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We show that time-reflection symmetry in periodically driven (Floquet) quantum systems enables an inherently nonequilibrium phenomenon structurally similar to quantum-mechanical sypersymmetry. In particular, we find Floquet analogues of the Witten index that place lower bounds on the degeneracies of states with quasienergies 0 and π. Moreover, we show that in some cases time reflection symmetry can also interchange fermions and bosons, leading to fermion/boson pairs with opposite quasienergy. We provide a simple class of disordered, interacting, and ergodic Floquet models with an exponentially large number of states at quasienergies 0 and π, which are robust as long as the time-reflection symmetry is preserved. Floquet supersymmetry manifests itself in the evolution of certain local observables as a period-doubling effect with dramatic finite-size scaling, providing a clear signature for experiments.

Quantum systems driven by time-periodic perturbations are ubiquitous in atomic, molecular, and optical physics [1–3]. In recent years, periodic driving has been exploited by theory [4–12] and experiment [13–17] as a resource for quantum simulation; by varying certain control parameters periodically in time, intricate effective Hamiltonians can be realized for synthetic quantum systems that might be outlandish in the context of solid-state physics. However, the analogy between static and periodically-driven (Floquet) quantum matter only goes so far. Being time-dependent, Floquet systems do not conserve energy and generically heat up to infinite temperature. They lose any discernible phase structure [18, 19] unless some notion of integrability [20–22], many-body localization (MBL) [9, 23–26], or prethermalization [27–34] is invoked.

With any of these three stabilizing mechanisms, Floquet systems have exhibited many new phases that lack equilibrium counterparts. These include a vast array of Floquet topological phases [35–39] and the so-called "π spin glass" (πSG) [26, 40] or "discrete time crystal" (DTC) [41, 42] phase, which are the objectives of recent experiments [43, 44]. These Floquet phases share qualitative features that stem from their nonequilibrium nature; for example, in all such phases there are certain operators whose dynamics synchronizes robustly with the periodic drive in a nontrivial way. This is especially striking in the πSG/DTC, where the local magnetization exhibits robust subharmonic response at half the driving frequency.

In this work, we introduce a distinct class of Floquet systems that also exhibits subharmonic response, but for fundamentally different reasons than the πSG/DTC. We dub this phenomenon "Floquet supersymmetry" (FSUSY), as the underlying structure has many close parallels to quantum-mechanical supersymmetry (SUSY). For instance, while SUSY exchanges bosons and fermions, FSUSY exchanges forward and backward time evolution—its generator is a *time-reflection symmetry*. Interestingly, as we will show, in some cases the time reflection operator can *also* interchange bosons and fermions, leading to pairs of bosonic and fermionic states *at opposite quasienergies*. SUSY models are characterized by an invariant, the Witten index, which provides a lower bound on the ground-state degeneracy; similarly, FSUSY models are characterized by *two* invariants, which place lower bounds on the degeneracies of the "quasienergies" 0 and π. We emphasize, however, that FSUSY is not simply a generalization of SUSY to the Floquet context; rather, as we will show, it is a distinct property of the time evolution operator of a Floquet system.

After establishing this general framework, we present a simple class of interacting, disordered, and *ergodic* Floquet models exhibiting FSUSY. In these models, the degeneracies of the $0, \pi$ quasienergies are exponentially large—at least $2^{L/2}$, where L is the system size. We show that this exponentially large degeneracy is robust to any disorder and interactions preserving the underlying time-reflection symmetry. Such models show a distinct experimental signature of FSUSY. Local observables exhibit a subharmonic response; however, in stark contrast to the πSG/DTC, the response is suppressed exponentially in system size. This finite-size scaling of the response serves as sharp evidence of FSUSY. It is remarkable that this subharmonic response occurs in an otherwise ergodic quantum system; FSUSY provides an example of a class of thermalizing Floquet systems which display nontrivial phenomena in a macroscopic subspace of the full Hilbert space. Nevertheless, there is no contradiction with ergodicity, as the subharmonic response scales to zero in the thermodynamic limit for generic initial states.

