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Michelle D. Maiden, Dalton V. Anderson, Nevil A. Franco, Gennady A. El, and Mark A. Hoefer

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Solitonic dispersive hydrodynamics: theory and observation

Michelle D. Maiden,[1] Dalton V. Anderson,[1] Nevil A. Franco,[1] Gennady A. El,[2] and Mark A. Hoefer[1,*]
[1] *Department of Applied Mathematics, University of Colorado, Boulder, Colorado 80309-0526, USA*
[2] *Department of Mathematical Sciences, Loughborough University, Loughborough, LE11 3TU, UK*
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Ubiquitous nonlinear waves in dispersive media include localized solitons and extended hydrodynamic states such as dispersive shock waves. Despite their physical prominence and the development of thorough theoretical and experimental investigations of each separately, experiments and a unified theory of solitons and dispersive hydrodynamics are lacking. Here, a general soliton-mean field theory is introduced and used to describe the propagation of solitons in macroscopic hydrodynamic flows. Two universal adiabatic invariants of motion are identified that predict trapping or transmission of solitons by hydrodynamic states. The result of solitons incident upon smooth expansion waves or compressive, rapidly oscillating dispersive shock waves is the same, an effect termed hydrodynamic reciprocity. Experiments on viscous fluid conduits quantitatively confirm the soliton-mean field theory with broader implications for nonlinear optics, superfluids, geophysical fluids, and other dispersive hydrodynamic media.

Long wavelength, hydrodynamic theories abound in physics, from fluids [1] to optics [2], condensed matter [3] to quantum mechanics [4], and beyond. Such theories describe expansion and compression waves until breaking. When the physics at shorter wavelengths are predominantly dispersive, dispersive hydrodynamic theories [5, 6] are used to describe shock waves of a spectacularly different character than their dissipative counterparts. Dispersive shock waves (DSWs) consist of coherent, rank-ordered, nonlinear oscillations that continually expand [6, 7]. Observations in a wide range of physical media that include quantum matter [8, 9], optics [10, 11], classical fluids [12, 13] and magnetic materials [14] demonstrate the prevalence of DSWs.

Another celebrated feature of dispersive hydrodynamic media are localized, nonlinear solitary waves. When they exhibit particle-like properties such as elastic, pairwise interactions, solitary waves are called solitons [15] and have been extensively studied both theoretically [16] and experimentally [17]. The focus here is on solitary waves that exhibit solitonic behavior, i.e., elastic or near-elastic interaction, henceforth we refer to them as solitons. Despite their common origins, solitons and dispersive hydrodynamics have been primarily studied independently.

Utilizing the scale separation between extended hydrodynamic states and localized solitons (see Fig. 1), we propose in this Letter a general theory of solitonic dispersive hydrodynamics encapsulated by a set of effective partial differential equations for the hydrodynamic mean field, the soliton's amplitude, and its phase. We identify two adiabatic invariants of motion and show that they lead to two pivotal predictions. First, the soliton trajectory is a characteristic of the governing equations that is directed by the mean field, a nonlinear analogue of wavepacket trajectories in quantum mechanics [4]. This implies that solitons are either trapped by or transmitted through a hydrodynamic state, depending on the relative amplitudes of the soliton and the hydrodynamic "barrier".

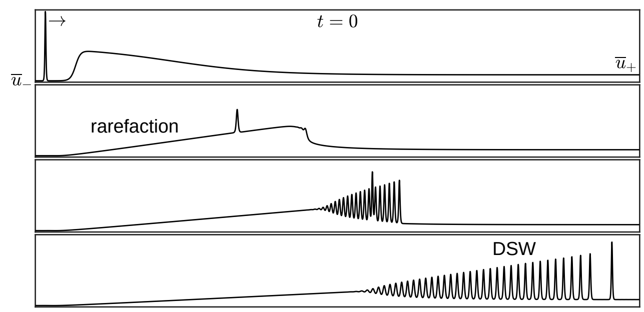

FIG. 1. Representative initial configuration and evolution (top to bottom) for solitonic dispersive hydrodynamics. The narrow soliton on the uniform mean field $\overline{u}_-$ is transmitted through the broad hydrodynamic flow if it reaches and propagates freely on the uniform mean field $\overline{u}_+$. The hydrodynamic flow exhibits expansion (rarefaction) and compression that leads to a dispersive shock wave.

