Latent Computational Complexity of Symmetry-Protected Topological Order with Fractional Symmetry

Jacob Miller and Akimasa Miyake

Phys. Rev. Lett. **120**, 170503 — Published 26 April 2018

DOI: 10.1103/PhysRevLett.120.170503

Latent Computational Complexity of Symmetry-Protected Topological Order with Fractional Symmetry

Jacob Miller* and Akimasa Miyake†
Center for Quantum Information and Control, Department of Physics and Astronomy, University of New Mexico, Albuquerque, NM 87131, USA

An emerging insight is that ground states of symmetry-protected topological orders (SPTO's) possess latent computational complexity in terms of their many-body entanglement. By introducing a fractional symmetry of SPTO, which requires the invariance under 3-colorable symmetries of a lattice, we prove that every renormalization fixed-point state of 2D $(\mathbb{Z}_2)^m$ SPTO with fractional symmetry can be utilized for universal quantum computation using only Pauli measurements, as long as it belongs to a nontrivial 2D SPTO phase. Our infinite family of fixed-point states may serve as a base model to demonstrate the idea of a "quantum computational phase" of matter, whose states share universal computational complexity ubiquitously.

Introduction.—Understanding the varied correspondence between quantum entanglement and quantum computation is one of the leading goals of quantum information science. Measurement-based quantum computation (MQC) [1–3], where computation is driven by single-spin measurements on a many-body resource state, lets us study this correspondence directly, in terms of the computations achievable with a fixed resource state. Of particular interest are the universal resource states, whose many-body entanglement lets us implement any quantum computation efficiently [4–6]. In trying to characterize the entanglement found in universal resource states, researchers have developed a long list of examples, from the 2D cluster state [7, 8] and certain tensor network states [4, 5, 9–13], to condensed matter models such as 2D Affleck-Kennedy-Lieb-Tasaki (AKLT) states [14–18] and renormalization fixed-point states of interacting bosonic quantum matter [19, 20].

An emergent insight from these examples has been the utility of symmetry-protected topological order (SPTO), a form of quantum order arising from nontrivial many-body entanglement protected by a symmetry [21–29]. This insight has led researchers to investigate a general correspondence between SPTO and MQC, with the ultimate aim of discovering a "universal computational phase" of quantum matter. In such a phase, the constituent states' SPTO and symmetry alone structure them as universal resource states. While this approach has uncovered increasingly general single-qubit computational phases in 1D spin chains [30–40], much less is known in the computationally important setting of 2D spin systems outside of variously perturbed phases containing the cluster state [41–45]. This disparity comes both from the increased complexity present in 2D many-body systems, as well as the existence of physically distinct forms of 2D SPTO with different operational capabilities [19]. For these reasons, we have yet to figure out even a base model for realizing the idea of a universal computational phase within the framework of SPTO.

Here, our key starting point is to focus on 2D model states representing renormalization group (RG) fixed-point states of SPTO. As described in detail in Appendix A, these "3-cocycle states" [23] define a coarse-grained, yet infinitely large, family of representative wavefunctions which are macroscopically distinct regarding their SPTO. In addition to the standard abelian, on-site symmetry groups $G = (\mathbb{Z}_2)^m$, we introduce an additional fractional $\frac{1}{3}$ symmetry of 2D lattice geometry, where symmetry operators are applied to only a certain fraction of spins on a 3-colorable lattice. It turns out that this fractional symmetry is powerful enough to establish a *one-to-one* correspondence between the computational universality of these states for MQC and the non-triviality of SPTO phases they represent in terms of cohomology classes. Our findings form compelling evidence pointing towards universal computational 2D phases among general fractionally symmetric SPTO states, with the expectation that an operational analysis of RG flows may be feasible along the same lines as the aforementioned success of RG methods in 1D spin chains.

Measurement-based Quantum Computation.—Measurement-based quantum computation (MQC) utilizes an entangled many-body resource state to perform quantum computation via local measurements on single lattice sites. An MQC protocol is adaptive if the choice of measurement basis depends on previous measurement outcomes. A universal resource state is one which allows any unitary quantum circuit to be efficiently implemented using single-site measurements.

While MQC has historically focused on the 2D cluster state [7], which has a peculiar nature regarding SPTO (see Appendix B), we are more interested here in its 1D spin chain cousin and the Union Jack state of [19] (see Figure 1). Within MQC, the 1D cluster state can implement all single-qubit operations, while the Union Jack state is universal using only Pauli measurements, a property called Pauli universality.