We begin with some definitions. Consider a periodically driven system with the time-dependent Hamiltonian $H(t+T)=H(t)$, with T the driving period hereafter set to 1 (along with $\hbar$). Define the Floquet unitary U_{F}, which evolves states by one period:

$$U_{\mathrm{F}}\,|\psi(t)\rangle = |\psi(t+1)\rangle\,. \tag{1}$$

U_{F} has eigenstates $\{|E\rangle\}$ with corresponding eigenvalues $\{e^{\mathrm{i}E}\}$; the quasienergies $\{E\}$ are defined modulo 2π.

We say that U_{F} has time-reflection symmetry if there exists a unitary operator R satisfying $R^2=\mathbb{1}$ and

$$RU_{\mathrm{F}}R^\dagger = e^{\mathrm{i}\theta}U_{\mathrm{F}}^\dagger. \tag{2}$$

We hereafter set $\theta\to 0$ by redefining $U_{\mathrm{F}}\to e^{\mathrm{i}\theta/2}U_{\mathrm{F}}$. Since R maps the "forward" Floquet evolution operator U_{F} to the "backward" Floquet evolution operator $U_{\mathrm{F}}^\dagger$, it can be interpreted as reversing the direction of time. However, unlike the usual time-reversal operator, R is unitary, hence the name "time-reflection symmetry." (The corresponding symmetry for the effective Hamiltonian is called chiral symmetry, see e.g. Ref. [45].) Using Eq. (2), we can deduce the action of R on the Floquet eigenbasis:

$$U_{\mathrm{F}}\,(R\,|E\rangle) = e^{-\mathrm{i}E}(R\,|E\rangle). \tag{3}$$

R thus maps eigenstates of U_{F} with quasienergy E to eigenstates of U_{F} with quasienergy $-E$. Hence,

$$\langle E|\,R\,|E\rangle = 0 \text{ if } E\neq 0,\pi. \tag{4}$$

In the $E=0,\pi$ eigenspaces, $U_{\mathrm{F}}=U_{\mathrm{F}}^\dagger$, so (2) implies that R and U_{F} share a common eigenbasis for the $E=0,\pi$ states. We will label the common eigenbasis for $E=0(\pi)$ as $\{|0(\pi),\alpha\rangle\}$ where $\alpha=1,\ldots,N_{0(\pi)}$ and $N_{0(\pi)}$ is the degeneracy of the $E=0(\pi)$ eigenspace. Because $R^2=\mathbb{1}$,

$$\begin{aligned}\langle 0,\alpha|\,R\,|0,\alpha\rangle &= \pm 1\\ \langle \pi,\alpha|\,R\,|\pi,\alpha\rangle &= \pm 1.\end{aligned} \tag{5}$$

These properties motivate the definition of two trace formulas which we will prove to be integers providing lower bounds for the degeneracies $N_{0,\pi}$. Define $\mathcal{I}_0,\mathcal{I}_\pi$ as

$$\begin{aligned}\mathcal{I}_{0(\pi)} &\equiv \mathrm{tr}\left(R\,\frac{U_{\mathrm{F}}\pm\mathbb{1}}{2}\right)\\ &= \sum_{\alpha=1}^{N_{0(\pi)}}\langle 0(\pi),\alpha|\,R\,|0(\pi),\alpha\rangle\,,\end{aligned} \tag{6}$$

where we have used (4). Moreover, (5) implies that both invariants are integers and $|\mathcal{I}_{0(\pi)}|\le N_{0(\pi)}$; these invariants thus provide a lower bound for the number of 0 and π quasienergy eigenstates, respectively.

Given time reflection symmetry, $\mathcal{I}_{0(\pi)}$ are topological invariants in the following sense. Consider any small perturbation to U_F which preserves the time-reflection symmetry (2). We expect that the time reflection operator R for which (2) holds will change continuously as U_F is perturbed; we illustrate this later in a concrete model. Since the trace is a continuous function of R and U_F, small changes in the arguments must lead to small changes in $\mathcal{I}_{0(\pi)}$. Because the latter are integers, they must remain invariant. Hence, any symmetry-respecting perturbation continuously connected to the identity operator will not change the trace invariants $\mathcal{I}_{0(\pi)}$. We emphasize that the existence of these invariants and the subsequent properties depends essentially on the presence of time-reflection symmetry.

