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# Integrable time-dependent quantum Hamiltonians

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We formulate a set of conditions under which the non-stationary Schrödinger equation with a time-dependent Hamiltonian is exactly solvable analytically. The main requirement is the existence of a nonabelian gauge field with zero curvature in the space of system parameters. Known solvable multistate Landau-Zener models satisfy these conditions. Our method provides a strategy to incorporate time-dependence into various quantum integrable models while maintaining their integrability. We also validate some prior conjectures, including the solution of the driven generalized Tavis-Cummings model.

Quantum coherent dynamics controlled by strong time-dependent fields can be realized and explored nowadays in systems of considerable complexity [1–8]. Time-dependent parameters play a critical role in NMR [9], quantum information processing [10–20], molecular dynamics [21–23] and cold atom experiments [24–26]. On the theory side, quantum dynamics of time-dependent many-body Hamiltonians, especially their exact analytical description, present considerable challenges. In contrast, exact solutions of significant relevance to experiment inform our understanding of stationary states, e.g., Bethe’s Ansatz solution of paradigmatic models [27–29]. Nontrivial exact results have been also obtained for quantum quenches, such as the Generalized Gibbs Ensemble description of the dynamics of the spin-1/2 Heisenberg chain [30] and quantum quench phase diagrams of BCS superconductors [31]. Such methods, unfortunately, do not apply to a Hamiltonian with continuous time-dependence.

In this letter, we propose an approach for solving the non-stationary Schrödinger equation exactly for a broad class of time-dependent Hamiltonians. This approach allows us to make parameters of a quantum integrable model, e.g., the BCS and generalized Tavis-Cummings Hamiltonians, vary in time in such a way that resulting dynamics are exactly solvable. **While here we primarily focus on the scattering problem for Hamiltonians linear in time, our method applies to a much broader class of problems, including Floquet Hamiltonians [32] and models with other non-polynomial time-dependence.**

Important examples of driven systems are matrix Hamiltonians **of the form**  $H(t) = A + Bt$ , where  $A$  and  $B$  are time-independent Hermitian  $N \times N$  matrices. The problem of finding the scattering matrix that relates the state of the system at  $t = +\infty$  to that at  $t = -\infty$  is called then the multistate Landau-Zener problem. The  $2 \times 2$  problem was solved by Landau, Zener, Majorana and Stückelberg in 1932 [33–36]. For  $N \geq 3$  the solution is known only for special choices of  $A$  and  $B$ . Earliest examples include Demkov-Osherov [37, 38], bow-tie [39], generalized bow-tie [40, 41], composite [42], and infinite chain [43] models. In a more recent work [44], it was shown that nontrivial solvable models belong to fami-

lies of mutually commuting Hamiltonians linear or quadratic in  $t$ . It was therefore conjectured that quantum integrability understood as the existence of nontrivial time-dependent commuting partners [45–48] is a necessary condition for the multistate Landau-Zener solvability. In a parallel development, **empirical** methods to solve and search for new models were discovered [49–51] and since then the number of such models has grown rapidly [52–55]. **However, these results still lack rigorous justification.**

Our approach provides a framework to **justify** exact solutions for all these models and **to identify and solve new ones**. It also supports the conjecture made in [44]. Below, we first formulate our approach and then discuss various many-body and matrix models that fit into it. To illustrate our technique, we solve the scattering problem for two nontrivial models – the generalized Tavis-Cummings Hamiltonian with a linear drive and a new 4-state Hamiltonian linear in  $t$ . **Moreover, we present a new class of solvable interacting spin systems with time-dependent couplings.** We conclude with several general observations and an outline of the idea of the solution for arbitrary  $t$ .