The second prediction we term hydrodynamic reciprocity. Given an incident soliton amplitude and the far-field mean conditions, the adiabatic invariants are used to predict when the soliton is trapped or transmitted and, in the latter case, what its transmitted amplitude and phase shift are. Hydrodynamic reciprocity means that the trapping, transmission amplitude/phase relations are the same for soliton interactions with smooth, expanding rarefaction waves (RWs) and compressive, oscillatory DSWs.

We confirm these predictions with experiments on the interfacial dynamics of a viscous fluid conduit, a model dispersive hydrodynamic medium [18] that has been used previously to investigate solitons [19–21] and DSWs [13]. Although soliton-DSW interaction has been observed previously [13], the nature and properties of the interaction were not explained. We stress that the theory presented is general and applies to a wide range of physical media [8–14].

Experiments are performed on the interfacial dynam-

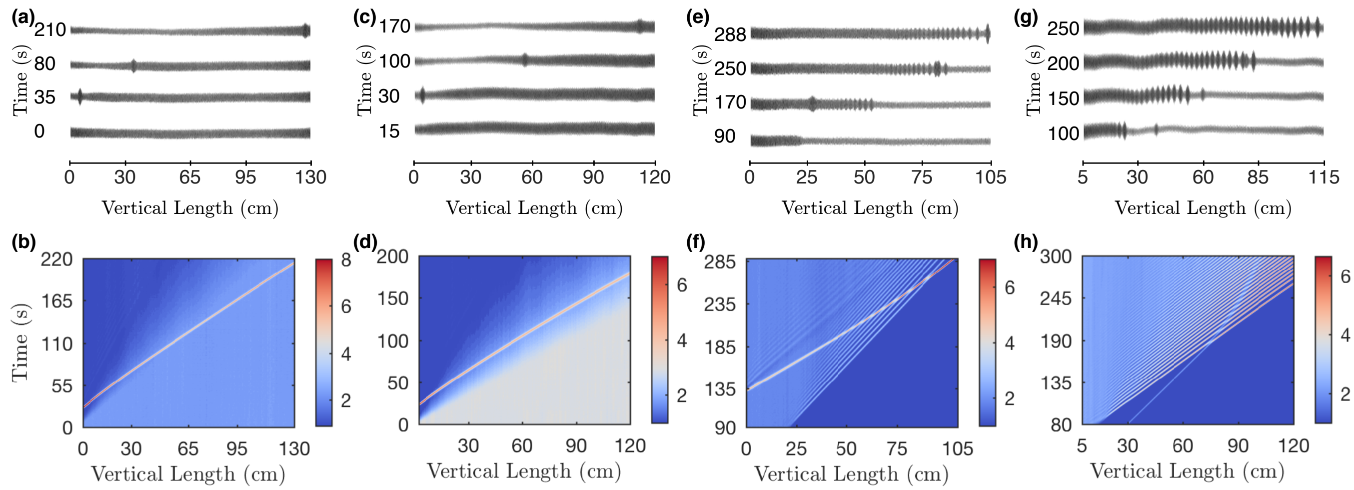

FIG. 2. Experiments demonstrating soliton transmission and trapping with hydrodynamic states. Representative image sequences (a,c,e,g) and space-time contours (b,d,f,h) extracted from image processing are shown. The contour intensity scale is the dimensionless conduit cross-sectional area relative to the smallest area. a,b) Soliton-RW transmission. c,d) Soliton-RW trapping. e,f) Soliton-DSW transmission. g,h) Soliton-DSW trapping.

ics of a buoyant, viscous fluid injected from below into a miscible, much more viscous fluid matrix. Due to negligible diffusion and high viscosity contrast, the two-fluid interface serves as the dispersive hydrodynamic medium [18, 19]. The experimental setup is similar to that described in [13] and consists of a tall acrylic column filled with glycerol (viscosity 1.2±0.2 P, density 1.2587±0.0001 g/cm^3). A nozzle at the column base serves as the injection point for the interior fluid (viscosity 0.51 ± 0.01 P, density 1.2286 ± 0.0001 g/cm^3), a miscible solution of glycerol, water, and black food coloring. By injecting at a constant rate (0.25 mL/min or 0.77 mL/min), the buoyant interior fluid establishes a vertically uniform fluid conduit. Although predicted to be unstable, our experiment operates in the convective regime [22]. By varying the injection rate, conduit solitons, RWs, and DSWs can be generated at the interface between the interior and exterior fluids.