At this point, it is useful to draw contrasts and comparisons with ground states of static systems. At first glance, this symmetry-protected "pinning" of quasienergy eigenvalues to 0 or π may be reminiscent of the protection of certain zero-energy modes in symmetry protected topological phases [46–50]. In systems with topological defects or boundaries, zero modes may appear as bound states protected by index theorems that define topological invariants similar to Eq. (6) [51, 52]. However, in our Floquet setting there is no such bulk-boundary correspondence nor defects; the symmetry-protected 0 and π quasienergy modes are bulk entities.

In fact, the closest static analogues of these protected many-body degeneracies arise in SUSY [53, 54], where the relevant topological invariant is the Witten index [55] $\mathrm{tr}[(-1)^F e^{-\beta H}]$, with H the Hamiltonian, F the fermion number, and β the inverse temperature. The (integer) Witten index places a lower bound on the number of eigenstates at zero energy, and thereby on the ground-state degeneracy of H. One remarkable phenomenon that can arise in certain SUSY models is "superfrustration," where the Witten index scales exponentially with system size [53, 56–58].

Within this algebraic framework, there is potential for another connection to SUSY. In systems with a conserved fermion parity $(-1)^F$, one possibility is that $\{(-1)^F, R\}=0$, in which case time-reflection changes the fermion parity of an eigenstate. In this case, any bosonic state at quasienergy E must have a fermionic partner with quasienergy $-E$. This is in stark contrast to conventional SUSY, which exhibits pairs of bosonic and fermionic states at *the same* energy.

We now present a simple model of FSUSY that features exponentially large degeneracy for even system sizes, and boson/fermion partners at equal and opposite quasienergy for odd system sizes. For the sake of exposition, we begin with the simplest model below and add interactions later. Consider a spin-$\frac{1}{2}$ chain with L sites and the two part drive

$$U_{\mathrm{F}} = U_{ZZ}\,U_X, \tag{7a}$$

$$U_{ZZ} \equiv \exp\left(\mathrm{i}\frac{\pi}{4}\sum_{i=1}^{L} Z_i Z_{i+1}\right) \tag{7b}$$

$$U_X \equiv \exp\left(-\mathrm{i}\sum_{i=1}^{L} h_i X_i\right). \tag{7c}$$

Here, X_i, Z_i are Pauli operators on the site i, and h_i are random couplings. We hereafter impose periodic boundary conditions (identifying sites 1 and $L+1$).

The model (7) has time-reflection symmetry, generated by the operator

$$R_1 = U_X^\dagger \prod_{i=1}^{L} Z_i. \tag{8}$$

To see that $R_1^2 = \mathbb{1}$, one can rewrite $R_1 = U_X^{1/2\dagger}\left(\prod_{i=1}^{L} Z_i\right) U_X^{1/2} \equiv \prod_{i=1}^{L}\widetilde{Z}_i$, where $\widetilde{Z}_i \equiv e^{\mathrm{i}\,h_i X_i/2}\, Z_i\, e^{-\mathrm{i}\,h_i X_i/2}$. Using the fact that $U_{ZZ} = \mathrm{i}^L\, U_{ZZ}^\dagger$, one verifies that Eq. (2) holds with $\theta = L\pi/2$. Observe that R_1 depends explicitly on U_{F}, just as the generator of SUSY depends explicitly on the Hamiltonian.

In fact, one can define another time-reflection operator

$$R_2 = U_{ZZ} \prod_{i=1}^{L} Z_i \tag{9}$$

with the requisite properties (setting $\theta = L\pi/2$), and $U_F = R_2 R_1$. Again, this parallels SUSY, in which the Hamiltonian is constructed from the SUSY generators [54].

Having established time-reflection symmetries in this model, we calculate the trace invariants (6) and find

$$|\mathcal{I}_{0(\pi)}| = \begin{cases} 2^{L/2} & L \text{ even} \\ 0 & L \text{ odd} \end{cases} \tag{10}$$

(for both $R_{1,2}$). Thus, for even system sizes, there is an exponentially large number of states with quasienergy $0, \pi$. (See also [59].)

