Black Hole Entropy from Bondi-Metzner-Sachs Symmetry at the Horizon

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Black Hole Entropy from BMS Symmetry at the Horizon

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Abstract

Near the horizon, the obvious symmetries of a black hole spacetime—the horizon-preserving diffeomorphisms—are enhanced to a larger symmetry group with a BMS_3 algebra. Using dimensional reduction and covariant phase space techniques, I investigate this augmented symmetry, and show that it is strong enough to determine the black hole entropy.

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1 Introduction

One of the most striking features of black hole thermodynamics is the universality of the Bekenstein-Hawking entropy. Black holes, black strings, black rings, black branes, black Saturns, in any dimension, with any charges and spins, with horizons arbitrary distorted by external fields, all have entropies

$$S_{BH} = \frac{A_{hor}}{4G\hbar} \tag{1.1}$$

where A_{hor} is the horizon area. Changing the action can change this formula, but only by another universal term. To deepen the mystery, many different models of the quantum black hole, from string theory to loop quantum gravity to induced gravity, all yield the same entropy, even though they seem to count very different microstates [1]. Even in the elegant analysis of BPS black holes in string theory [2], the entropy and the area are determined separately in terms of a set of charges, and the relation (1.1) requires a new computation for each choice of charges. It seems clear that some underlying structure is missing.

A first guess for this deeper structure is that the relevant degrees of freedom live on the horizon [3]. But this is not enough: while it may explain the proportionality of entropy to area, there is no obvious reason for the coefficient of 1/4 to be universal. A more elaborate idea, first suggested (I believe) in [4], is that the entropy is governed by a horizon symmetry. Two-dimensional conformal symmetry, in particular, has similar universal properties—the Cardy formula fixes the asymptotic behavior of the density of states in terms of a few parameters, independent of the details of the theory [5]—and the possibility of a connection is appealing.

This possibility was first confirmed for the (2+1)-dimensional BTZ black hole in 1998 [6,7], and attempts to extend it to higher dimensions soon followed [8,9]. These efforts have had significant success; see [10] for a review. But they are plagued by serious limitations:

- The symmetries are normally imposed either at infinity or at a timelike "stretched horizon" (although with rare exceptions [11]). Physics at infinity is very powerful, especially for asymptotically anti-de Sitter spaces, but the symmetries by themselves cannot distinguish a black hole from, for instance, a star. The stretched horizon more directly captures the properties of the black hole, but while the entropy has a well-defined limit at the horizon, other parameters typically blow up [12, 13] (again with occasional exceptions [14]), and different definitions of the stretched horizon can lead to different entropies [15, 16].

- The approach fails in what should be the simplest case, two-dimensional dilaton gravity. There are ad hoc fixes—lifting the theory to three dimensions [17] or artificially introducing an integral over time [18]—but none is convincing.

- In higher dimensions, the relevant symmetries are those of the "r–t plane" picked out by the horizon. But to obtain a well-behaved symmetry algebra, one must introduce extra ad hoc angular dependence of the parameters that has no clear physical justification.

Here* I show how to fix these problems. The basic mistake, I argue, has been to try to force the horizon symmetry into the form of a two-dimensional conformal symmetry. This was an understandable choice: until recently, such a symmetry was the only one known to be powerful

*An expanded version of this work will appear in [23].

enough to control the density of states. But it has now been shown that a BMS_3 symmetry has similar universal properties, including a generalized Cardy formula for the entropy [19].

Using covariant phase space methods [20, 21], I show that the symmetry generators can be expressed as integrals along the horizon [22], with no "stretching." I then demonstrate that a BMS_3 symmetry appears in a natural way on the horizon, circumventing the problems of previous efforts, and that it gives the correct counting of states.

2 Quantum states from classical symmetries

Before proceeding, it is worth understanding how a classical symmetry can control the number of quantum states. The basis assumption is that the symmetry, perhaps deformed, is realized in the quantum theory. As we shall see below, the classical symmetry of the horizon of a dilaton black hole is a BMS_3 symmetry, with generators L_n and M_n satisfying a Poisson algebra

$$\begin{aligned} &i\{L_m, L_n\} = (m-n)L_{m+n}, \quad i\{M_m, M_n\} = 0, \\ &i\{L_m, M_n\} = (m-n)M_{m+n} + c_{LM}m(m^2-1)\delta_{m+n,0} \end{aligned} \tag{2.1}$$