Consider a Hamiltonian  $\hat{H}(t, \vec{x})$  that, in addition to time, depends on  $M$  real parameters  $(x^1, \dots, x^M) = \vec{x}$ . For example, in the multistate Landau-Zener problem these can be certain matrix elements of  $A$  and  $B$ . The main idea is to embed the non-stationary Schrödinger equation for  $\hat{H}(t, \vec{x})$  into a set of multi-time Schrödinger equations

$$i\partial_j \Psi(\mathbf{x}) = \hat{H}_j \Psi(\mathbf{x}), \quad j = 0, 1, \dots, M, \quad (1)$$

where  $\mathbf{x} = (t, \vec{x})$ ,  $\partial_j \equiv \partial/\partial x^j$ ,  $x^0 = t$ ,  $\hat{H}_0 \equiv \hat{H}(t, \vec{x})$ , and Hamiltonians  $\hat{H}_j$  are independent. In other words, the first equation ( $j = 0$ ) is the non-stationary Schrödinger equation, while the rest are auxiliary Schrödinger equations that help us solve it exactly. Taking the derivative of Eq. (1) with respect to  $x^k$ , we derive consistency conditions

$$\partial_j \hat{H}_k - \partial_k \hat{H}_j - i[\hat{H}_k, \hat{H}_j] = 0, \quad k, j = 0, \dots, M. \quad (2)$$

These conditions are sufficient and necessary for system (1) to possess a joint solution for any initial condition [56, 57]. We

may view them as a generalization of the notion of integrals of motion for time-dependent quantum Hamiltonians.

A formal solution of Eq. (1) along a path in the space of real parameters  $\mathbf{x}$  that starts at a reference point  $\mathbf{x}_0$  is an ordered exponential

$$\Psi(\mathbf{x}) = T \exp \left( -i \int_{\mathcal{P}} \hat{H}_j d\mathbf{x}^j \right) \Psi(\mathbf{x}_0), \quad (3)$$

where we assume summation over repeated indices. Treating Hamiltonians  $\hat{H}_j$  as matrix components of a nonabelian gauge field  $\mathcal{A}(\mathbf{x})$ ,  $\mathcal{A}_j = -i\hat{H}_j$ , we interpret Eq. (2) as the zero curvature condition  $\mathcal{F}_{jk} \equiv \partial_j \mathcal{A}_k - \partial_k \mathcal{A}_j - [\mathcal{A}_j, \mathcal{A}_k] = 0$ , so that  $\Psi(\mathbf{x})$  in Eq. (3) is independent of the integration path  $\mathcal{P}$  as long as its endpoints are fixed. It is precisely this freedom to choose a suitable path that enables us to explicitly solve the scattering problem. Similar zero curvature integrability condition is also well known in soliton physics [58].

Further, consider a path  $\mathcal{P}_\tau$  parameterized by a variable  $\tau$

$$\mathcal{P}_\tau : \quad x^j(\tau) = v^j \tau + x_0^j, \quad j = 0, \dots, M, \quad (4)$$

where  $v^j$  and  $x_0^j$  are constants. The state vector  $\Psi(\tau) = \Psi(\mathbf{x}(\tau))$  along this path satisfies

$$i \frac{d\Psi(\tau)}{d\tau} = \hat{h}(\tau) \Psi(\tau), \quad (5)$$

$$\hat{h}(\tau) = \sum_j v^j \hat{H}_j(\mathbf{x}(\tau)). \quad (6)$$

Solutions of Eq. (5) follow from those of Eq. (1). Therefore,  $\hat{h}(\tau)$  – an arbitrary linear combination of  $\hat{H}_j$  – is also a solvable time-dependent model just like a linear combination of integrals of motion of a time-independent model is also an integral. Note however that the coefficients  $v^j$  of this linear combination dictate the time-dependence of  $\hat{h}(\tau)$ . **More generally, one can choose nonlinear  $x^j(\tau)$  instead of the linear path in Eq. (4), leading to, e.g., integrable Hamiltonians  $\hat{h}(\tau)$  with exponential or oscillatory behavior of the couplings.**

An important observation is that complex looking Eq. (1) simplifies considerably when the matrix elements of the Hamiltonians are real. Then, the real and imaginary parts of Eq. (2) yield two separate conditions

$$\begin{aligned} [\hat{H}_j, \hat{H}_k] &= 0, \\ \partial_j \hat{H}_k - \partial_k \hat{H}_j &= 0, \quad j, k = 0, 1, \dots, M. \end{aligned} \quad (7)$$