Symmetry-Protected Topological Order.—Symmetry-protected topological order (SPTO) is a quantum phe-

* jmjacobmiller@gmail.com
† amiyake@unm.edu

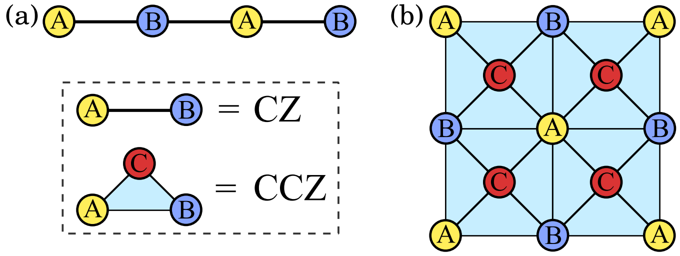

FIG. 1. The 1D cluster state $|\psi_{1C}\rangle$ (a), and 2D Union Jack state $|\psi_{UJ}\rangle$ (b), canonical examples of the entangled many-body states we investigate. (a) The 1D cluster state is formed from qubit $|+\rangle$ states (with $|+\rangle := \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)$) on a 1D spin chain, which are entangled with nearest-neighbors CZ gates acting as $CZ\,|\alpha,\beta\rangle = (-1)^{\alpha\cdot\beta}\,|\alpha,\beta\rangle$. (b) The Union Jack state is obtained from $|+\rangle$ states on a 2D Union Jack lattice, which are entangling with nearest-neighbor triple CCZ gates acting as $CCZ\,|\alpha,\beta,\gamma\rangle = (-1)^{\alpha\cdot\beta\cdot\gamma}\,|\alpha,\beta,\gamma\rangle$. Both $|\psi_{1C}\rangle$ and $|\psi_{UJ}\rangle$ possess distinctive "fractional symmetries", leaving them invariant under X applied to all qubits on sites of a single color (A, B, or C). Replacing the $(d-1)$-controlled Z gates by unitaries $U(\omega_d)$ parameterized by d-cocycles of a group G, we obtain the cocycle states of [23]. Cocycle states with fractional symmetry can be graphically represented by expanding every vertex into a collection of virtual qubits, and expanding every entangling gate into a product of CZ's or CCZ's (see Figure 2).

nomenon in many-body systems with global symmetry G, which will always be abelian here. An SPTO phase is the collection of all many-body states connected to some fiducial short-range entangled state using only constant depth quantum circuits built from constant range, symmetry-respecting gates. The trivial SPTO phase is the unique phase containing unentangled product states. Nontrivial SPTO consequently represents a form of persistent many-body entanglement, protected by a symmetry group G.

SPTO phases can be classified using group cohomology theory [23]. For 2D states, SPTO phases relative to G correspond to elements of the third cohomology group of G, $\mathcal{H}^3(G,U(1))$. We can analyze $\mathcal{H}^3(G,U(1))$ using 3-cocycles, complex-valued functions $\omega_3(g_1,g_2,g_3) : G^3 \to U(1)$ which satisfy the condition $\partial_3\omega_3(g_0,g_1,g_2,g_3) := \omega_3(g_1,g_2,g_3)\,\omega_3^*(g_0g_1,g_2,g_3)\,\omega_3(g_0,g_1g_2,g_3)\,\omega_3^*(g_0,g_1,g_2g_3)\,\omega_3(g_0,g_1,g_2) = 1$, for all $g_0,g_1,g_2,g_3 \in G$. Each 3-cocycle ω_3 lies in a unique "cohomology class", $[\omega_3]_G \in \mathcal{H}^3(G,U(1))$, where the cohomology class of the function $\omega_3(g_1,g_2,g_3)=1$ is the trivial SPTO phase. In general $d \geq 1$ spatial dimensions, SPTO phases are classified by $\mathcal{H}^{d+1}(G,U(1))$.

Cocycle States.—While the correspondence between SPTO phases and cohomology classes may appear obscure, it lets us construct useful SPTO fixed-point states using the cocycle state model of [23]. This model converts abstract d-cocycles ω_d of G into d-body unitary gates $U(\omega_d)$, which then form many-body states $|\psi(\omega_d)\rangle$ in $d-1$ spatial dimensions. These states have global symmetry G, and belong to the SPTO phase associated with $[\omega_d]_G$. We discuss only the 2D case ($d=3$), but this method extends to any $d \geq 1$ spatial dimensions.

For any G, $|\psi(\omega_3)\rangle$ is made of $|G|$-dimensional qudits on a 2D lattice Λ without boundaries. On-site symmetry operators X_g act in a generalized computational basis as $X_g|h\rangle = |gh\rangle$, $\forall g,h \in G$. When $G=(\mathbb{Z}_2)^m$, a generating set for G (explained below) lets us represent each qudit as m "virtual" qubits, on which $X_g = \bigotimes_{i=1}^m (X_i)^{g_i}$. We visualize these qubits stacked in vertical layers, from $i=1$ (top) to $i=m$ (bottom). The state $|+_G\rangle = |+\rangle^{\otimes m}$ is the unique $+1$ eigenstate state of every X_g.