Observations of the hydrodynamic transmission and trapping of solitons resulting from their interaction with RWs and DSWs are depicted in Fig. 2. The contour plots in 2(b,f) show that transmitted solitons exhibit a smaller (larger) amplitude and faster (slower) speed post interaction with a RW (DSW). The transmitted solitons experience a phase shift due to hydrodynamic interaction, defined as the difference between the post and pre interaction spatial intercept. Our measurements show a negative (positive) phase shift for the soliton transmitted through a RW (DSW). Sufficiently small incident solitons in Fig. 2(d,h) do not emerge from the RW or DSW interior during the course of experiment, remaining trapped inside the hydrodynamic state.

We now present a theory to explain these observations by considering a general dispersive hydrodynamic medium with nondimensional scalar quantity $u(x,t)$ (e.g., conduit cross-sectional area) governed by

$$u_t + V(u)u_x = D[u]_x, \quad x \in \mathbb{R}, \quad t > 0. \tag{1}$$

$V(u)$ is the long-wave speed, $D[u]$ is an integro-differential operator, and Eq. (1) admits a real-valued, linear dispersion relation with frequency $\omega(k,\overline{u})$ where k is the wavenumber and $\overline{u}$ is the background mean field. We assume $V'(u) > 0$ so that the dispersive hydrodynamic system has convex flux [23]. The dispersion is assumed negative ($\omega_{kk} < 0$) for definiteness. We also assume that equation (1) satisfies the prerequisites for Whitham theory, an approximate description of modulated nonlinear waves that accurately characterizes dispersive hydrodynamics in a wide-range of physical systems [5, 6].

Many models can be expressed in the form (1). In the Supplemental Material [24], we perform calculations for the Korteweg-de Vries (KdV) equation $V(u) = u$, $D[u] = -u_{xx}$, a universal model of weakly nonlinear, dispersive waves, and the conduit equation $V(u) = 2u$, $D[u] = u^2(u^{-1}u_t)_x$, an accurate model for our experiments [18].

The dynamics of DSWs, RWs, and solitons for Eq. (1) can be described using Whitham theory [5], where a nonlinear periodic wave's mean $\overline{u}$, amplitude a, and wavenumber k are assumed to vary slowly via modulation equations. The modulation equations admit an asymptotic reduction in the non-interacting soliton wavetrain regime $0 < k \ll 1$ [25, 26]

$$\begin{aligned} &\overline{u}_t + V(\overline{u})\overline{u}_x = 0, \quad a_t + c(a,\overline{u})a_x + f(a,\overline{u})\overline{u}_x = 0, \\ &k_t + [c(a,\overline{u})\,k]_x = 0. \end{aligned} \tag{2}$$

The first equation is for the decoupled mean field, which is governed by the dispersionless, $D \to 0$, equation (1).

The second equation describes the soliton amplitude a, which is advected by the mean field according to the soliton amplitude-speed relation $c(a,\overline{u})$ and the coupling function $f(a,\overline{u})$. The final equation expresses wave conservation [5] and describes a train of solitons with spacing $2\pi/k \gg 1$. The soliton train here is a useful, yet fictitious construct because we will only consider the soliton limit $k \to 0$ of solutions to Eq. (2). Equation (2) with $c(a,\overline{u}) = a/3 + \overline{u}$ and $f(a,\overline{u}) = 2a/3$ corresponds to the soliton limit of the KdV-Whitham system of modulation equations, shown in [27] to be equivalent to the soliton modulation equations determined by other means [25] with application to shallow water soliton propagation over topography in [25, 28–30]. The general case of Eq. (2) was derived in [26] and can be interpreted as a mean field approximation for the interaction of a soliton with the hydrodynamic flow. In contrast to standard soliton perturbation theory where the soliton's parameters evolve temporally [31], solitonic dispersive hydrodynamics require the soliton amplitude $a(x,t)$ be treated as a spatio-temporal field. We note that the equations in (2) can be solved sequentially by the method of characteristics [25].

