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Momentum Space Josephson Effects

Junpeng Hou,¹ Xi-Wang Luo,¹ Kuei Sun,¹ Thomas Bersano,² Vandna Gokhroo,² Sean Mossman,² Peter Engels,² and Chuanwei Zhang^{1,*}

¹*Department of Physics, The University of Texas at Dallas, Richardson, Texas 75080-3021, USA*

²*Department of Physics and Astronomy, Washington State University, Pullman, WA 99164, USA*

The Josephson effect is a prominent phenomenon of quantum supercurrents that has been widely studied in superconductors and superfluids. Typical Josephson junctions consist of two real-space superconductors (superfluids) coupled through a weak tunneling barrier. Here we propose a momentum-space Josephson junction in a spin-orbit coupled Bose-Einstein condensate, where states with two different momenta are coupled through Raman-assisted tunneling. We show that Josephson currents can be induced not only by applying the equivalent of “voltages”, but also by tuning tunneling phases. Such *tunneling-phase-driven Josephson junctions* in momentum space are characterized through both full mean field analysis and a concise two-level model, demonstrating the important role of interactions between atoms. Our scheme provides a platform for experimentally realizing momentum-space Josephson junctions and exploring their applications in quantum-mechanical circuits.

Introduction. The Josephson effect [1, 2] is an intriguing quantum phenomenon of supercurrents across a device known as a Josephson junction (JJ). A typical JJ consists of two macroscopic quantum systems [e.g., superconductors, superfluids, or Bose-Einstein condensates (BECs)] that are separated in real or spin space and weakly coupled by quantum tunneling through a thin barrier [Fig. 1(a)] or by Rabi coupling between different spins. Because of quantum tunneling of particles across the junction, JJs have found important applications in quantum-mechanical circuits, such as SQUIDS [3, 4], superconducting qubits [5–8], and precision measurements [3]. In experiments, JJs have been widely realized in solid state superconductors [9, 10], superfluid Helium [11–14], and recently, in ultra-cold atomic gases [15–26], where oscillating supercurrents were generated by applying a voltage drop (or its equivalent) across JJs while maintaining a constant weak coupling (*i.e.*, a.c. Josephson effect [27]).

While JJs have been well studied in real space, a natural and important question is whether Josephson effects can also be observed in momentum space. In this paper, we address this question and propose a scheme for realizing momentum-space JJs (MSJJs). In analogy to bosonic JJs in a real-space double well [22, 23], a MSJJ may be realized with a momentum-space double-well dispersion [see Fig. 1(a)], which is an essential property of spin-orbit coupled systems [28, 29]. Spin-orbit coupling (SOC) is ubiquitous in solid state materials and has recently been realized experimentally in ultracold atomic gases [29–41]. In the presence of SOC, condensates at distinct band minima can be considered as two distinct independent quantum systems. However, unlike quantum tunneling between two wells in real space, two BECs at distinct momenta are not directly coupled.

Here we propose a MSJJ facilitated by a tunable interwell coupling in an spin-orbit coupled BEC [42, 43], where the coupling is generated by an additional pair of counter-propagating Raman lasers. Such Raman-assisted

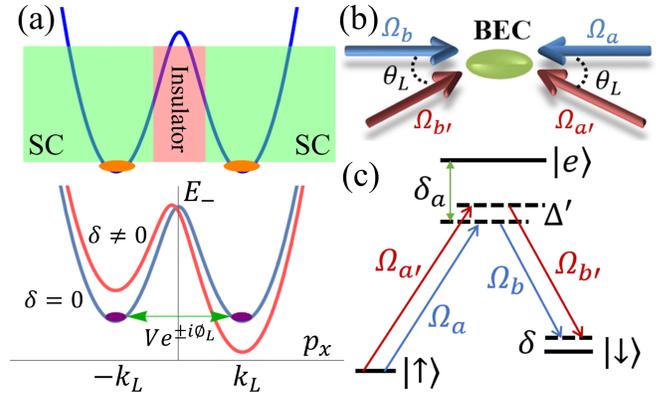


FIG. 1: (a) Illustration of conventional JJ for real-space superconductors (top) versus MSJJ (bottom), where the double well band dispersion is generated using a spin-orbit coupled BEC. (b,c) Experimental setup for realizing a MSJJ. Two pairs of Raman lasers realize SOC (blue) and weak coupling (red) between two band minima, respectively.