where c_{LM} is a classical central charge. With the usual substitutions $\{\bullet,\bullet\} \to \frac{1}{i\hbar}[\bullet,\bullet]$, $\frac{1}{\hbar}L \to \hat{L}$, $\frac{1}{\hbar}M \to \hat{M}$, $\frac{1}{\hbar}c \to \hat{c}$, we obtain a quantum operator algebra

$$\begin{aligned} &[\hat{L}_m, \hat{L}_n] = (m-n)\hat{L}_{m+n}, \quad [\hat{M}_m, \hat{M}_n] = 0, \\ &[\hat{L}_m, \hat{M}_n] = (m-n)\hat{M}_{m+n} + \hat{c}_{LM}m(m^2-1)\delta_{m+n,0} \end{aligned} \tag{2.2}$$

The factors of $\hbar$ in $\hat{L}$ and $\hat{M}$ ensure that the operators are dimensionless symmetry generators. Classical values of the zero modes L_0 and M_0 now become eigenvalues $h_L = L_0/\hbar$, $h_M = M_0/\hbar$ of the corresponding operators. The true quantum symmetry may be a deformation of (2.2)—other central terms may appear, for example—but any differences will be suppressed by factors of $\hbar$.

Now, it has been known since 1986 that a two-dimensional conformal symmetry determines the asymptotic density of states [5].[†] For the simplest case of free bosons and fermions, the result is just the Hardy-Ramanujan formula from number theory for partitions of an integer [25]. I know of no elementary explanation for the general case, but, roughly, the exact symmetry is strong enough to prevent any exponential growth of states at all; growth occurs only because of the anomalous symmetry-breaking characterized by the central charge c.

The symmetry (2.2) is not quite a conformal symmetry, but it is a group contraction, and as Bagchi et al. have shown [19], it has a version of the Cardy formula for the density of states,

$$S \sim 2\pi h_L\sqrt{\frac{\hat{c}_{LM}}{2h_M}} = \frac{2\pi}{\hbar}L_0\sqrt{\frac{c_{LM}}{2M_0}} \tag{2.3}$$

where L_0, M_0, and c_{LM} in the last equality are the classical values. As might have been expected, the entropy is a classical "phase space volume" divided by $\hbar$.

3 Dilaton gravity with null dyads

We now apply this argument to the black hole. The horizon Δ of a stationary black hole in any dimension has a preferred null direction, determined by the geodesics that generate the horizon.

[†]For a careful derivation, see [24].

A neighborhood of the horizon also has a preferred spatial coordinate, the proper distance from Δ. Together, these define a two-dimensional r–t plane, in which most of the interesting physics is expected to take place, since transverse derivatives are red-shifted away near the horizon. Hawking radiation, for instance, can be obtained by dimensional reduction to this plane [26].

Upon dimensional reduction and a field redefinition, the Einstein-Hilbert action becomes [27]

$$I = \frac{1}{16\pi G}\int_M (\varphi R + V[\varphi])\,\epsilon \tag{3.1}$$

where ϵ is the volume two-form. The scalar field φ, the dilaton, is the remnant of the transverse geometry, essentially the transverse area. The resulting equations of motion are

$$E_{ab} = \nabla_a\nabla_b\varphi - g_{ab}\Box\varphi + \frac{1}{2}g_{ab}V = 0 \tag{3.2a}$$

$$R + \frac{dV}{d\varphi} = 0 \tag{3.2b}$$

where the second equation is not independent, but follows from the divergence of the first.

Let us choose a null dyad (ℓ_a, n_a), with $\ell^2 = n^2 = 0$ and $\ell\cdot n = -1$. For notational convenience, define $D = \ell^a\nabla_a$, $\bar{D} = n^a\nabla_a$. The metric and Levi-Civita tensor are then

$$g_{ab} = -\left(\ell_a n_b + n_a\ell_b\right) \qquad\qquad \epsilon_{ab} = \left(\ell_a n_b - n_a\ell_b\right) \tag{3.3}$$