It will be physically revealing to diagonalize the system of equations in (2) by identifying its Riemann invariants [5]. Owing to the special structure of (2) with just two characteristic velocities $V < c$, it is always possible to find a change of variables to Riemann invariant form that diagonalizes the system. The mean field equation is already diagonalized with $\overline{u}$ the Riemann invariant associated to the velocity V. The second Riemann invariant, $q = q(a,\overline{u})$ is associated with the velocity c. q can be found by integrating the differential form $f\mathrm{d}\overline{u} + (c - V)\mathrm{d}a$ [5] (see the Supplemental Material [24]). For KdV, $q(a,\overline{u}) = a/2 + \overline{u}$, whereas for the conduit equation

$$\begin{aligned} c(a,\overline{u}) &= [\overline{u}^2 + (a+\overline{u})^2(2\ln(1+a/\overline{u}) - 1)]\overline{u}/a^2, \\ q(a,\overline{u}) &= c(a,\overline{u})[c(a,\overline{u}) + 2\overline{u}]/\overline{u}. \end{aligned} \tag{3}$$

The third Riemann invariant is found by direct integration of the wavenumber equation to be the quantity $kp(q,\overline{u})$ given by

$$p(q,\overline{u}) = \exp\left(-\int_{\overline{u}_0}^{\overline{u}} \frac{C_u(q,u)}{V(u) - C(q,u)}\,\mathrm{d}u\right), \tag{4}$$

where $C(q(a,\overline{u}),\overline{u}) \equiv c(a,\overline{u})$. For KdV, $p(q,\overline{u}) = (q - \overline{u})^{-1/2}$. The change of variables $q = q(a,\overline{u})$ and $p = p(q,\overline{u})$ diagonalizes (2)

$$\begin{aligned} &\overline{u}_t + V(\overline{u})\overline{u}_x = 0, \quad q_t + C(q,\overline{u})q_x = 0, \\ &(kp)_t + C(q,\overline{u})(kp)_x = 0. \end{aligned} \tag{5}$$

We seek solutions to Eq. (5) subject to an initial mean field profile $\overline{u}(x,0) = \overline{u}_0(x)$ and an initial soliton of amplitude a_0 located at $x = x_0$. But we require initial soliton and wavenumber fields $a(x,0)$ and $k(x,0)$ for all x in order to give a properly posed problem for (2). Admissible initial conditions are obtained by recognizing this as a special solution, a simple wave in which all but one of the Riemann invariants are constant [5]. The non-constant Riemann invariant must be $\overline{u}$ to satisfy the initial condition and therefore satisfies $\overline{u} = \overline{u}_0(x - V(\overline{u})t)$. The initial soliton amplitude and position determine the constant Riemann invariant $q_0 = q(a_0, \overline{u}_0(x_0))$. An initial wavenumber k_0 determines the other constant Riemann invariant $k_0p_0 = k_0p(q_0, \overline{u}_0(x_0))$. As we will show, the value of $k_0 > 0$ is not relevant so can be arbitrarily chosen. We now show how this solution physically describes soliton-mean field interaction.

A smooth, initial mean field, e.g., in Fig. 1, will evolve according to the obtained implicit solution until wavebreaking occurs. Our interest is in the interaction of a soliton with the expansion and compression waves that result. In dispersive hydrodynamics, the simplest examples of these are RWs and DSWs, respectively, which are most conveniently generated from step initial data. We therefore analyze the obtained general solution subject to step initial data

$$\overline{u}(x,0) = \overline{u}_\pm, \; a(x,0) = a_\pm, \; k(x,0) = k_\pm, \; \pm x > 0, \tag{6}$$

that model incident and transmitted soliton amplitudes a_- and a_+ through the mean field transition $\overline{u}_-$ to $\overline{u}_+$ for soliton train wavenumbers k_- and k_+. The mean field dynamics depend upon the ordering of $\overline{u}_-$ and $\overline{u}_+$. When $\overline{u}_- < \overline{u}_+$, the mean field equation admits a RW solution, otherwise an unphysical, multi-valued solution. Short-wave dispersion regularizes such behavior and leads to the generation of a DSW. We consider each case in turn.

The transmission of a soliton through a RW is shown experimentally in Fig. 2(a,b). The incident soliton "climbs" the RW and emerges from the interaction with altered amplitude and speed. The mean field is the self-similar, RW solution with $u(x,t) = \overline{u}_\pm$ for $\pm x > \pm V_\pm t$ and

$$\overline{u}(x,t) = V^{-1}(x/t), \quad V_- t \le x \le V_+ t, \tag{7}$$

where $V_\pm = V(\overline{u}_\pm)$ and V^{-1} is the inverse of V. Constant q and kp correspond to adiabatic invariants of the soliton-mean field dynamics that yield constraints on the amplitude, mean field, and wavenumber parameters we call the *transmission and phase conditions*

$$q(a_-,\overline{u}_-) = q(a_+,\overline{u}_+), \quad \frac{k_-}{k_+} = \frac{p(q_+,\overline{u}_+)}{p(q_-,\overline{u}_-)}. \tag{8}$$

The first adiabatic invariant $q(a,\overline{u})$ determines the transmitted soliton amplitude a_+ in terms of the incident soliton amplitude a_- and the mean fields $\overline{u}_\pm$. The second adiabatic invariant determines the ratio k_-/k_+, which in turn yields the soliton's phase shift due to hydrodynamic interaction. Equation (8) is the main theoretical result of this Letter and describes the trapping or transmission of a soliton through a RW and a DSW.