tunneling between two momentum states changes both the atomic spin and momentum, and thus couples the condensates at the two band minima. The SOC coupling strength dictates the height of the insulating barrier while the Raman detuning serves as an effective voltage between the two band minima. Suddenly changing the detuning (*i.e.*, applying a voltage) induces a coherent oscillation of the BECs between the two band minima (*i.e.*, supercurrent oscillations), similar to traditional a.c. Josephson effects in superconductors. More interestingly, the phase of the Raman-assisted tunneling between BECs at the two band minima is highly tunable [44], in contrast to real tunneling coefficients for real-space JJs in superconductors [9, 10] and double-well BECs [22, 23]. We show that a sudden change of the tunneling phase (while keeping the effective voltage unchanged) can also induce Josephson effects of supercurrents, a phenomenon

that we name as “*tunneling-phase-driven JJ*”. We focus on this new type of Josephson effect and study its properties through both full mean-field simulation with the Gross-Pitaevskii equation (GPE) [15, 45] and the development of an effective two-level model. Our results present rich physics in this system with different types of supercurrent oscillations (Josephson, plasmonic [17], self-trapping [17, 22], etc.) and display the important role of many-body interactions between atoms. Due to their stability and high controllability, the proposed MSJJs and tunneling-phase-driven JJs may have potential applications for building novel quantum mechanical circuits.

Experimental setup and theoretical modeling. We consider a BEC confined in an elongated trap. Two internal states $|\uparrow\rangle$ and $|\downarrow\rangle$ are coupled by two counter-propagating Raman lasers with Rabi frequencies Ω_a and Ω_b , forming an effective one-dimensional (1D) SOC dispersion relation along the x direction [see Fig. 1(b, c)]. Hereafter we choose recoil momentum $\hbar k_R$ and recoil energy $E_R = \hbar^2 k_R^2 / 2m$ for the Raman lasers as the units of momentum and energy. Consequently, we have length and time in units of $2\pi/k_R$ and \hbar/E_R . The 1D SOC displays a double-well band dispersion in momentum space with two band minima located at $\pm k_L = \pm\sqrt{1 - (\Omega/4)^2}$, where Ω is the Raman coupling strength [46]. The tunneling between BECs at $\pm k_L$ requires simultaneous change of spin and momentum, which can be realized using another independent pair of Raman lasers $\Omega_{a'}$ and $\Omega_{b'}$ incident at an angle $\theta_L = \arccos(1 - k_L)$ to the x axis [Fig. 1(b)]. The frequencies of the pair (a', b') are shifted from those of the pair (a, b) by $\Delta' \sim 100$ MHz so that the interference between them is negligible. The frequency difference between a' and b' should match that between a and b to generate a time-independent coupling.

Since only the x direction is relevant for the SOC dynamics, the other two directions can be integrated out, yielding an effective 1D system. The dynamics of the system can be described by the GPE

$$i \frac{\partial}{\partial t} \psi = (H_0 + \frac{1}{2} \omega_x^2 x^2 + \frac{g}{2} |\psi|^2) \psi \quad (1)$$

under the mean-field approximation, where $\psi = (\psi_\uparrow, \psi_\downarrow)^T$ is the two component condensate wavefunction normalized by the average particle number density $n = \int dx \psi^\dagger \psi$, ω_x represents trapping frequency of harmonic trap. For a typical ^{87}Rb BEC, the effective density interaction $ng \sim 0.1$ with $\sim 10^4$ atoms (see “Experimental consideration” section) and the spin interaction is negligible. The Raman coupling does not affect atomic interactions. The single particle Hamiltonian can be written as [47, 49]

$$H_0 = \begin{pmatrix} (p_x - 1)^2 - \frac{\delta}{2} & \frac{\Omega}{2} + e^{i\phi_L} \Omega_L e^{2ik_L x} \\ \frac{\Omega}{2} + e^{-i\phi_L} \Omega_L e^{-2ik_L x} & (p_x + 1)^2 + \frac{\delta}{2} \end{pmatrix}, \quad (2)$$