These equations suggest a strategy for identifying solvable time-dependent models. First, we note that Eq. (7) is to be supplemented with a notion of a nontrivial commuting partner that weeds out trivial partners (e.g., projectors onto the eigenstates of  $\hat{H}$ ). One way is to consider parameter-dependent  $\hat{H}_j$  and to fix their dependence on the parameter [59]. This leads to a systematic classification and explicit construction of commuting families of parameter-dependent matrix Hamiltonians [44–47], which are interesting candidates for our approach. More generally, any quantum integrable model that contains two or more real parameters is a potential candidate. Such

models have an extensive number of integrals of motion that satisfy Eq. (7). If no initial subset of integrals satisfies Eq. (8), we attempt to redefine them by taking various combinations and similarly redefine the parameters to make Eq. (8) work for at least  $M = 1$ . Note that once we declare one of the variables  $x^j$  to be the physical time, commuting partners  $\hat{H}_j$  cease to be integrals of motion.

For example, take the generalized Tavis-Cummings model

$$\hat{H}_{\text{TC}} = \sum_{j=1}^{N_s} \varepsilon_j \hat{s}_j^z - \omega \hat{a}^\dagger \hat{a} + g \sum_{j=1}^{N_s} (\hat{a}^\dagger \hat{s}_j^- + \hat{a} \hat{s}_j^+), \quad (9)$$

where  $\hat{a}$  is the boson annihilation operator and  $\hat{s}_j^z, \hat{s}_j^\pm$  are spin-1/2 operators. Its commuting partners are [60]

$$\hat{H}_j = (\varepsilon_j + \omega) \hat{s}_j^z + g(\hat{a}^\dagger \hat{s}_j^- + \hat{a} \hat{s}_j^+) + 2g^2 \sum_{k \neq j} \frac{\hat{\mathbf{s}}_j \cdot \hat{\mathbf{s}}_k}{\varepsilon_j - \varepsilon_k}. \quad (10)$$

Equations (7,8) hold with  $M = N_s$ ,  $\hat{H}_0 = \hat{H}_{\text{TC}}$ ,  $\mathbf{x} = (\omega, \varepsilon_1, \dots, \varepsilon_{N_s})$ . **To derive a new solvable time-dependent model**, consider the BCS Hamiltonian. In terms of Anderson pseudospin-1/2 operators it reads

$$\hat{H}_{\text{BCS}} = \sum_{j=1}^{N_s} 2\varepsilon_j \hat{s}_j^z - \frac{1}{2B} \sum_{j,k} \hat{s}_j^+ \hat{s}_k^-, \quad (11)$$

where  $(2B)^{-1}$  stands for the BCS coupling constant. Its commuting partners are Gaudin magnets [27, 61]

$$\hat{H}_j = 2B \hat{s}_j^z - \sum_{k \neq j} \frac{\hat{\mathbf{s}}_j \cdot \hat{\mathbf{s}}_k}{\varepsilon_j - \varepsilon_k}. \quad (12)$$

Now  $\hat{H}_0 = \hat{H}_{\text{BCS}}$  and  $x^0 = B$ . Thus, the BCS Hamiltonian with coupling  $\propto 1/t$  fits into our construction. Similarly, using the commuting partners derived in [44], we verified that **many known solvable models**, including the Demkov-Osherov, bow-tie, and generalized bow-tie, as well as Landau-Zener-Coulomb models [62–65], **also** fit into our construction.

A key point of this letter is that zero curvature condition (1) leads to an explicit exact solution of the scattering problem. Consider, e.g., the multistate Landau-Zener model  $\hat{H}(t, \vec{x}) = \hat{A}(\vec{x}) + t\hat{B}(\vec{x})$  for which we need to determine the matrix of transition probabilities  $P$  with elements  $P_{nn'} \equiv P_{n' \rightarrow n} \equiv |S_{nn'}|^2$ . Here  $S$  is the scattering matrix between eigenstates at  $t = -\infty$  and  $t = +\infty$  at some fixed values of the parameters,  $\vec{x} = \vec{c}$  [38]. As discussed above, we are free to choose any path in the space  $\mathbf{x} = (t, \vec{x})$  that connects the points  $(-\infty, \vec{c})$  and  $(+\infty, \vec{c})$ . It is convenient to choose a path  $\mathcal{P}_\infty$ , such that  $|\mathbf{x}|$  is always large and the time-evolution is adiabatic everywhere, except the neighborhood of isolated points, where scattering takes place. The corresponding scattering problem is typically simple thanks to large  $|\mathbf{x}|$ , e.g., it reduces to a  $2 \times 2$  Landau-Zener problem in the two nontrivial examples we consider below. In general, Eqs. (7,8) enable one to construct a multidimensional version of WKB