The necessary and sufficient condition for soliton transmission is a positive transmitted soliton amplitude $a_+ > 0$, which places a restriction on the incident soliton amplitude a_-. For the conduit equation, Eq. (3) implies $c_- > c_{\rm cr} = -\overline{u}_- + (\overline{u}_-^2 + 8\overline{u}_+\overline{u}_-)^{1/2}$. For KdV, $a_- > a_{\rm cr} = 2(\overline{u}_+ - \overline{u}_-)$. In both cases, we find that the transmitted soliton's amplitude is decreased, $a_+ < a_-$ and its speed is increased, $c_+ > c_-$. More generally, $\mathrm{sgn}(a_+ - a_-) = -\mathrm{sgn}(q_{\overline{u}} q_a)$ and $\mathrm{sgn}(c_+ - c_-) = \mathrm{sgn}(C_{\overline{u}})$ (see Supplemental Material [24]).

The soliton phase shift is $\Delta = x_+ - x_-$ where $x_\pm$ are the x-intercepts of the soliton pre $(-)$ and post $(+)$ hydrodynamic interaction. Given the initial soliton position x_-, the contraction/expansion of the soliton train determines the phase shift as $\Delta/x_- = k_-/k_+ - 1 = p_+/p_- - 1$. Hence, the ratio k_-/k_+ in the phase condition (8), not the arbitrary initial wavenumber k_-, determines the soliton phase shift. Our use of a fictitious soliton train is therefore justified.

We also determine the soliton-RW trajectory. A soliton with position $x(t)$ propagates through the mean field along a characteristic of the modulation system (2)

$$\frac{dx}{dt} = C(q, \overline{u}(x,t)), \quad x(0) = x_-, \tag{9}$$

where the soliton amplitude $a(x,t)$ varies along the trajectory according to the adiabatic invariant $q(a(x,t), \overline{u}(x,t)) = q(a_-, \overline{u}_-)$. The phase shift from integration of (9) equals Δ from the adiabatic invariant in (8), as expected.

When $a_+ \le 0$ in (8), the soliton is trapped by the RW, as in experiment, Fig. 2(c,d).

If $\overline{u}_- > \overline{u}_+$, a DSW is generated. Soliton-DSW transmission is experimentally depicted in Fig. 2(e,f). An incident soliton propagates through the DSW, exhibiting a highly non-trivial interaction, ultimately emerging with altered amplitude and speed.

In contrast to the soliton-RW problem, the modulation equations (2) are no longer valid throughout the soliton-DSW interaction. Instead, the mean field equation is replaced by the DSW modulation equations [6, 7]. We seek a simple wave solution for soliton-DSW modulation. Because DSW generation occurs only for $t > 0$, the soliton-DSW modulation system for $t < 0$ reduces exactly to Eq. (2), i.e., that of soliton-RW modulation. For $t < 0$, the adiabatic invariants (8) hold. By continuity of the modulation solution, these conditions must hold for $t \ge 0$ as well. In particular, soliton-RW and soliton-DSW interaction both satisfy *the same transmission and phase conditions* (8). This fact, termed hydrodynamic reciprocity, is due to time reversibility of the governing equation (1) and is depicted graphically in Fig. 3.

Equations (3) and (8) for the conduit equation indicate that solitons incident upon DSWs exhibit a decreased transmitted speed $c_{\rm cr} < c_+ < c_-$ and an increased transmitted amplitude $a_+ > a_{\rm cr} > a_-$. $a_{\rm cr}$ and $c_{\rm cr}$ are precisely the amplitude and speed of the DSW's soliton leading edge [32]. Hydrodynamic reciprocity therefore implies that the transmitted soliton's amplitude is decreased (increased), its speed is increased (decreased), and its phase shift is negative (positive) relative to the soliton incident upon the RW (DSW), as observed experimentally in Fig. 2. Using the transmission and phase conditions (8), we accurately predict the conduit soliton trajectory post DSW interaction without any detailed knowledge of soliton-DSW interaction (see Supplemental Material [24]).

