Fan, M. Gu, A. Hamma, and V. Vedral, "Local characterization of one-dimensional topologically ordered states," Phys. Rev. B **88**, 125117 (2013).

[62] A. Kitaev, "Anyons in an exactly solved model and beyond," Ann. Phys. **321**, 2–111 (2006).

[63] M. Suzuki, "Relationship among exactly soluble models of critical phenomena .1. 2D Ising model, dimer problem and generalized XY-model," Prog. Theor. Phys. **46**, 1337 (1971).

[64] Y. Z. Niu, S. B. Chung, C. H. Hsu, I. Mandal, S. Raghu, and S. Chakravarty, "Majorana zero modes in a quantum Ising chain with longer-ranged interactions," Phys. Rev. B **85**, 035110 (2012).

[65] G. Zhang and Z. Song, "Topological characterization of extended quantum Ising models," Phys. Rev. Lett. **115**, 177204 (2015).

[66] G. Zhang, C. Li, and Z. Song, "Majorana charges, winding numbers and Chern numbers in quantum Ising models," Sci. Rep. **7**, 8176 (2017).

[67] X. Z. Zhang and J. L. Guo, "Quantum correlation and quantum phase transition in the one-dimensional extended Ising model," Quantum Inf. Process. **16**, 223 (2017).

[68] See Supplemental Material [url] for detailed derivations of our main results, which includes Ref. [89].

[69] L. V. Ahlfors, *Complex analysis: an introduction to the theory of analytic functions of one complex variable* (New York, London, 1953).

[70] P. Fendley, "Parafermionic edge zero modes in Z(n)-invariant spin chains," J. Stat. Mech. **2012**, P11020 (2012).

[71] O. Viyuela, D. Vodola, G. Pupillo, and M. A. Martin-Delgado, "Topological massive Dirac edge modes and long-range superconducting Hamiltonians," Phys. Rev. B **94**, 125121 (2016).

[72] M. Kitagawa and M. Ueda, "Squeezed spin states," Phys. Rev. A **47**, 5138–5143 (1993).

[73] J. Ma, X. G. Wang, C. P. Sun, and F. Nori, "Quantum spin

squeezing," Phys. Rep. **509**, 89–165 (2011).
[74] P. Hauke, M. Heyl, L. Tagliacozzo, and P. Zoller, "Measuring multipartite entanglement through dynamic susceptibilities," Nat. Phys. **12**, 778–782 (2016).
[75] S. Suzuki, J.-I. Inoue, and B. K. Chakrabarti, *Quantum Ising phases and transitions in transverse Ising models*, Vol. 862 (Springer, 2012).
[76] E. Cobanera, G. Ortiz, and Z. Nussinov, "The bond-algebraic approach to dualities," Adv. Phys. **60**, 679–798 (2011).
[77] L. Fidkowski and A. Kitaev, "Topological phases of fermions in one dimension," Phys. Rev. B **83**, 075103 (2011).
[78] A. Yu Kitaev, "Unpaired Majorana fermions in quantum wires," Phys. Usp. **44**, 131 (2001).
[79] T. Stoferle, H. Moritz, C. Schori, M. Kohl, and T. Esslinger, "Transition from a strongly interacting 1D superfluid to a Mott insulator," Phys. Rev. Lett. **92**, 130403 (2004).
[80] P. T. Ernst, S. Gotze, J. S. Krauser, K. Pyka, D. S. Luhmann, D. Pfannkuche, and K. Sengstock, "Probing superfluids in optical lattices by momentum-resolved Bragg spectroscopy," Nat. Phys. **6**, 56–61 (2010).
[81] G. Shirane, S. M. Shapiro, and J. M. Tranquada, *Neutron Scattering with a Triple-Axis Spectrometer, Basic Techniques* (Cambridge Univ. Press, 2002).
[82] S. Yang, S. J. Gu, C. P. Sun, and H. Q. Lin, "Fidelity susceptibility and long-range correlation in the Kitaev honeycomb model," Phys. Rev. A **78**, 012304 (2008).
[83] D. F. Abasto and P. Zanardi, "Thermal states of the Kitaev honeycomb model: Bures metric analysis," Phys. Rev. A **79**, 012321 (2009).
[84] X. F. Shi, Y. Yu, J. Q. You, and F. Nori, "Topological quantum phase transition in the extended Kitaev spin model," Phys. Rev. B **79**, 134431 (2009).
[85] J. J. Chen, J. Cui, Y. R. Zhang, and H. Fan, "Coherence susceptibility as a probe of quantum phase transitions," Phys. Rev. A **94**, 022112 (2016).
[86] E. H. Lieb, "Flux phase of the half-filled band," Phys. Rev. Lett. **73**, 2158–2161 (1994).
[87] H. L. Stormer, D. C. Tsui, and A. C. Gossard, "The fractional quantum Hall effect," Rev. Mod. Phys. **71**, S298–S305 (1999).
[88] D. Leykam, K. Y. Bliokh, C. L. Huang, Y. D. Chong, and F. Nori, "Edge modes, degeneracies, and topological numbers in non-Hermitian systems," Phys. Rev. Lett. **118**, 040401 (2017).
[89] E. Barouch and B. M. Mccoy, "Statistical mechanics of XY-model .2. spin-correlation functions," Phys. Rev. A **3**, 786–804 (1971).