The above model can be rewritten in terms of free fermions via a Jordan-Wigner transformation. Interestingly, the fermion parity operator $(-1)^F = \prod_{i=1}^{L} X_i$ (anti)commutes with the time reflection operators for even (odd) L. Thus, while the odd-L case does not host an exponentially large $|\mathcal{I}_{0(\pi)}|$, it does exhibit an unconventional pairing of bosonic and fermionic states at equal and opposite quasienergy.

Crucially, the above properties are *not* artifacts of free fermions; the invariants $\mathcal{I}_{0,\pi}$ are robust to any interaction that preserves time-reflection symmetry, while the pairing of bosonic and fermionic states additionally requires maintaining fermion parity conservation (Ising symmetry in the spin language). To illustrate this, we add interac-

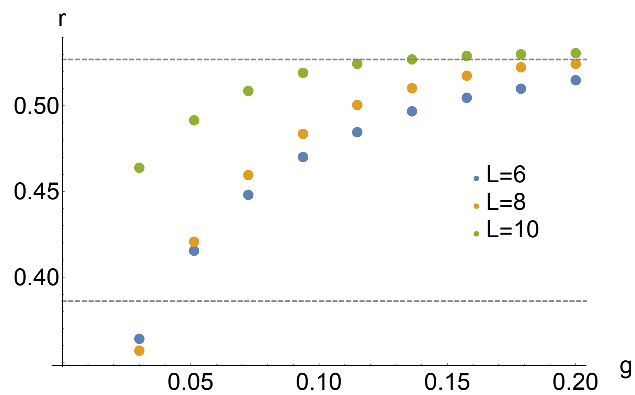

FIG. 1. (Color online) Statistics of quasienergy levels outside of the degenerate subspaces in the model (7), as measured by the parameter r defined in Ref. [60]. All data points are averaged over disorder realizations and quasienergy (see main text). Gray dashed lines at $r = 0.386$ and 0.527 indicate the expected values for the Poisson and Wigner-Dyson distributions, respectively.

tions to the transverse-field part of the drive:

$$U_X \to U_H \equiv \exp\left[-\mathrm{i}\left(\sum_{i=1}^{L} h_i X_i + g H_{\mathrm{int}}\right)\right], \tag{11}$$

where g parameterizes the strength of interaction and we demand that H_{int} anticommutes with $\prod_{i=1}^{L} Z_i$. The modified system then maintains time-reflection symmetry, with the modified time-reflection operator $R = U_H^\dagger \prod_{i=1}^{L} Z_i$. (If H_{int} additionally commutes with $\prod_{i=1}^{L} X_i$, then the unconventional boson/fermion pairing for odd L also remains.) As a result, the trace invariants (and exponentially large degeneracies) remain the same even in the presence of these interactions. In [59], we provide an alternative way to understand the degeneracy as arising from the intersection of two large subspaces; this derivation also explains why the model's properties are robust to interactions of the above type.

We now focus on the case of even L, and investigate some consequences of the macroscopic degeneracies protected by the indices $\mathcal{I}_{0,\pi}$ For the purpose of numerics, we specify to the choice

$$\begin{aligned} H_{\mathrm{int}} = \sum_{i=1}^{L} \big(& J_i^{xz} X_i Z_{i+1} + J_i^{xxx} X_{i-1} X_i X_{i+1} \\ & + J_i^{zxz} Z_{i-1} X_i Z_{i+1} \big), \end{aligned} \tag{12}$$

and we draw the random couplings h_i, J_i^{xz}, J_i^{xxx}, and J_i^{zxz} uniformly from the interval $[0, \pi/2]$.