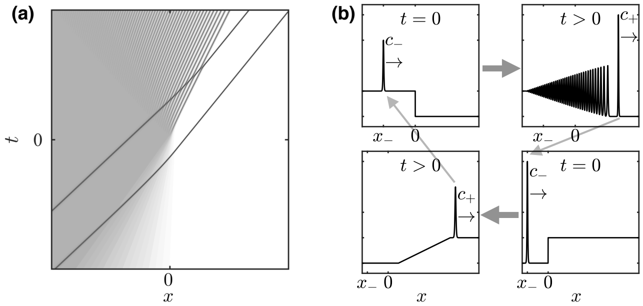

FIG. 3. Graphical depictions of hydrodynamic reciprocity. (a) Space-time contour plot of soliton-DSW ($t > 0$) and soliton-RW ($t < 0$) interaction with two solitons satisfying the transmission condition (8). For $|t|$ sufficiently large, the soliton speeds are the same. (b) If the soliton post DSW interaction (top, left to right) is used to initialize soliton-RW interaction (bottom, right to left), the post RW interaction soliton has the same properties as the pre DSW interaction soliton.

In contrast to soliton-RW transmission, solitons with amplitude a_+ initially placed to the right of the step will interact with the DSW if $a_+ < a_{\rm cr}$. Then the transmission condition (8) implies $a_- < 0$, i.e., the soliton cannot transmit back through the DSW. Instead, the soliton is effectively trapped as a localized defect in the DSW interior as observed experimentally in Fig. 2(g,h).

The transmission and phase conditions (8) for the conduit equation are shown in Fig. 4. For soliton-RW interaction, the abscissa and ordinate are a_- and a_+, respectively reversed for soliton-DSW interaction. Hydrodynamic reciprocity implies that the transmission condition on these axes is the same for soliton-RW and DSW transmission. Reciprocity is confirmed by experiment and numerical simulations of the conduit equation in Fig. 4(a), that slightly deviate from soliton-mean field theory as the amplitudes increase, consistent with previously observed discrepancies [13, 32]. Reciprocity of the phase shift is also confirmed by conduit equation numerics in Fig. 4(b). Our experiments provide definitive evidence of soliton-hydrodynamic transmission, trapping, reciprocity, and the theory's efficacy.

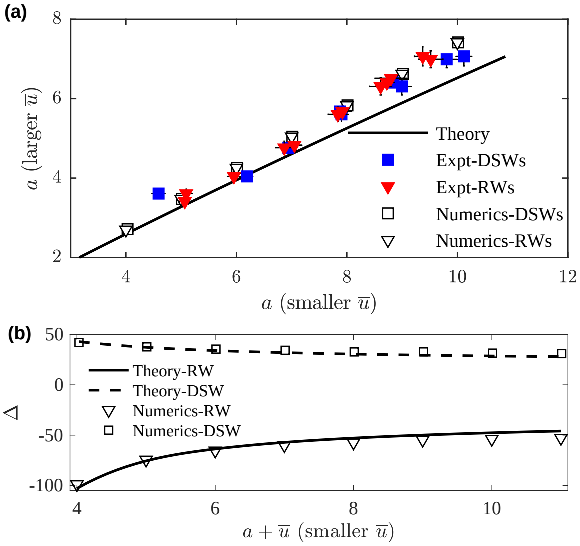

FIG. 4. Transmitted soliton properties due to conduit soliton-RW and DSW interaction for a hydrodynamic transition from $\overline{u} = 1$ to $\overline{u} = 1.75$. a) Soliton amplitude from eq. (8) (curve), experiment (filled squares, triangles), and numerical simulations (open squares, triangles). b) Soliton phase shift from eq. (8) (curves) and numerical simulations (symbols).

We have introduced a general framework for soliton-mean field interaction. The dynamics exhibit two adiabatic invariants that describe soliton trapping or transmission. The existence of the same adiabatic invariants for soliton-mean field interactions of compression (DSW) and expansion (RW) imply hydrodynamic reciprocity. This describes a conceptually new notion of hydrodynamic soliton "tunneling" where the potential barrier is the mean field, obeying the same equations as the soliton [33].

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* hoefer@colorado.edu

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