ω_3 sets the eigenvalues of our entangling unitary $U(\omega_3)$, as $U(\omega_3) = \sum_{g,h,f\in G}\omega_3(g,g^{-1}h,h^{-1}f)\,|g,h,f\rangle\langle g,h,f|$. We form $|\psi(\omega_3)\rangle$ from $|+_G\rangle$ states on every vertex of a 3-colorable lattice, with $U(\omega_3)$ (or $U(\omega_3)^\dagger$) applied to all nearest-neighbor triples of qudits Δ_3. The three arguments g,h,f match the three qudits in Δ_3 according to their lattice colors. Overall, $|\psi(\omega_3)\rangle = \left(\prod_{\Delta_3\in\Lambda} U(\omega_3)^{\pm 1}_{\Delta_3}\right)|+_G\rangle^{\otimes n}$, where the alternation of $U(\omega_3)$ and $U(\omega_3)^\dagger$ is described in [23].

The 1D cluster state and Union Jack state are both $G=\mathbb{Z}_2$ cocycle states, with respective cocycles $\omega_2^{(1C)}(g,h) = (-1)^{g\cdot h}$ and $\omega_3^{(UJ)}(g,h,f) = (-1)^{g\cdot h\cdot f}$ (c.f. Appendix B of [19]). However, these states both possess additional "$\frac{1}{d}$" fractional symmetry, arising from X_g applied to spins of a single vertex color on a d-colorable lattice. As we show below, this fractional symmetry is connected to each cocycle being a d-linear function, something we define explicitly for $d=3$.

A function $\tau_3(g,h,f) : G^3 \to U(1)$ is 3-linear (trilinear) when it satisfies $\tau_3(gg',h,f) = \tau_3(g,h,f)\tau_3(g',h,f)$, and similarly for its other two arguments. Every trilinear function is a 3-cocycle, but one possessing additional algebraic structure. This lets us efficiently describe τ_3 by choosing a generating set for $G=(\mathbb{Z}_2)^m$, namely a collection of m elements $\{e_i\}_{i=1}^m \subseteq G$ by which every $g \in G$ is $g = \prod_{i=1}^m (e_i)^{g_i}$ for a unique choice of binary coordinates g_i. Given a fixed generating set, we have $\tau_3(g,h,f) = (-1)^{\sum_{i,j,k=1}^m \hat{\tau}_3(i,j,k)\cdot g_i\cdot h_j\cdot f_k}$, where i,j,k index the generators of $(\mathbb{Z}_2)^m$, and $\hat{\tau}_3(i,j,k)$ is a binary "component" of τ_3 encoding the value of $\tau_3(e_i,e_j,e_k)$. These components form an $m\times m\times m$ binary tensor $\hat{\tau}_3$, whose transformation under index-dependent changes of generating set will concern us below. We can similarly define 2-linear (bilinear) functions $\tau_2(g,h)$, described by $m\times m$ binary component matrices $\hat{\tau}_2(i,j)$. For more information on group cohomology, the cocycle state model, and the formulation of so-called stabilizer states as examples of cocycle states, see Appendix B.

Cocycle States with Fractional Symmetry.—Given the fractional symmetry of the 1D cluster state and Union Jack state, we ask how this symmetry orders the entanglement of general many-body states. Our main results form a largely exhaustive answer to this question for 1D 2-cocycle states and 2D 3-cocycle states. We first show that any $\frac{1}{d}$-symmetric cocycle state with $d=2$ or 3 and

$G = (\mathbb{Z}_2)^m$ is either a trivial product state, or is reducible by local operations to several disjoint copies of the 1D cluster state or the Union Jack state, respectively. For $d = 2$, this characterization is complete, in that every nontrivial $\frac{1}{2}$-symmetric cocycle state $|\psi(\omega_2)\rangle$ is isomorphic to r copies of $|\psi_{1C}\rangle$, for an ω_2-dependent $r \geq 1$. When $d = 3$ however, we show that general $\frac{1}{3}$-symmetric cocycle states with $G = (\mathbb{Z}_2)^m$ are isomorphic to r "irreducible" 3-cocycle states, of which the Union Jack state is the simplest. This proves that all nontrivial 3-cocycle states with $\frac{1}{3}$-symmetry and $G = (\mathbb{Z}_2)^m$ are Pauli universal MQC resource states, identifying a robust correspondence between fractional symmetry and the utility of many-body states for quantum computation.

We first characterize the algebraic properties of cocycle states with $\frac{1}{d}$-symmetry. We show that for $d = 2$ and 3, d-cocycle state with $\frac{1}{d}$-symmetry are precisely those generated by d-linear functions (Lemma 1).