The dyad is determined only up to a local Lorentz transformation, $\ell^a \to e^\lambda\ell^a$, $n^a \to e^{-\lambda}n^a$. We can partially fix this freedom by choosing n_a to have vanishing acceleration, $n^b\nabla_b n^a = 0$; the remaining transformations are those for which $n^a\nabla_a\lambda = 0$. With this choice,

$$\nabla_a\ell_b = -\kappa n_a\ell_b \qquad\qquad \nabla_a n_b = \kappa n_a n_b \tag{3.4}$$

where κ will be the surface gravity at a horizon. Under variation of the dyad, (3.4) is preserved if

$$\begin{aligned}&\bar{D}(\ell^c\delta n_c) = (D+\kappa)(n^c\delta n_c)\\&\delta\kappa = -D(n^c\delta\ell_c) + \kappa\ell^c\delta n_c + \bar{D}(\ell^c\delta\ell_c)\end{aligned} \tag{3.5}$$

By considering the commutator $[\nabla_a, \nabla_b]\ell^b$, one may easily show that

$$R = 2\bar{D}\kappa \tag{3.6}$$

To integrate by parts along the horizon, I will also frequently use the identity

$$df = -Df\,n_a - \bar{D}f\,\ell_a \quad \text{for any function } f \tag{3.7}$$

where I am treating n_a and ℓ_a as one-forms.

4 The covariant canonical formalism and symplectic structure

The idea underlying the covariant canonical formalism is that for a theory with a unique time evolution, the phase space, viewed as the space of initial data, can be identified with the space of classical solutions [28, 29]. This observation, which can be traced back to Lagrange (see [28]), means that we can formulate all the usual ingredients of a Hamiltonian approach without ever having to break general covariance by choosing a time slicing.

Consider a theory in an n-dimensional spacetime with fields Φ^A (for us, φ and g) and a Lagrangian density $L[\Phi]$, which we view as an n-form. Under a general variation of the fields,

$$\delta L = E_A \delta\Phi^A + d\Theta[\Phi, \delta\Phi] \tag{4.1}$$

where the equations of motion are $E_A = 0$ and the last "boundary" term comes from integration by parts. We normally ignore the boundary term, but in the covariant canonical formalism it plays an essential role. The symplectic current ω is defined by a second variation,

$$\omega[\Phi; \delta_1\Phi, \delta_2\Phi] = \delta_1\Theta[\Phi, \delta_2\Phi] - \delta_2\Theta[\Phi, \delta_1\Phi] \tag{4.2}$$

and the symplectic form is

$$\Omega[\Phi; \delta_1\Phi, \delta_2\Phi] = \int_\Sigma \omega[\Phi; \delta_1\Phi, \delta_2\Phi] = \int_\Sigma \omega_{AB}\delta_1\Phi^A \wedge \delta_2\Phi^B \tag{4.3}$$

where Σ is a Cauchy surface.

In keeping with the covariant phase space philosophy, $\Omega[\Phi; \delta_1\Phi, \delta_2\Phi]$ depends on a classical solution Φ, which fixes a point in phase space, and is a two-form on the phase space. The variations $\delta\Phi$ are thus tangent vectors to the space of classical solutions, that is, solutions of the linearized equations of motion. For a field theory in flat spacetime, it is easy to check that when Σ is a surface of constant time, (4.3) is the ordinary symplectic form. The integral (4.3) may depend on the choice of Cauchy surface, but only weakly: the symplectic current is a closed form, so integrals over two Cauchy surfaces Σ_1 and Σ_2 differ only by boundary terms that may arise if $\partial\Sigma_1 \neq \partial\Sigma_2$.

As in ordinary mechanics, the symplectic form determines Poisson brackets and Hamiltonians. In particular, given a family of transformations $\delta_\tau\Phi^A$ labeled by a parameter τ, the Hamiltonian $H[\tau]$ is determined by the condition

$$\delta H[\tau] = \Omega[\delta\Phi, \delta_\tau\Phi] \tag{4.4}$$

for an arbitrary variation $\delta\Phi$. This is just a disguised form of Hamilton's equations of motion,

$$\delta_\tau\Phi^A = (\omega^{-1})^{AB}\frac{\delta H[\tau]}{\delta\Phi^B} \tag{4.5}$$

The Poisson bracket of two Hamiltonians is then

$$\{H[\tau_1], H[\tau_2]\} = \delta_{\tau_1} H[\tau_2] = \Omega[\delta_{\tau_1}\Phi, \delta_{\tau_2}\Phi] \tag{4.6}$$

Specializing to dilaton gravity and using (3.6), it is straightforward to show that

$$\Omega[(\varphi, g); \delta_1(\varphi, g), \delta_2(\varphi, g)] = \frac{1}{8\pi G}\int_\Sigma \left[\delta_1\varphi\, \delta_2(\kappa n_a) + \delta_1(\bar{D}\varphi)\delta_2\ell_a\right] - (1 \leftrightarrow 2) \tag{4.7}$$

again treating ℓ_a and n_a as one-forms on the (one-dimensional) Cauchy surface Σ.