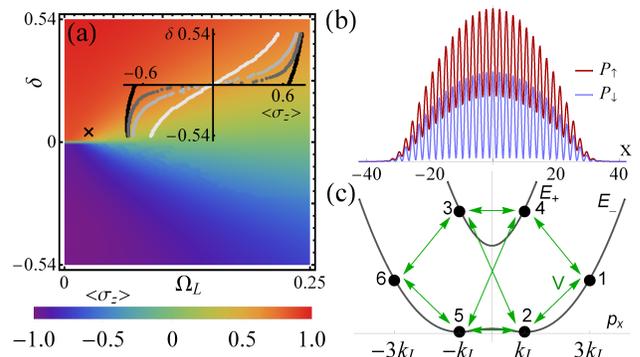


FIG. 2: (a) Ground state phase diagram, where $\Omega = 2.7$, $\phi_L = 0$ and $ng = 0.07$. The inset shows the first-order phase transition for small Ω_L . Black, dark gray, light gray and white lines correspond to $\Omega_L = 0.01, 0.1, 0.2$ and 0.5 respectively. (b) Real space density modulation for the ground state with parameters $\delta = 0.054$ and $\Omega_L = 0.015$ as denoted by the black cross in (a). (c) Illustration of induced couplings between six most relevant momentum states.

where Ω_L is the coupling strength generated by the tunneling lasers, ϕ_L is the relative phase between the two Raman couplings, and δ is the detuning.

The ground state of the BEC is obtained from the imaginary time evolution of the GPE [47, 50] using a time-split-operator method, resulting in the phase diagram shown in Fig. 2(a) in the Ω_L - δ plane, where the color represents spin polarization $\langle \sigma_z \rangle$. For weak Ω_L , interactions lock the condensate to one momentum minimum, yielding a plane-wave phase at large detunings. There is a first-order phase transition [black line in the inset of Fig 2(a)] when δ crosses 0. With increasing Ω_L , the single-particle coupling dominates over the interaction, hence the ground state is in a stripe-like phase with a real-space density modulation [Fig. 2(b)], and $\langle \sigma_z \rangle$ varies continuously and smoothly with respect to δ (white line in the inset of Fig. 2(a)). While a supersolid stripe phase is defined through spontaneous breaking of both continuous translational and gauge symmetries [51, 52], here continuous translational symmetry is synthetically broken by the periodic potential $e^{2ik_L x}$. Nevertheless, the ground state is the superposition of two band minima, similar to an authentic stripe phase induced by interactions.

The additional Raman lasers Ω_L couple not only the two band minima, but also other states from both lower and upper bands. The six most relevant momentum states ψ_i are shown in Fig. 2(c). Expanding the wavefunction $\psi = \sum_{i=1}^6 C_i \psi_i$ in this six-state basis, we obtain a 6×6 effective Hamiltonian [47]. The direct coupling between the two band minima at 2 and 5 is $-V_0 e^{\mp i\phi_L}$ with $V_0 = \frac{1}{2} \Omega_L (1 + k_L)$, while the couplings with other neighboring high-energy states are $-\sqrt{\frac{1-k_L}{2}} e^{\pm i\phi_L}$ and $\frac{1}{2} \sqrt{1 - k_L^2} e^{\mp i\phi_L}$, which approach 0 when $k_L \rightarrow 1$, leav-