Lemma 1. *Let $|\psi(\omega_d)\rangle$ be a d-cocycle state defined on a d-colorable $(d-1)$-dimensional lattice without boundaries, generated by a d-cocycle ω_d with $d = 2, 3$. $|\psi(\omega_d)\rangle$ is $\frac{1}{d}$-symmetric, i.e. is invariant under the application of G to all sites of any one of the d lattice colors, if and only if it is generated by a unique d-linear function τ_d, so that $|\psi(\omega_d)\rangle = |\psi(\tau_d)\rangle$.*

Lemma 1's statement can be generalized to arbitrary d, but due to our focus on low-dimensional MQC resource states, this generalized version remains a conjecture. Proving that d-linear cocycle states possess $\frac{1}{d}$-symmetry is trivial, so we focus on the reverse implication. Our proof analyzes the action of fractional symmetry operators on local regions of a d-cocycle state $|\psi(\omega_d)\rangle$, and iteratively builds up necessary conditions for $|\psi(\omega_d)\rangle$ to possess $\frac{1}{d}$-symmetry. This shows that ω_d is the product of a unique d-linear τ_d with additional terms acting on the boundaries of our system, proving our result. The full proof of Lemma 1 is given in Appendix C.

The specification of d-linear τ_d's using component tensors $\hat{\tau}_d$ lets us decompose $U(\tau_d)$ into a product of d-qubit component unitary gates, one for each nonzero component of $\hat{\tau}_d$. When $G = (\mathbb{Z}_2)^m$ and $d = 2$ or 3, these component gates are CZ or CCZ, which shows each $|\psi(\tau_d)\rangle$ to be a so-called hypergraph state [46–48]. This decomposition of $U(\tau_d)$ into CZ or CCZ gates requires a choice of generating set for each vertex color of our d-colorable lattice, with changes of generating set acting as gauge freedoms in the description of $|\psi(\tau_d)\rangle$. We can fix these spurious degrees of freedom by enumerating the local unitary orbits of $|\psi(\tau_d)\rangle$ under color-dependent changes of basis, which reduces to finding a normal form for our component tensor $\hat{\tau}_d$.

For 1D and 2D states, this classification reduces to that of irreducible $\frac{1}{d}$-symmetric cocycle states $|\psi(\gamma_i)\rangle$ (defined below), as given in Theorem 1.

Theorem 1. *Let $|\psi(\tau_d)\rangle$ be a nontrivial $\frac{1}{d}$-symmetric d-cocycle state without boundaries in $d-1$ spatial dimensions, with global symmetry $G = (\mathbb{Z}_2)^m$ and $d = 2, 3$. By an appropriate color-dependent change of basis, there is a unique r with $1 \leq r \leq m$ such that the nontrivial portion of $|\psi(\tau_d)\rangle$ is isomorphic to r disjoint irreducible $\frac{1}{d}$-symmetric cocycle states, i.e. $\bigotimes_{i=1}^{r} |\psi(\gamma_i)\rangle$.*

We let $\zeta_d(m)$ denote the number of distinct irreducible d-cocycle states in $G = (\mathbb{Z}_2)^m$, which is calculated using the component tensors $\hat{\tau}_d$. When $d = 2$, we reduce $\hat{\tau}_2$ to normal form using color-dependent changes of generating set on lattice colors A, B, transforming $\hat{\tau}_2$ to $\chi_A^T \hat{\tau}_2 \chi_B$ with invertible binary matrices χ_A, χ_B. Choosing χ_A and χ_B to implement elementary row and column operations, we can transform $\hat{\tau}_2$ into a diagonal normal form using Gaussian elimination. This gives $U(\tau_2)$ as a product of disjoint CZ gates forming r disjoint copies of $|\psi_{1C}\rangle$, with r the rank of $\hat{\tau}_2$ (see Figure 2a). This proves Theorem 1 for $d = 2$, and shows also that $\zeta_2(m) = 1$ for all m, meaning the 1D cluster state is the unique irreducible cocycle state in 1D.

When $d = 3$, our formation unitaries $U(\tau_3)$ correspond to 3-index component tensors $\hat{\tau}_3$, which are harder to characterize. Similar to our $d = 2$ proof, color-dependent changes of basis let us rewrite $\hat{\tau}_3$ as a collection of r irreducible tensors, which form the r irreducible $\frac{1}{3}$-symmetric cocycle states in Theorem 1. More precisely, $\hat{\tau}_3$ is irreducible when it cannot be written as the sum of two nonzero tensors with disjoint supports at every index. In $d = 3$ however, there is no known analog of Gaussian elimination to efficiently decompose $\hat{\tau}_3$ into irreducible tensors. Nonetheless, we show in Appendix D3 that there is still a normal form letting us prove Theorem 1 for $d = 3$. Consequently, the behavior of general $\frac{1}{3}$-symmetric cocycle states depends only on the behavior of general irreducible cocycle states.

In the simplest case of $m = 1$, the only nontrivial trilinear function is $\omega_3^{(UJ)}$ (defined previously), showing that $\zeta_3(1) = 1$. In contrast to the 1D case though, in 2D we find many different irreducible cocycle states, the simplest being shown in Figure 2c. A numerical search shows that $\zeta_3(2) = 4$ and $\zeta_3(3) = 50$, and we expect infinitely many irreducible states to appear in general $(\mathbb{Z}_2)^m$. Despite this difficulty, every irreducible $\frac{1}{3}$-symmetric cocycle state should clearly contain at least as much usable entanglement as the Union Jack state, which lets us prove a useful operational corollary to Theorem 1 for $d = 3$.