Figure 1: Penrose diagram for the exterior of a black hole

5 Horizons and boundary conditions

For dilaton models obtained by dimensional reduction, φ is essentially the transverse area, and the natural definition of a local "nonexpanding horizon" Δ—a null surface with vanishing expansion [30]—is that $D\varphi = 0$ on Δ. This correctly determines the horizon from the purely two-dimensional viewpoint as well: on shell, Δ is a Killing horizon [27] and the boundary of a trapped region [31]. Exact black hole solutions in two dimensions have such horizons, with essentially the same Penrose diagrams as those in higher dimensions [32].

To study horizon symmetries in the covariant phase space formalism, we incorporate Δ as part of our Cauchy surface. Focus on the exterior region of an asymptotically flat black hole, with a Penrose diagram of figure 1, and take Σ to be the union of the future horizon Δ and future null infinity $\mathscr{I}^+$, with ends at the bifurcation point B and spacelike infinity. The details of $\mathscr{I}^+$ are unimportant; the analysis will still hold for asymptotically de Sitter or anti-de Sitter space.

Let us define $\triangleq$ to mean "equal on Δ," where the horizon Δ is now defined by the requirement that $D\varphi \triangleq 0$. We shall impose three "boundary conditions" at this horizon:

1. $DR \triangleq 0$. This is a requirement of stationary geometry on Δ. In higher dimensions, this condition follows automatically from the Raychaudhuri equation; here it must be imposed by hand, although it holds identically on shell.

2. The conformal class of the metric is fixed on the horizon—that is, only variations of the form $\delta g_{ab} \triangleq \delta\omega\, g_{ab}$ are allowed—in keeping with the physical picture that conformal fluctuations are the relevant degrees of freedom. This implies that $\ell^a\delta\ell_a \triangleq 0$ and $n^a\delta n_a \triangleq 0$.

3. The integration measure n_a is fixed on Δ. In view of condition 2, the additional requirement is that $\ell^a\delta n_a \triangleq 0$. This is really a gauge-fixing condition, which can always be achieved by a suitable local Lorentz transformation. I believe conditions 2 and 3 can be relaxed, at the cost of some complication [23].

These conditions simplify the symplectic form (4.7) considerably. For the portion lying on the horizon,

$$\Omega_\Delta[(\varphi,g);\delta_1(\varphi,g),\delta_2(\varphi,g)] = \frac{1}{8\pi G}\int_\Delta [\delta_1\varphi\,\delta_2\kappa - \delta_1\varphi\,\delta_2\kappa]\, n_a \tag{5.1}$$

a version of the known fact that the area φ and the surface gravity κ are canonically conjugate [33]. One subtlety remains, though. A variation of φ will typically "move the horizon," changing the locus of points $D\varphi = 0$. This will not matter for the symplectic form, since Ω_Δ is independent of the integration contour.[‡] For the variation of an object such as a Hamiltonian defined as an integral over Δ, however, we shall have to take this change into account. The diffeomorphism needed to "move the horizon back" is determined by the condition that

$$(\delta+\delta_\zeta)(D\varphi) = \delta(D\varphi) + \zeta^a\nabla_a(D\varphi) \triangleq 0 \;\Rightarrow\; \zeta^a = \bar\zeta n^a = -\frac{D\delta\varphi}{\bar D D\varphi}n^a \tag{5.2}$$

[‡]As noted in section 4, Ω_Δ *can* change by boundary terms if the ends of the Cauchy surface move. Avoiding this requires that $\delta(D\varphi) = 0$ at the bifurcation point B of figure 1.

and hence

$$\delta \int_\Delta \mathscr{H}\, n_a = \int_\Delta (\delta\mathscr{H} + \zeta^a\nabla_a\mathscr{H})n_a \tag{5.3}$$