One might wonder whether the symmetry constraint (2), which is evidently strong enough to protect exponentially large degeneracies, is also strong enough to constrain the many-body spectrum outside of the degenerate subspaces. Given the presence of strong dis-

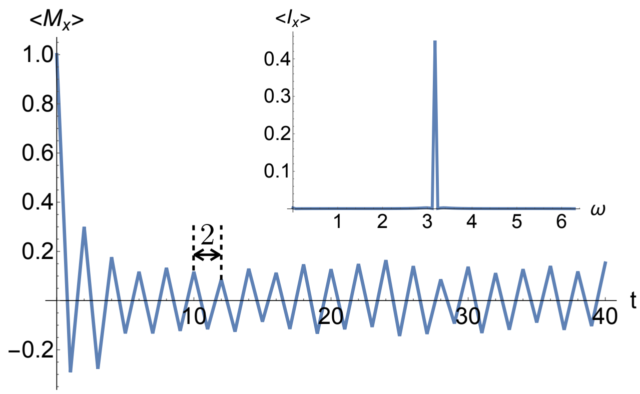

FIG. 2. (Color online) A representative time-series of the magnetization M_X starting from an initial state with all spins polarized in the $X = +1$ direction, for a single disorder realization in the interacting version of the model (7) with $L = 8$ and $g = 1$ [see Eqs. (11) and (12)]. Period-2 oscillations around the expected infinite-temperature value $\langle M_X \rangle = 0$ are clearly visible at late times. Inset: power spectrum $\langle I_X(\omega)\rangle$ for the same L and g, averaged over 20000 disorder realizations. The dominant coherent structure in the power spectrum is the peak at $\omega = \pi$, which results from the period-2 oscillations.

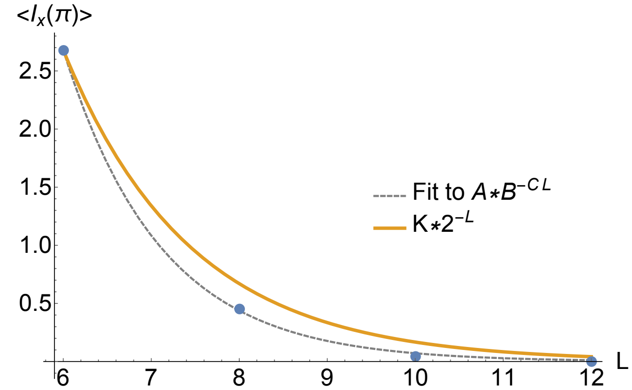

FIG. 3. (Color online) Magnitude of the peak in $\langle I_X(\omega)\rangle$ at $\omega = \pi$ as a function of system size, with an exponential fit (gray, dashed line) and the estimate (13) (orange) plotted for reference. The model used to generate the data is that used in Fig. 2. Data are averaged over 40000, 20000, 10000, and 5000 disorder realizations for $L = 6$, 8, 10, and 12, respectively.

order, are there signatures of many-body localization in this system? To answer these questions, we performed exact diagonalization at system sizes $L = 6, 8$, and 10 (for 20000, 10000, and 5000 disorder realizations, respectively) and computed the disorder-averaged level statistics of the states outside the $0, \pi$ subspaces. We computed the parameter r [60]; given an ordered list $\{E_j\}$ of quasienergies, r is defined in terms of the quasienergy gaps $\delta_j \equiv E_{j+1} - E_j$ as the average of the quantity $r_j = \min(\delta_j, \delta_{j+1})/\max(\delta_j, \delta_{j+1})$ over quasienergy (j) and disorder realizations. Even at these very small system sizes, we see level statistics consistent with the Wigner-Dyson distribution for $g \gtrsim 0.2$ (see Fig. 1). Thus, apart from the protected degeneracies, the model (7) appears to be a generic ergodic system.

Nonetheless, we now show that the protected degeneracies give rise to a distinct subharmonic response which serves as a direct signature of Floquet supersymmetry. In particular, the time evolution of the expectation values of certain operators exhibit period-2 oscillations. This follows directly from the existence of $0, \pi$ states, which are protected by FSUSY. Assume there is at least one protected pair of states with quasienergy $0, \pi$, and denote by D the space spanned by the two states. Then the Floquet operator restricted to D can be represented by ZP_D, where P_D is the projection onto D and Z is a Pauli-Z operator in the basis of the $0, \pi$ quasienergy states. Hence, the operator XP_D will flip sign every period, as $\{Z, X\} = 0$. Note that in this general discussion ZP_D and XP_D may be nonlocal operators; however, in the model (7), there is a local operator, namely the on-site X_i, which flips sign under the Floquet evolution restricted to the degenerate subspaces (see [59]).