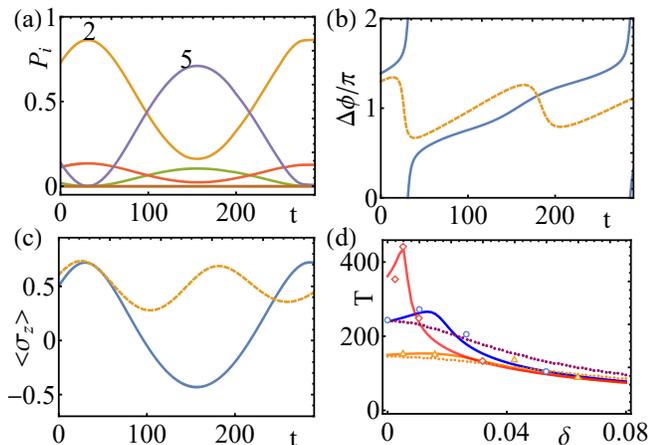


FIG. 3: (a) Time evolution of the populations at different momentum states for tunneling-phase-driven MSJJ for $\Omega = 2.7$, $\delta = 0.014$, $\Omega_L = 0.015$ and $ng = 0.07$. (b) and (c): Evolutions of phase difference (b) and polarization (c) for Josephson oscillation (solid blue) and plasma oscillation (dashed orange). (d) Oscillation period T versus δ when ϕ_L is quenched from ϕ_{L0} to 0 at $\Omega = 2.7$. Circles are results from the GPE simulation, while solid ($ng = 0.07$) and dashed (single particle) lines are from the two-level model. Different colors correspond to different parameter sets: $\Omega_L = 0.015$, $\phi_{L0} = 0.4\pi$ (blue); $\Omega_L = 0.025$, $\phi_{L0} = 0.4\pi$ (orange); and $\Omega_L = 0.015$, $\phi_{L0} = 0.2\pi$ (red). Blue and red dashed lines overlap (purple) since T is independent of ϕ_{L0} for the single particle case.

ing V_0 as the dominant tunneling term. We focus on the region $\Omega_L \ll \Omega$ to avoid significant modification of the original SOC band dispersion and also for the observation of Josephson effects with weak tunneling.

Tunneling-phase-driven MSJJ. In an a.c. JJ, a suddenly applied voltage can induce an oscillation of supercurrents between two superconductors. In our system, BECs at the two band minima marked 2 and 5 are considered as two superfluids and the detuning between them corresponds to a voltage. A sudden change of δ induces an oscillation of the BEC between the two minima, yielding a MSJJ whose properties are described in the supplementary materials [47]. Here we focus on the relative phase ϕ_L for the tunneling element between 2 and 5, which is highly tunable in experiments [44]. In contrast, such tunneling is a real number for a real space JJ between two superconductors or double well BECs. A sudden change of the phase ϕ_L (keeping δ constant) can induce a different type of Josephson effect, *i.e.*, *tunneling-phase-driven JJ*.

In Figs. 3(a)–(c) we show dynamics from simulations of the GPE with a sudden change of the phase ϕ_L from an initial ϕ_{L0} to $\phi_{Lf} = 0$. In panel (a) we plot the population $P_i(t)$ at each momentum state for $\phi_{L0} = 0.4\pi$. Clearly only the states 2 and 5 at the two band minima are largely populated while all other states can be neglected due to their small initial populations, weak cou-

pling to states 2 and 5, and high energies. Panel (b) shows the relative phase between BECs in states 2 and 5. For $\phi_{L0} = 0.4\pi$ (blue solid line), the phase varies through $[0, 2\pi)$, representing a Josephson type of oscillation; while for $\phi_{L0} = 0.3\pi$ (yellow dashed line), the phase oscillates in a small range, showing a plasma oscillation. The polarization $\langle\sigma_z\rangle$ exhibits sinusoidal oscillations for both cases [panel (c)].

Because the population of the BEC stays mainly at the two band minima 2 and 5, we can neglect the other states to derive an effective two-level model, yielding an equation of motion [46, 47]

$$i\partial_t \begin{pmatrix} C_2 \\ C_5 \end{pmatrix} = (H_0^{\text{eff}} + H_I^{\text{eff}}) \begin{pmatrix} C_2 \\ C_5 \end{pmatrix}, \quad (3)$$

where $H_0^{\text{eff}} = \begin{pmatrix} -k_L\delta & -V_0e^{-i\phi_L} \\ -V_0e^{i\phi_L} & k_L\delta \end{pmatrix}$ is the effective single-particle Hamiltonian, and $H_I^{\text{eff}} = 2g_G \begin{pmatrix} |C_5|^2 & 0 \\ 0 & |C_2|^2 \end{pmatrix}$ is the effective interaction term obtained through a variational approximation of the GPE. Generally, g_G depends on $|C_2|^2|C_5|^2$ but is approximately a constant when the interaction strength is weak compared to E_R , yielding $g_G = ng(1 - k_L^2)$. Note that the coupling phase ϕ_L in Eq. (3) can be incorporated into the relative phase between C_2 and C_5 through a simple phase transformation, therefore the quench of ϕ_L is mathematically equivalent to a quench of the relative phase between condensates at two minima (2, 5), although the latter is experimentally impractical.