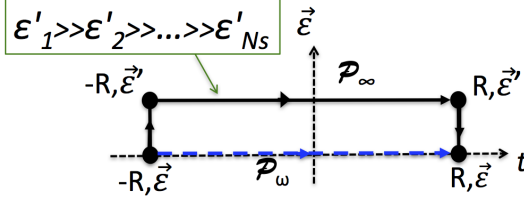


FIG. 1. (color online) Paths in the space of parameters  $\varepsilon_j$  and time used to evaluate transition probabilities for the driven generalized Tavis-Cummings model (9),  $\omega = t$ . On  $\mathcal{P}_\omega$ ,  $\omega$  changes from  $-R$  to  $+R$ , all  $\varepsilon_j$  are fixed. Since this model is a part of a commuting family of  $\hat{H}_j$  that satisfies the zero curvature condition, we can deform  $\mathcal{P}_\omega$  into a new path  $\mathcal{P}_\infty$  without modifying the scattering matrix.

with simple scattering matrices connecting adiabatic (WKB) solutions in different adiabatic regions [66].

Our first example is the Tavis-Cummings model (9) with linear drive,  $\omega = t$ . Let  $\varepsilon_1 > \varepsilon_2 > \dots > \varepsilon_{N_s}$ . We are interested in the evolution along the path  $\mathcal{P}_\omega$  shown in Fig. 1. On this path  $\varepsilon_j = \text{const}$ , while  $\omega$  changes from  $-R$  to  $+R$ . At the end we take the limit  $R \rightarrow \infty$ . This scattering problem was solved in [53] under the assumption that  $\varepsilon_j$  are well separated, i.e.  $\varepsilon_1 \gg \varepsilon_2 \gg \dots \gg \varepsilon_{N_s}$ . It was further conjectured in [53] that this is the general solution. We are now in the position to prove this conjecture. To do so, consider the path  $\mathcal{P}_\infty$  in Fig. 1 that has the same endpoints as  $\mathcal{P}_\omega$ . On the first vertical leg of  $\mathcal{P}_\infty$ ,  $\varepsilon_j$  evolve, keeping the ordering of  $\varepsilon_j$ , until the condition  $\varepsilon_1 \gg \varepsilon_2 \gg \dots \gg \varepsilon_{N_s}$  is met. On the second vertical leg, they evolve back to their initial values. Since  $|\omega|$  is large and  $\varepsilon_j$  are distinct, this evolution is purely adiabatic and does not affect the transition probabilities. On the horizontal leg of  $\mathcal{P}_\infty$  the problem is precisely the one solved in [53]. This proves the above conjecture.

In our second example, we take a previously solved  $4 \times 4$  multistate Landau-Zener problem [51, 52], which was originally introduced to describe hysteresis in molecular nanomagnets [67], and derive from it a new, more general Hamiltonian by the prescription outlined below Eq. (6). We then proceed to determine the transition probabilities for this new model. Let

$$H(t, e) = \begin{pmatrix} b_1 t + e & 0 & g & -\gamma \\ 0 & -b_1 t + e & \gamma & g \\ g & \gamma & b_2 t & 0 \\ -\gamma & g & 0 & -b_2 t \end{pmatrix}, \quad (13)$$

where  $b_1, b_2, e, g$  and  $\gamma$  are constants. To determine if this Hamiltonian fits into our approach, we first search for a non-trivial commuting partner  $H_1$  linear in  $t$ . This reduces to a set of linear algebraic equations for parameters of  $H_1$  [47]. We find three linearly independent commuting operators. Two of them are trivial – the unit matrix and  $H$  itself. Therefore, there is a single nontrivial commuting partner, which we determine explicitly. When both  $H_0 \equiv H$  and  $H_1$  are linear in  $t$ , Eq. (7) implies that their time-dependent parts are diagonal in the same basis. So, to satisfy (8), the parameter  $x^1$  must be constructed from diagonal time-independent elements of  $H$ .