Therefore, in the time evolution of $X = XP_D + X(1 - P_D)$, the latter piece will decay to zero because the complement of D is generically ergodic, while the former piece contributes the period-2 oscillations. However, the ratio of the size of D to that of the entire Hilbert space decreases exponentially with system size L. Hence, if one evolves from a random initial state, then the amplitude of such period-2 oscillations will decrease exponentially with L, a phenomenon that distinguishes FSUSY from the πSG/DTC phase. In fact, such dependence on system size also occurs in signatures of SUSY in Majorana models with translation symmetry [61].

We have checked these properties in the above model by computing the time evolution of the total magnetization $M_X = \frac{1}{L}\sum_{i=1}^{L} X_i$ starting from an initial state with all spins polarized in the $X = +1$ direction. A representative time series for $L = 8$ is shown in Fig. 2. Plots of the expectation values of single-site X_i operators look similar. A useful figure of merit for quantifying this subharmonic response is the power spectrum $\langle I_X(\omega)\rangle$, obtained by taking the modulus-squared of the Fourier transform of $\langle M_X(t)\rangle$, which displays a peak at $\omega = \pi$ if $\langle M_X(t)\rangle$ exhibits period-2 oscillations. We indeed find such behavior in the power spectrum; averaging over disorder realizations, we find a single peak at $\omega = \pi$, and all other structure washes out (see Fig. 2 inset).

For a typical initial state, which has overlap with all eigenstates of U_{F}, we can estimate (up to a multiplicative prefactor) that

$$\begin{aligned}\langle I_X(\pi)\rangle &= \Big|\sum_{\substack{E,E',\\ \alpha,\alpha'}} c^*_{E,\alpha} c_{E',\alpha'} \langle E,\alpha|M_X|E',\alpha'\rangle\, \delta(E-E'-\pi)\Big|^2 \\ &\lesssim 2^{-L}, \end{aligned} \tag{13}$$

where $c_{E,\alpha} = \langle E,\alpha|\psi\rangle$ is the overlap of eigenstates with the initial state $|\psi\rangle$. This exponential upper bound on the finite-size scaling of $\langle I_X(\pi)\rangle$ results from the fact that the degenerate quasienergy eigenstates constitute a fraction of order $2^{-L/2}$ of all eigenstates of U_F. We see finite-size scaling of the disorder-average of $\langle I_X(\pi)\rangle$ in exact diagonalization that is consistent with this estimate (see Fig. 3). Our simulations were carried out at $g = 1$, so that the energy levels outside the degenerate subspaces are approximately Wigner-Dyson-distributed. It is interesting that even in this chaotic regime, there are still coherent period-2 oscillations. Although this effect disappears in the thermodynamic limit due to the exponential suppression described above, it should be accessible in quantum simulation experiments, which are performed at a variable finite size.

We note that the persistence of the oscillations described above depend crucially on the presence of time-reflection symmetry; without it, the oscillations acquire a finite lifetime. However, in [59] we show that, for sufficiently small time-reflection breaking, the oscillations can persist long enough to be experimentally observable.

There are several interesting avenues to pursue regarding both Floquet supersymmetry and the particular class of models presented. FSUSY provides an alternative mechanism for achieving subharmonic response; whereas the robustness in the discrete time crystal relies on the rigidity of eigenstates (long-range correlations in space), the robustness in FSUSY relies on the rigidity of the eigenvalues pinned to $0, \pi$, a consequence of the underlying time-reflection symmetry. Moreover, FSUSY provides a mechanism whereby a protected subspace can exhibit nontrivial phenomena (e.g., period-2 oscillations) despite being embedded in a thermal system. Thus, even though non-integrable systems without many-body localization may heat to infinite temperature, it may be possible that a subspace (whose dimension can grow exponentially with system size) can behave nontrivially, as FSUSY illustrates.