Corollary 1. *Let $|\psi(\tau_3)\rangle$ be a nontrivial $\frac{1}{3}$-symmetric 3-cocycle state with global symmetry group $(\mathbb{Z}_2)^m$ defined on a Union Jack lattice. By appropriate color-dependent changes of basis and non-adaptive single-qubit Z measurements, $|\psi(\tau_3)\rangle$ can be reduced to r disjoint copies of the Union Jack state, for the same state-dependent $r \geq 1$ as in Theorem 1. Consequently, $|\psi(\tau_3)\rangle$ is a Pauli universal resource state for MQC.*

We prove Corollary 1 by showing that any irreducible $|\psi(\gamma_i)\rangle$ is equal in some color-dependent change of generating set to a single copy of the Union Jack state, which is disjoint or "vertex entangled" with all other virtual

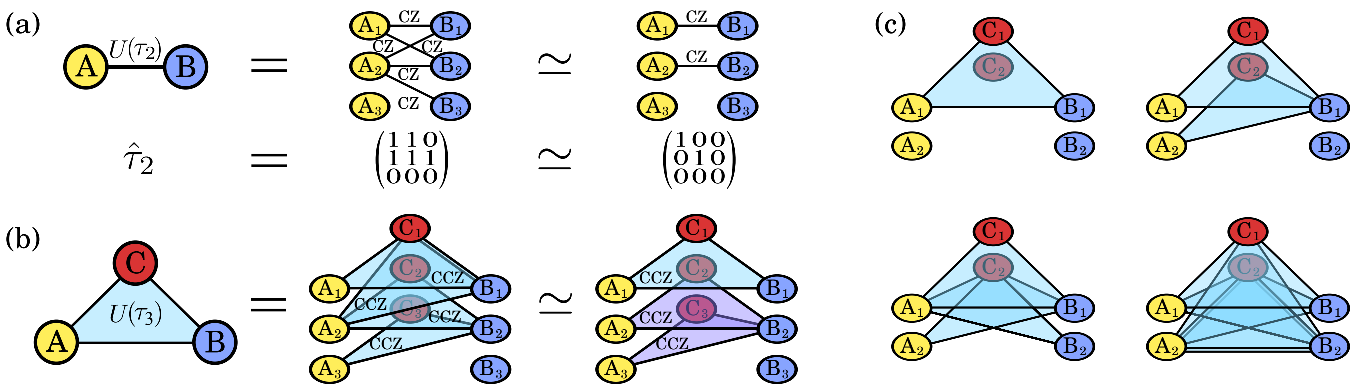

FIG. 2. (a) Fixing a $G = (\mathbb{Z}_2)^m$ generating set at sites A and B lets us represent the entangling gates $U(\tau_2)$ forming our $\frac{1}{2}$-symmetric 2-cocycle state using an $m \times m$ binary component matrix, $\hat{\tau}_2$. Nonzero entries of $\hat{\tau}_2$ give CZ gates between adjacent virtual qubits. Color-dependent changes of generating set (corresponding to color-dependent changes of basis on the single-spin Hilbert spaces), enact Gaussian elimination, reducing $\hat{\tau}_2$ to a diagonal normal form in which our state is simply $r = \text{rank}(\hat{\tau}_2)$ disjoint 1D cluster states. Here, $m = 3$ and $r = 2$. (b) In 2D, we again use color-dependent changes of generating set to simplify our state, but now represent 3-body entangling gates $U(\tau_3)$ as 3-index binary component tensors, $\hat{\tau}_3$ (not shown). Nonzero entries of $\hat{\tau}_3$ give CCZ gates between triples of virtual qubits. Our normal form reduces this state to r disjoint irreducible 3-cocycle states, where again $m = 3$ and $r = 2$. (c) Representatives of the $\zeta_3(2) = 4$ irreducible 3-cocycle states which exist in $G = (\mathbb{Z}_2)^2$. Theorem 1 proves that any $\frac{1}{3}$-symmetric 3-cocycle state with $G = (\mathbb{Z}_2)^m$ is either trivial, isomorphic to one of these states (up to permutation of lattice colors), or isomorphic to two disjoint copies of the Union Jack state (the only irreducible state in $\mathbb{Z}_2$). An exhaustive numerical search shows that of the $2^{m^3} = 2^{27}$ possible $\frac{1}{3}$-symmetric cocycle states in $G = (\mathbb{Z}_2)^3$, there exist only $\zeta_3(3) = 50$ distinct irreducible states up to local changes of basis. However, a precise classification of irreducible cocycle states is unnecessary for our purposes, since every irreducible state is a Pauli universal resource state (Corollary 1).

qubits. This guarantees that measuring Z on the other virtual qubits leaves only the Union Jack state, up to trivial Pauli byproduct operators. Applying this protocol to each irreducible $|\psi(\gamma_i)\rangle$ in Theorem 1 then proves Corollary 1. Further details are given in Appendix D2.