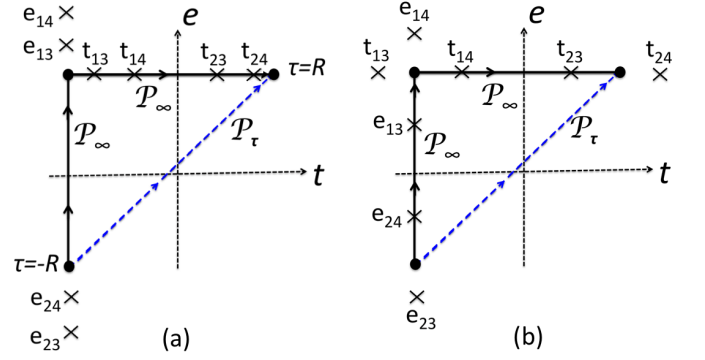


FIG. 2. (color online) Paths in the space  $(t, e)$  for evaluating transition probabilities for the model (16). On  $\mathcal{P}_\tau$ ,  $\tau$  changes from  $-R$  to  $+R$ ; all other parameters are fixed. We deform  $\mathcal{P}_\tau$  into  $\mathcal{P}_\infty$  without affecting the scattering matrix. Points  $e_{ij}$  and  $t_{ij}$  marked with crosses indicate nonadiabatic Landau-Zener transitions between levels  $i$  and  $j$  for (a)  $v < b_1 - b_2$  and (b)  $b_1 + b_2 > v > b_1 - b_2$ .

A natural candidate is  $x^1 = e$ . Searching then for  $H_1$  that satisfies (8) in the form of a linear combination of the three commuting operators, we find

$$H_1(t, e) = \begin{pmatrix} t + \frac{b_1 e}{b_1^2 - b_2^2} & 0 & \frac{g}{b_1 - b_2} & \frac{-\gamma}{b_1 + b_2} \\ 0 & t - \frac{b_1 e}{b_1^2 - b_2^2} & \frac{-\gamma}{b_1 + b_2} & \frac{g}{b_1 - b_2} \\ \frac{g}{b_1 - b_2} & \frac{-\gamma}{b_1 + b_2} & -\frac{b_2 e}{b_1^2 - b_2^2} & 0 \\ \frac{-\gamma}{b_1 + b_2} & \frac{g}{b_1 - b_2} & 0 & \frac{b_2 e}{b_1^2 - b_2^2} \end{pmatrix}. \quad (14)$$

Let the evolution path be

$$\mathcal{P}_\tau : t = \tau, e = v\tau + e_0, \quad (15)$$

with constant  $v$  and  $e_0$ . The Hamiltonian (5) for  $\mathcal{P}_\tau$  is

$$h(\tau) = \begin{pmatrix} \beta_1 \tau + e_1 & 0 & g(1+x) & -\gamma(1+y) \\ 0 & \beta_2 \tau + e_2 & \gamma(1-y) & g(1-x) \\ g(1+x) & \gamma(1-y) & \beta_3 \tau + e_3 & 0 \\ -\gamma(1+y) & g(1-x) & 0 & \beta_4 \tau + e_4 \end{pmatrix}, \quad (16)$$

$$x = \frac{v}{b_1 - b_2}, y = \frac{v}{b_1 + b_2}, \beta_{1,2} = 2v \pm b_1(1+xy), \beta_{3,4} = \pm b_2(1-xy), e_{1,2} = e_0(1 \pm b_1 xy/v), e_{3,4} = \mp e_0 b_2 xy/v.$$

This is a new, previously unsolved model more general than (13), e.g., all couplings (off-diagonal matrix elements) in (16) are distinct. We proceed to solve it with our method.