The most pressing question concerning the model (7) at even L is that of the nature of the degenerate states—aside from their fixed quasienergy, do they have any special properties that are not shared by the rest of the eigenstates of U_F? The derivation of the degenerate states as the intersection of two large subspaces in [59] suggests that the degenerate states may be highly entangled, but it would be useful to quantify the amount of entanglement. It would also be interesting to consider whether the protected macroscopic degeneracy could be useful for quantum information processing. Having access to an exponentially large number of exactly degenerate eigenstates could aid in the coherent storage and manipulation of quantum information.

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[1] J. H. Shirley, Phys. Rev. **138**, B979 (1965).
[2] H. Sambe, Phys. Rev. A **7**, 2203 (1973).
[3] M. Bukov, L. D'Alessio, and A. Polkovnikov, Advances in Physics **64**, 139 (2015).
[4] S. Rahav, I. Gilary, and S. Fishman, Phys. Rev. A **68**, 013820 (2003).
[5] A. Eckardt, C. Weiss, and M. Holthaus, Phys. Rev. Lett. **95**, 260404 (2005).
[6] T. Oka and H. Aoki, Phys. Rev. B **79**, 081406 (2009).
[7] N. H. Lindner, G. Refael, and V. Galitski, Nature Phys. **7**, 490 (2011).
[8] J. Dalibard, F. Gerbier, G. Juzeliunas, and P. Öhberg, Rev. Mod. Phys. **83**, 1523 (2011).
[9] L. D'Alessio and A. Polkovnikov, Ann. Phys. **333**, 19 (2013).
[10] N. Goldman and J. Dalibard, Phys. Rev. X **4**, 031027 (2014).
[11] A. G. Grushin, A. Gómez-León, and T. Neupert, Phys. Rev. Lett. **112**, 156801 (2014).
[12] T. Iadecola, L. H. Santos, and C. Chamon, Phys. Rev. B **92**, 125107 (2015).
[13] Y.-J. Lin, R. L. Compton, A. R. Perry, W. D. Phillips, J. V. Porto, and I. B. Spielman, Phys. Rev. Lett. **102**, 130401 (2009).
[14] M. Aidelsburger, M. Atala, M. Lohse, J. Barreiro, B. Paredes, and I. Bloch, Phys. Rev. Lett. **111**, 185301 (2013).
[15] H. Miyake, G. A. Siviloglou, C. J. Kennedy, W. C. Burton, and W. Ketterle, Phys. Rev. Lett. **111**, 185302 (2013).
[16] G. Jotzu, M. Messer, R. Desbuquois, M. Lebrat, T. Uehlinger, D. Greif, and T. Esslinger, Nature **515**, 237 (2014).
[17] P. Bordia, H. Lüschen, U. Schneider, M. Knap, and I. Bloch, Nat. Phys. **13**, 460 (2017).
[18] L. D'Alessio and M. Rigol, Phys. Rev. X **4**, 041048 (2014).
[19] A. Lazarides, A. Das, and R. Moessner, Phys. Rev. E **90**, 012110 (2014).