Having discussed the general classification and computational power of low-dimensional $\frac{1}{3}$-symmetric cocycle states $|\psi(\tau_3)\rangle$, we now study their SPTO phases relative to the fractional symmetry group $G_{\frac{1}{3}}$. This classification relative to $G_{\frac{1}{3}}$ then determines the SPTO phase of $|\psi(\tau_3)\rangle$ relative to any subgroup of $G_{\frac{1}{3}}$, including the usual global symmetry G. While $G_{\frac{1}{3}} \simeq G^3$ as groups, they differ operationally by the former arranging each copy of G on a distinct vertex color ("horizontally"), and the latter arranging each copy on a distinct layer of a single vertex ("vertically"). This allows a simple characterization of the SPTO present in these states (Theorem 2).

Theorem 2. *Let $|\psi(\tau_3)\rangle, |\psi(\tau_3')\rangle$ be two $\frac{1}{3}$-symmetric 2D 3-cocycle states with global symmetry group G, where τ_3 and τ_3' are trilinear functions. If $\tau_3 \neq \tau_3'$, then $|\psi(\tau_3)\rangle$ and $|\psi(\tau_3')\rangle$ belong to different SPTO phases relative to $G_{\frac{1}{3}}$. In particular, if τ_3 is nontrivial, then $|\psi(\tau_3)\rangle$ possesses nontrivial SPTO relative to $G_{\frac{1}{3}}$.*

We prove Theorem 2 by embedding each 3-cocycle state $|\psi(\tau_3)\rangle$ into a larger Hilbert space associated with G^3, where the original $G_{\frac{1}{3}}$ fractional symmetry of $|\psi(\tau_3)\rangle$ is simulated using an operationally equivalent G^3 global symmetry. This lets us use a known classification of 2D SPTO phases relative to global G^3 symmetry to identify each component of $\hat{\tau}_3$ as a unique label of the SPTO phase of $|\psi(\tau_3)\rangle$, relative to $G_{\frac{1}{3}}$. Consequently, two states $|\psi(\tau_3)\rangle, |\psi(\tau_3')\rangle$ are in the same SPTO phase only when their associated tensors $\hat{\tau}_3, \hat{\tau}_3'$ are identical, which proves Theorem 2. Further details of our proof are given in Appendix D3.

Outlook.—We have shown that computationally universal entanglement is a ubiquitous property of fixed-point states of SPTO with fractional symmetry. While we were able to obtain "exact" universal resource states in our simple setting of fixed-point model states, more general states with SPTO may require renormalization-style techniques like those of [34, 37, 39, 40] to extract their usefulness for MQC, as discussed in more detail in Appendix A. Overall, we expect fractional symmetry to be a powerful tool for guaranteeing certain operational capabilities in more general quantum information processing tasks, such as quantum simulation [49, 50] and fault-tolerant quantum computation [51–53].

Acknowledgements.—This work was supported in part by National Science Foundation grants PHY-1314955, PHY-1521016, and PHY-1620651.

[1] R. Raussendorf and H.J. Briegel, *A One-Way Quantum Computer*, Phys. Rev. Lett. **86**, 5188 (2001).