Let  $b_1 > b_2 > 0$  and  $v > 0$ . We are interested in the evolution matrix for  $h(\tau)$  along the path  $\mathcal{P}_\tau$  from  $\tau = -R$  to  $\tau = R$ , see Fig. 2(a), in the limit  $R \rightarrow \infty$ . Because  $H_0(t, e)$  and  $H_1(t, e)$  satisfy the zero curvature condition, the evolution matrix is the same as that for the path  $\mathcal{P}_\infty$ . The latter has two pieces. In the vertical one, we set  $t = -R$  and vary  $e$  from  $-vR + e_0$  to  $vR + e_0$ . In the horizontal piece, we fix  $e = vR + e_0$  and vary  $t$  from  $-R$  to  $R$ . According to Eq. (3), only  $H_1$  contributes on the first piece and only  $H_0$  on the second. Along  $\mathcal{P}_\infty$ , diagonal matrix elements of  $H_0$  and  $H_1$  (adiabatic levels) are large compared to the couplings. Therefore, the levels are well separated, except on disjointed

small segments of  $\mathcal{P}_\infty$  near points where a pair of the diagonal elements is degenerate. These segments connect adiabatic parts of  $\mathcal{P}_\infty$  where the adiabatic approximation is exact in the limit  $R \rightarrow \infty$ . Let us write the state of the system as  $\Psi(t, e) = \sum_k a_k |k\rangle$ , where  $|k\rangle$  are the eigenstates of the diagonal parts of  $H_0$  and  $H_1$  (diabatic eigenstates). Diabatic and adiabatic (instantaneous) eigenstates coincide in adiabatic parts of  $\mathcal{P}_\infty$  when  $R \rightarrow \infty$ . In the adiabatic approximation, absolute values of  $a_k$  remain the same, while their phases evolve with  $t$  and  $e$ .

In the vicinity of degeneracy points, two levels come close and transitions between them become locally possible. The other two levels, however, remain far remote and do not affect these nonadiabatic transitions. Suppose  $v < b_1 - b_2$ . For this case, we mark the points of diabatic level crossings with crosses in Fig. 2(a). Along  $\mathcal{P}_\infty$ , adiabatic approximation brakes near four points that all have  $e = vR + e_0$  and

$$t_{13/24} = \mp \frac{vR + e_0}{b_1 - b_2}, \quad t_{14/23} = \mp \frac{vR + e_0}{b_1 + b_2}.$$

The distances between these points are  $\propto R$ , which means that regions of pairwise nonadiabatic transitions along  $\mathcal{P}_\infty$  are well apart. Consider, e.g., the evolution of the amplitudes  $a_1$  and  $a_3$  near  $t_{13}$  that is governed by  $H_0$ . Writing  $t = t' + t_{13}$  and disregarding the other two levels, we find

$$i \frac{da_1}{dt'} = b_1 t' a_1 + g a_3, \quad i \frac{da_3}{dt'} = b_2 t' a_3 + g a_1, \quad (17)$$

which is a  $2 \times 2$  Landau-Zener problem, whose scattering matrix is known explicitly [33–36]. Since the other two levels do not experience nonadiabatic transitions here, they produce only diagonal unit entries in the scattering matrix for evolution through  $t_{13}$ . The total evolution matrix  $S$  for the path  $\mathcal{P}_\infty$  factorizes into an ordered product of such pairwise scattering matrices  $S^{ab}$ , where  $a, b$  label states experiencing nonadiabatic transitions and diagonal matrices  $U^{\alpha, \beta}$  describe adiabatic evolution between points  $\alpha$  and  $\beta$  on this path, i.e.

$$S = U^{R, t_{24}} S^{24} U^{t_{24}, t_{23}} S^{23} U^{t_{23}, t_{14}} S^{14} U^{t_{14}, t_{13}} S^{13} U^{t_{13}, -R}.$$

Trivial phases resulting from the adiabatic evolution drop out from the matrix of transition probabilities and we obtain [66]

$$P^{v < b_1 - b_2} = \begin{pmatrix} p_1 p_2 & 0 & p_2 q_1 & q_2 \\ 0 & p_1 p_2 & q_2 & p_2 q_1 \\ p_2 q_1 & q_2 & p_1 p_2 & 0 \\ q_2 & p_2 q_1 & 0 & p_1 p_2 \end{pmatrix}, \quad (18)$$

$$p_1 = e^{-2\pi g^2/(b_1 - b_2)}, \quad p_2 = e^{-2\pi \gamma^2/(b_1 + b_2)}, \quad q_{1,2} = 1 - p_{1,2}.$$

This result does not depend on  $v$ , so it coincides with the solution for the model (13) found in [51, 52].