[20] A. Lazarides, A. Das, and R. Moessner, Phys. Rev. Lett. **112**, 150401 (2014).
[21] A. Chandran and S. L. Sondhi, Phys. Rev. B **93**, 174305 (2016).
[22] V. Gritsev and A. Polkovnikov, SciPost Phys. **2**, 021 (2017).
[23] D. A. Huse, R. Nandkishore, V. Oganesyan, A. Pal, and S. L. Sondhi, Phys. Rev. B **88**, 014206 (2013).
[24] A. Chandran, V. Khemani, C. R. Laumann, and S. L. Sondhi, Phys. Rev. B **89**, 144201 (2014).
[25] P. Ponte, A. Chandran, Z. Papić, and D. A. Abanin, Ann. Phys. **353**, 196 (2015).
[26] V. Khemani, A. Lazarides, R. Moessner, and S. L. Sondhi, Phys. Rev. Lett. **116**, 250401 (2016).
[27] D. A. Abanin, W. De Roeck, and F. Huveneers, Phys. Rev. Lett. **115**, 256803 (2015).
[28] M. Bukov, S. Gopalakrishnan, M. Knap, and E. Demler, Phys. Rev. Lett. **115**, 205301 (2015).
[29] T. Mori, T. Kuwahara, and K. Saito, Phys. Rev. Lett. **116**, 120401 (2016).
[30] T. Kuwahara, T. Mori, and K. Saito, Ann. Phys. **367**, 96 (2016).
[31] E. Canovi, M. Kollar, and M. Eckstein, Phys. Rev. E **93**, 012130 (2016).
[32] D. A. Abanin, W. De Roeck, W. W. Ho, and F. Huveneers, Phys. Rev. B **95**, 014112 (2017).
[33] D. V. Else, B. Bauer, and C. Nayak, Phys. Rev. X **7**, 011026 (2017).
[34] S. Vajna, K. Klobas, T. Prosen, and A. Polkovnikov, arXiv:1707.08987 (unpublished).
[35] C. W. von Keyserlingk and S. L. Sondhi, Phys. Rev. B **93**, 245145 (2016).
[36] D. V. Else and C. Nayak, Phys. Rev. B **93**, 201103 (2016).
[37] R. Roy and F. Harper, Phys. Rev. B **94**, 125105 (2016).
[38] A. C. Potter, T. Morimoto, and A. Vishwanath, Phys. Rev. X **6**, 041001 (2016).
[39] I.-D. Potirniche, A. C. Potter, M. Schleier-Smith, A. Vishwanath, and N. Y. Yao, Phys. Rev. Lett. **119**, 123601 (2017).
[40] C. W. von Keyserlingk, V. Khemani, and S. L. Sondhi, Phys. Rev. B **94**, 085112 (2016).
[41] D. V. Else, B. Bauer, and C. Nayak, Phys. Rev. Lett. **117**, 090402 (2016).
[42] N. Y. Yao, A. C. Potter, I.-D. Potirniche, and A. Vishwanath, Phys. Rev. Lett. **118**, 030401 (2017).
[43] J. Zhang, P. Hess, A. Kyprianidis, P. Becker, A. Lee, J. Smith, G. Pagano, I. Potirniche, A. Potter, A. Vishwanath, et al., Nature **543**, 217 (2017).
[44] S. Choi, J. Choi, R. Landig, G. Kucsko, H. Zhou, J. Isoya, F. Jelezko, S. Onoda, H. Sumiya, V. Khemani, et al., Nature **543**, 221 (2017).
[45] J. K. Asbóth, B. Tarasinski, and P. Delplace, Phys. Rev. B **90**, 125143 (2014).
[46] R. Jackiw and C. Rebbi, Phys. Rev. D **13**, 3398 (1976).
[47] W. P. Su, J. R. Schrieffer, and A. J. Heeger, Phys. Rev. Lett. **42**, 1698 (1979).
[48] R. Jackiw and P. Rossi, Nuclear Physics B **190**, 681 (1981).
[49] N. Read and D. Green, Phys. Rev. B **61**, 10267 (2000).
[50] J. C. Y. Teo and C. L. Kane, Phys. Rev. B **82**, 115120 (2010).
[51] M. F. Atiyah and I. M. Singer, Bull. Amer. Math. Soc. **69**, 422 (1963).
[52] E. J. Weinberg, Phys. Rev. D **24**, 2669 (1981).
[53] H. Nicolai, Journal of Physics A: Mathematical and General **9**, 1497 (1976).
[54] E. Witten, Nucl. Phys. B **188**, 513 (1981).
[55] E. Witten, Nucl. Phys. B **202**, 253 (1982).
[56] P. Fendley and K. Schoutens, Phys. Rev. Lett. **95**, 046403 (2005).
[57] H. van Eerten, J. Math. Phys. **46**, 123302 (2005).
[58] H. Katsura, H. Moriya, and Y. Nakayama, arXiv:1710.04385 (unpublished).
[59] See Supplementary Material.
[60] A. Pal and D. A. Huse, Phys. Rev. B **82**, 174411 (2010).
[61] T. H. Hsieh, G. B. Halász, and T. Grover, Phys, Rev. Lett. **117**, 166802 (2016).