[2] R. Raussendorf, D.E. Browne, and H.J. Briegel,

Measurement-Based Quantum Computation on Cluster States, Phys. Rev. A **68**, 022312 (2003).
[3] R. Jozsa, *An Introduction to Measurement Based Quantum Computation*, arXiv:quant-ph/0508124, (2005).
[4] M. Van den Nest, A. Miyake, W. Dür, and H.J. Briegel, *Universal Resources for Measurement-Based Quantum Computation*, Phys. Rev. Lett. **97**, 150504 (2006).
[5] M. Van den Nest, W. Dür, A. Miyake and H.J. Briegel, *Fundamentals of universality in one-way quantum computation*, New J. Phys. **9**, (2007).
[6] X. Chen, R. Duan, Z. Ji, and B. Zeng, *Quantum State Reduction for Universal Measurement Based Computation*, Phys. Rev. Lett. **105**, 020502 (2010).
[7] H.J. Briegel and R. Raussendorf, *Persistent Entanglement in Arrays of Interacting Particles*, Phys. Rev. Lett. **86**, 910 (2001).
[8] D.-L. Zhou, B. Zeng, Z. Xu, and C.-P. Sun, *Quantum computation based on d-level cluster state*, Phys. Rev. A **68**, 062303 (2003).
[9] F. Verstraete and J.I. Cirac, *Valence-Bond States for Quantum Computation*, Phys. Rev. A **70**, 060302(R) (2004).
[10] D. Gross and J. Eisert, *Novel Schemes for Measurement-Based Quantum Computation*, Phys. Rev. Lett. **98**, 220503 (2007).
[11] D. Gross, J. Eisert, N. Schuch, and D. Pérez-García, *Measurement-Based Quantum Computation Beyond The One-way Model*, Phys. Rev. A **76**, 052315 (2007).
[12] X. Chen, B. Zeng, Z.-C. Gu, B. Yoshida, and I.L. Chuang, *Gapped Two-Body Hamiltonian Whose Unique Ground State Is Universal for One-Way Quantum Computation*, Phys. Rev. Lett. **102** 220501 (2009).
[13] J.-M. Cai, A. Miyake, W. Dür, and H.J. Briegel, *Universal Quantum Computer from a Quantum Magnet*, Phys. Rev. A, **82**, 052309 (2010).
[14] A. Miyake, *Quantum Computational Capability of a 2D Valence Bond Solid Phase*, Ann. Phys. **326**, 1656 (2011).
[15] T.-C. Wei, I. Affleck, and R. Raussendorf, *Affleck-Kennedy-Lieb-Tasaki State on a Honeycomb Lattice is a Universal Quantum Computational Resource*, Phys. Rev. Lett. **106**, 070501 (2011).
[16] A.S. Darmawan, G.K. Brennen, and S.D. Bartlett, *Measurement-Based Quantum Computation in a 2D Phase of Matter*, New J. Phys. **14**, 013023 (2012).
[17] T.-C. Wei, *Quantum Computational Universality of Affleck-Kennedy-Lieb-Tasaki States Beyond the Honeycomb Lattice*, Phys. Rev. A **88**, 062307 (2013).
[18] T.-C. Wei and R. Raussendorf, *Universal Measurement-based Quantum Computation with Spin-2 Affleck-Kennedy-Lieb-Tasaki States*, Phys. Rev. A **92**, 012310 (2015).
[19] J. Miller and A. Miyake, *Hierarchy of Universal Entanglement in 2D Measurement-based Quantum Computation*, npj Quantum Information **2**, 16036 (2016).
[20] H.P. Nautrup and T.-C. Wei, *Symmetry-protected topologically ordered states for universal quantum computation*, Phys. Rev. A **92**, 052309 (2015).
[21] Z.-C. Gu and X.-G. Wen, *Tensor-Entanglement-Filtering Renormalization Approach and Symmetry-Protected Topological Order*, Phys. Rev. B **80**, 155131 (2009).
[22] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, *Symmetry Protected Topological Orders in Interacting Bosonic Systems*, Science **338**, 1604 (2012).
[23] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, *Symmetry Protected Topological Orders and the Group Cohomology of Their Symmetry Group*, Phys. Rev. B **87**, 155114 (2013).
[24] A. Kitaev, *Periodic Table for Topological Insulators and Superconductors*, AIP Conf. Proc. **1134**, 22 (2009).
[25] S. Ryu, A.P. Schnyder, A. Furusaki and A.W. Ludwig, *Topological Insulators and Superconductors: Ten-Fold Way and Dimensional Hierarchy*, New J. Phys. **12**, 065010 (2010).
[26] F. Pollmann, E. Berg, A.M. Turner, and M. Oshikawa, *Symmetry Protection of Topological Phases in One-Dimensional Quantum Spin Systems*, Phys. Rev. B **85**, 075125 (2012).
[27] X. Chen, Z.-C. Gu, and X.-G. Wen, *Classification of Gapped Symmetric Phases in One-Dimensional Spin Systems*, Phys. Rev. B **83**, 035107 (2011).
[28] N. Schuch, D. Pérez-García, and J.I. Cirac, *Classifying Quantum Phases Using Matrix Product States and Projected Entangled Pair States*, Phys. Rev. B **84**, 165139 (2011).
[29] F. Pollmann, A.M. Turner, E. Berg, and M. Oshikawa, *Entanglement Spectrum of a Topological Phase in One Dimension*, Phys. Rev. B **81**, 064439 (2010).
[30] G. K. Brennen and A. Miyake, *Measurement-Based Quantum Computer in the Gapped Ground State of a Two-Body Hamiltonian*, Phys. Rev. Lett. **101**, 010502 (2008).
[31] A.C. Doherty and S.D. Bartlett, *Identifying phases of quantum many-body systems that are universal for quantum computation*, Phys. Rev. Lett. **103**, 020506 (2009).
[32] S.O. Skrøvseth and S.D. Bartlett, *Phase transitions and localizable entanglement in cluster-state spin chains with Ising couplings and local fields*, Phys. Rev. A **80**, 022316 (2009).
[33] A. Miyake, *Quantum Computation on the Edge of a Symmetry-Protected Topological Order*, Phys. Rev. Lett. **105**, 040501 (2010).
[34] S.D. Bartlett, G.K. Brennen, A. Miyake, and J.M. Renes, *Quantum Computational Renormalization in the Haldane Phase*, Phys. Rev. Lett. **105**, 110502 (2010).
[35] D.V. Else, I. Schwarz, S.D. Bartlett, and A.C. Doherty, *Symmetry-Protected Phases for Measurement-Based Quantum Computation*, Phys. Rev. Lett. **108**, 240505 (2012).
[36] D.V. Else, S.D. Bartlett, and A.C. Doherty, *Symmetry Protection of Measurement-Based Quantum Computation in Ground States*, New J. Phys. **14**, 113016 (2012).
[37] J. Miller and A. Miyake, *Resource Quality of a Symmetry-Protected Topologically Ordered Phase for Quantum Computation*, Phys. Rev. Lett. **114**, 120506 (2015).
[38] A. Prakash and T.-C. Wei, *Ground States of One-Dimensional Symmetry-Protected Topological Phases and Their Utility as Resource States for Quantum Computation*, Phys. Rev. A **92**, 022310 (2015).
[39] R. Raussendorf, D.-S. Wang, A. Prakash, T-C. Wei, and D.T. Stephen, *Symmetry-protected topological phases with uniform computational power in one dimension*, Phys. Rev. A **96**, 012302 (2017).
[40] D.T. Stephen, D.-S. Wang, A. Prakash, T-C. Wei, and R. Raussendorf, *Computational Power of Symmetry-Protected Topological Phases*, Phys. Rev. Lett. **119**, 010504 (2017).