When the coupling V_0 is strong, the dynamics of the BEC are governed by single particle physics, yielding a linear Rabi oscillation with period $T = \pi/\omega$, where the Rabi frequency $\omega = \sqrt{(k_L\delta)^2 + |V_0|^2}$. Such a simple formula for the period does not apply when the tunneling V_0 is comparable to or weaker than the inter-particle interactions, although the two-level model still agrees reasonably well with the GPE simulations, as shown in Fig. 3(d). We see that the period is similar for interacting and single-particle cases for a large coupling $\Omega_L = 0.025$, but shows strong deviations [see the sharp peak for the solid red line in Fig. 3(d)] from the single particle curve for $\Omega_L = 0.015$. For a very large detuning δ (*i.e.*, voltage), all T collapse to the same line as the single particle case, as expected.

In the two-level approximation, we can choose the normalization $|C_2|^2 + |C_5|^2 = 1$, and recast the equation of motion (3) as [47]

$$\begin{aligned} \partial_t z &= -\sqrt{1 - z^2} \sin(\phi - \phi_{Lf}), \\ \partial_t \phi &= \frac{g_G}{V_0} z + \frac{z}{\sqrt{1 - z^2}} \cos(\phi - \phi_{Lf}) + \frac{k_L\delta}{V_0}, \end{aligned} \quad (4)$$

using the population difference $z = (N_2 - N_5)/N$ and relative phase $\phi = \theta_2 - \theta_5$, where N_i and θ_i are defined

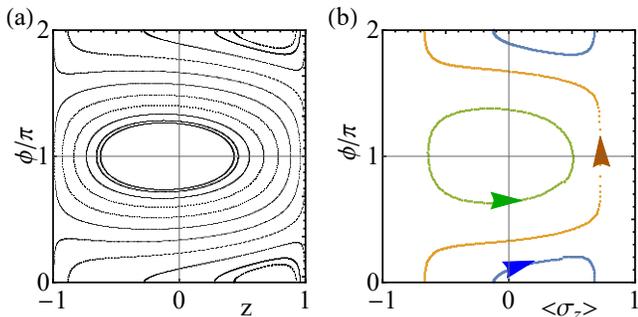


FIG. 4: (a) Classical trajectories in z - ϕ plane for $0 < \phi_{L0} \leq \pi$, with initial value of z at 0.434. (b) Same as (a) but generated through the GPE simulation. Parameters are $ng = 0.07$, $\delta = 0.008$ (corresponding to initial polarization 0.434), $\Omega = 2.7$ and $\Omega_L = 0.015$. The three colors correspond to $\phi_{L0} = 0.2$ (blue), 0.4 (orange), 0.8 (green), respectively. The arrows denote the direction of each trajectory.

through $C_2 = \sqrt{N_2}e^{i\theta_2}$ and $C_5 = \sqrt{N_5}e^{i\theta_5}$. These two classical equations characterize the essential dynamics of MSJJs.

Fig. 4(a) shows how the initial value ϕ_{L0} affects the dynamics. For a relatively small ϕ_{L0} , the classical trajectory is a closed loop around a fixed point with a small amplitude of z and a confined range of phase change $\Delta\phi$, showing a plasma oscillation [17]. With increasing ϕ_{L0} , the amplitudes for both ϕ and z increase. Beyond a critical ϕ_{L0} , ϕ varies through $[0, 2\pi)$, showing a Josephson oscillation. The system returns to plasma oscillation around another fixed point when ϕ_{L0} exceeds another critical point. These classical trajectories from the two-level model agree with those from the GPE simulations in Fig. 4(b). Note that the trajectories around two fixed points have opposite directions. In the single-particle case, these two fixed points correspond to two opposite Zeeman fields for spin precession of the Rabi oscillation [47].

Strong interaction between atoms can dramatically change the BEC dynamics and lead to a self-trapping effect [17, 22], where the oscillation amplitude of z is strongly suppressed. We consider a symmetric oscillation with $\delta = 0$. For a weak interaction of $ng = 0.07$, the oscillation of $\langle\sigma_z\rangle$ shows a perfect sinusoidal pattern (blue line), as seen by the blue line Fig. 5(a) obtained from the GPE simulation. When the interaction is doubled $ng = 0.14$, the oscillation amplitude is reduced and the average $\langle\sigma_z\rangle$ in one period changes from 0 to a finite value (orange line). For a larger but still practicable interaction of $ng = 0.35$, the oscillatory behavior disappears and the condensate is locked at the initial band minimum because of strong density interaction. Such nonlinear self-trapping effects can also be captured in the classical trajectories in the two-level model [Fig. 5 (b)]. With increasing ng , the initial plasma oscillation with a