The situation changes for  $b_2 + b_2 > v > b_1 - b_2$ . Now the points of adiabaticity violation  $e_{24}$  and  $e_{13}$  lie on the first leg of the path  $\mathcal{P}_\infty$  as shown in Fig. 2(b). Pairwise transitions near these points are now governed by the Hamiltonian  $H_1$

and the transition probability matrix in this case is different

$$P^{v > b_1 - b_2} = \begin{pmatrix} p_1 p_2 & q_1 q_2 & p_2 q_1 & p_1 q_2 \\ q_1 q_2 & p_1 p_2 & p_1 q_2 & p_2 q_1 \\ p_2 q_1 & p_1 q_2 & p_1 p_2 & q_1 q_2 \\ p_1 q_2 & p_2 q_1 & q_1 q_2 & p_1 p_2 \end{pmatrix}. \quad (19)$$

For  $v > b_1 + b_2$ , all four points with Landau-Zener transitions lie on the first leg of  $\mathcal{P}_\infty$  and

$$P^{v > b_1 + b_2} = \begin{pmatrix} p_1 p_2 & 0 & q_1 & p_1 q_2 \\ 0 & p_1 p_2 & p_1 q_2 & q_1 \\ q_1 & p_1 q_2 & p_1 p_2 & 0 \\ p_1 q_2 & q_1 & 0 & p_1 p_2 \end{pmatrix}. \quad (20)$$

We see that our approach not only reproduces the previously known solution for the Hamiltonian (13), but also solves a more complex model (16).

Thus, we have identified a symmetry – multi-time evolution with commuting Hamiltonians – that leads to the integrability of unitary dynamics with time-dependent Hamiltonians. Our approach generates numerous new solvable multistate Landau-Zener models. As examples, we solved a four-state model (16), and proved the previously conjectured solution of a combinatorially complex driven Tavis-Cummings Hamiltonian (9), which models detuning sweeps in atom-molecule condensates [68, 69]. The time-dependent BCS Hamiltonian (11) describes the effect of turning off or on the superconducting coupling as  $t$  varies from  $0^+$  to  $+\infty$  or vice versa, which can be experimentally realized in cold atomic fermions with tunable interactions. We believe, this integrability is behind most if not all nontrivial exactly solvable multistate Landau-Zener and Landau-Zener-Coulomb models [62–65]. It explains why in such problems the scattering matrix factorizes into a product of two-state scattering matrices [49] – since Eq. (1) allows a choice of an integration path that bypasses the region of complex nonadiabatic dynamics. It also explains why basic known solvable models have commuting partners with simple linear or quadratic dependence on  $t$  [44]. Indeed, pairs of such operators that also satisfy Eq. (8) lead to relatively simple versions of the WKB approximation necessary to determine scattering matrices. Further, Eq. (6) shows how certain distortions of the parameters [50] give rise to entire families of solvable models.

Finally, we note that when  $\hat{H}_j$  are isotropic Gaudin magnets [Eq. (12) at  $B = 0$ ], the  $j = 1, \dots, M$  subsystem of Eq. (1) is the famous Knizhnik-Zamolodchikov equation [70]. Its solutions have been obtained using off-shell Bethe's Ansatz [71]. This was generalized to  $B \neq 0$  in [72] (see also [73]) and exploited in [74] to obtain the dynamics of an isotropic Gaudin magnet with time-dependent  $\varepsilon_i$ . We believe, solutions to Eq. (1), i.e. exact inexplicit solutions of the non-stationary Schrödinger equation at arbitrary  $t$ , for all time-dependent Hamiltonians discussed in this letter can be obtained by further extending this technique.

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