[41] W. Son, L. Amico, R. Fazio, A. Hamma, S. Pascazio, and V. Vedral, *Quantum phase transition between cluster and antiferromagnetic states*, Europhys. Lett. **95**, 50001 (2011).
[42] H. Kalis, D. Klagges, R. Orús, and K.P. Schmidt, *Fate of the cluster state on the square lattice in a magnetic field*, Phys. Rev. A **86**, 022317 (2012).
[43] R. Orús, H. Kalis, M. Bornemann, and K.P. Schmidt, *Bounds on universal quantum computation with perturbed two-dimensional cluster states*, Phys. Rev. A **87**, 062312 (2013).
[44] K. Fujii, Y. Nakata, M. Ohzeki, and M. Murao, *Measurement-Based Quantum Computation on Symmetry Breaking Thermal States*, Phys. Rev. Lett. **110**, 120502 (2013).
[45] T.-C. Wei, Y. Li, and L.C. Kwek, *Transitions in the quantum computational power*, Phys. Rev. A **89**, 052315 (2014).
[46] M. Rossi, M. Huber, D. Bruß, and C. Macchiavello, *Quantum Hypergraph States*, New J. Phys. **15**, 113022 (2013).
[47] O. Gühne, M. Cuquet, F.E.S. Steinhoff, T. Moroder, M. Rossi, D. Bruß, B. Kraus, and C. Macchiavello, *Entanglement and Nonclassical Properties of Hypergraph States*, J. Phys. A **47**, 335303 (2014).
[48] F.E.S. Steinhoff, C. Ritz, N. Miklin, and O. Gühne, *Qudit Hypergraph States*, arXiv:1612.06418, (2016).
[49] J.I. Cirac and P. Zoller, *Goals and opportunities in quantum simulation*, Nat. Phys. **8**, 264266 (2012).
[50] I.M. Georgescu, S. Ashhab, and F. Nori, *Quantum simulation*, Rev. Mod. Phys. **86**, 153185 (2014).
[51] B. Yoshida, *Topological Color Code and Symmetry-Protected Topological Phases*, Phys. Rev. B **91**, 245131 (2015).
[52] B. Yoshida, *Gapped Boundaries, Group Cohomology and Fault-Tolerant Logical Gates*, arXiv:1509.03626, (2015).
[53] S. Roberts, B. Yoshida, A. Kubica, S.D. Bartlett, *Symmetry protected topological order at nonzero temperature*, arXiv:1611.05450, (2016).
[54] D. Gottesman, *Stabilizer Codes and Quantum Error Correction*, PhD Thesis, arXiv:quant-ph/9705052, (1997).
[55] M. Hein *et al.* in *Quantum Computers, Algorithms and Chaos* Vol. 162, Int. School of Physics Enrico Fermi (eds Casati, G *et al.*), 115–218 (IOS Press, 2006). Preprint at https://arxiv.org/abs/quant-ph/0602096.
[56] D. Schlingemann, *Stabilizer codes can be realized as graph codes*, Quant. Inf. Comp. **2**, 307 (2002).
[57] M. de Wild Propitius, *Topological interactions in broken gauge theories*, PhD Thesis, arXiv:hep-th/9511195, (1995).
[58] R. Raussendorf, S. Bravyi, and J. Harrington, *Long-range quantum entanglement in noisy cluster states*, Phys. Rev. A **71**, 062313 (2005).
[59] M.P. Zaletel, *Detecting two-dimensional symmetry-protected topological order in a ground-state wave function*, Phys. Rev. B **90**, 235113 (2014).