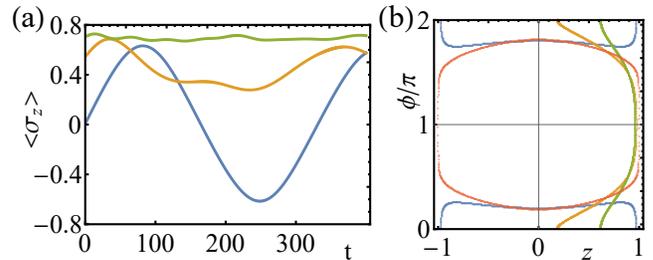


FIG. 5: (a) Self-trapping effects from the GPE simulation. The curves correspond to $ng = 0.07$ (blue), 0.14 (orange) and 0.35 (green), for $\Omega = 2.7$, $\Omega_L = 0.03$, $\phi_{L0} = 0.2\pi$ and $\delta = 0$. (b) Classical phase-plane from the two-level model. The colors are the same as in (a) except $ng = 0.1$ for the red curve.

large variation of z becomes the self-trapped Josephson oscillation with a small z change.

Experimental consideration. The periodic density modulation for the stripe-like ground state can be measured using Bragg scattering, similar to the recent experiments for observing supersolid stripe phases [53]. Consider a ^{87}Rb BEC confined in a quasi-1D harmonic trap. The Raman lasers for generating SOC are incident at 45° with the x axis, yielding an effective wavevector $k_R = \frac{2\pi}{\sqrt{2}\lambda}$ with $\lambda = 784$ nm. The corresponding recoil energy $E_R = 2\pi\hbar \times 1.8$ kHz, thus the time and length units are $\hbar/E_R = 0.088$ ms and $2\pi/k_R = 1109$ μm , respectively. The Raman coupling strength for SOC $\Omega = 2.7E_R$, thus $k_L = 0.738k_R$ and the second pair of Raman lasers should be incident at an angle $\theta_L = 58.6^\circ$ with respect to the x axis. The s-wave scattering length of ^{87}Rb is $a_s = 100.86a_0$, where a_0 is the Bohr radius. Considering a particle number 10^4 to 10^6 and typical trapping frequencies $\omega_x \sim 2\pi \times 5$ Hz and $\omega_y = \omega_z \sim 2\pi \times 75$ Hz, one has the average particle density $n \sim 10^{13}$ to 10^{14} cm^{-3} under Thomas-Fermi approximation [45]. The effective interaction strength can be evaluated through $ng = 4\pi\hbar^2 a_s n / m \sim 0.07$ to $0.48 E_R$, resulting in the time period $T \sim 10$ ms for tunneling-phase-driven Josephson oscillations [Fig. 3(d)].

Discussion and Conclusion. Our two major proposed concepts, momentum-space JJ and tunneling-phase-driven JJ, may also be realized in other physical systems where a double well band dispersion with two almost degenerate local band minima can be generated to ensure the long life time of the BEC at different momenta [54]. For instance, the double-well band dispersion may be realized in optical superlattices with Raman assisted tunneling [55], where two momentum minima can be coupled with additional Raman transitions. The double-well band dispersion can be generalized to triple-well or even more multiple-degenerate momentum states, and the coupling between neighboring minima may form a momentum-space optical lattice [54], which

can be considered as a Josephson junction array [20] in momentum space. The linear momentum discussed here can be generalized to orbital angular momentum (OAM), and an OAM-space JJ may be realized for a BEC on a ring utilizing recent proposals for spin-OAM coupling [56–58] for cold atoms. The discreteness of OAM states may induce interesting Josephson effects that are different from those in continuous real or momentum space. Finally, although absent in solid-state superconductors, the proposed tunneling-phase-induced JJ may be realized in real-space optical superlattices with Raman assisted tunneling [55], where the phase for the Raman tunneling may also be tuned.

In conclusion, we propose a new category of Josephson effects in momentum space, which can be built in a spin-orbit coupled BEC. In addition to traditional voltage-driven Josephson effects, we introduce quenching of the tunneling phase as a novel driving mechanism. Our work may motivate further experimental and theoretical works for studying MSJJs and provides a platform for exploring their applications in building novel quantum mechanical circuits.

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* Corresponding author.

Email: chuanwei.zhang@utdallas.edu

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