6 Symmetries and approximate symmetries

The action (3.1) is, of course, invariant under diffeomorphisms, including horizon "supertranslations" [34] generated by vector fields $\xi^a = \xi\ell^a$. Such diffeomorphisms fail to respect condition 3 of the preceding section, however, since $\ell^a\delta_\xi n_a \neq 0$. This is easily cured, by supplementing each diffeomorphism with a local Lorentz transformation $\delta\ell^a = (\delta\lambda)\ell^a$, $\delta n^a = -(\delta\lambda)n^a$ with $\delta\lambda = D\xi$. By (3.5), this requires that $\bar{D}\xi \triangleq 0$. We thus have an invariance

$$\begin{aligned}
&\delta_\xi\ell^a = 0, \quad \delta_\xi n^a = -(D+\kappa)\xi\, n^a\\
&\delta_\xi g_{ab} = -(D+\kappa)\xi\, g_{ab}\\
&\delta_\xi\varphi = \xi D\varphi \quad \text{with } \bar{D}\xi \triangleq 0
\end{aligned} \tag{6.1}$$

As noted some time ago [22], the action also has an *approximate* invariance under certain shifts of the dilaton near a black hole horizon, with an approximation that can be made arbitrarily good by restricting the transformation to a small enough neighborhood of Δ. Consider a variation

$$\hat{\delta}_\eta\varphi = \nabla_a(\eta\ell^a) = (D+\kappa)\eta \quad \text{with } \bar{D}\eta \triangleq 0 \tag{6.2}$$

(where the hat on $\hat{\delta}$ distinguishes it from a diffeomorphism). The action transforms as

$$\hat{\delta}_\eta I = \frac{1}{16\pi G}\int_M \left(R + \frac{dV}{d\varphi}\right)\hat{\delta}_\eta\varphi\,\epsilon = -\frac{1}{16\pi G}\int_M \eta\left[DR + \frac{d^2V}{d\varphi^2}D\varphi\right]\epsilon \tag{6.3}$$

But $D\varphi$ and DR both vanish at the horizon, so the variation (6.3) can be made as small as one wishes by choosing η to fall off fast enough away from Δ.

There is one remaining subtlety. While the transformation (6.2) does not directly act on the curvature, the change of φ moves the horizon, and DR may no longer vanish at the new location. The displacement of the horizon is characterized by the diffeomorphism (5.2), and can be compensated with a "small" (order $D\varphi$) Weyl transformation of the metric to restore the condition $DR \triangleq 0$. It may be shown that on shell, the required transformation is

$$\hat{\delta}_\eta g_{ab} = \hat{\delta}\omega_\eta\, g_{ab} \quad \text{with } \hat{\delta}\omega_\eta = -\frac{1}{2\bar{D}D\varphi}\frac{d^2V}{d\varphi^2}D\varphi\,\eta \tag{6.4}$$

Like (6.2), the Weyl transformation (6.4) changes the action only by terms proportional to $\eta D\varphi$, which can be made arbitrarily small by choosing η to fall off fast enough away from Δ.

We should also check the variation of the equations of motion (3.2a)–(3.2b). These are, of course, preserved by diffeomorphisms, so we need only consider the transformations (6.2) and (6.4). Since we are assuming that η falls off rapidly away from the horizon, it is enough to check the variations at Δ. By a straightforward computation, most of the equations of motion are preserved: up to terms that are themselves proportional to the equations of motion,

$$g^{ab}\hat{\delta}_\eta E_{ab} \triangleq 2(D+\kappa)\bar{D}\hat{\delta}_\eta\varphi + \frac{dV}{d\varphi}\hat{\delta}_\eta\varphi \triangleq 0 \tag{6.5a}$$

$$n^a n^b\hat{\delta}_\eta E_{ab} \triangleq \bar{D}^2\hat{\delta}_\eta\varphi - \bar{D}\varphi\bar{D}\hat{\delta}_\eta\omega \triangleq 0 \tag{6.5b}$$

$$\hat{\delta}_\eta\left(R + \frac{dV}{d\varphi}\right) \triangleq \hat{\delta}_\eta R + \frac{d^2V}{d\varphi^2}\hat{\delta}_\eta\varphi \triangleq 0 \tag{6.5c}$$

The remaining variation, $\ell^a\ell^b\hat{\delta}_\eta E_{ab}$, is not zero. But this is actually a familiar occurrence in conformal field theory. If we set $E_{ab} = 8\pi G T_{ab}$, we find that

$$\ell^a\ell^b\hat{\delta}_\eta T_{ab} \triangleq \frac{1}{8\pi G}(D-\kappa)D(D+\kappa)\eta \tag{6.6}$$

which is essentially the usual anomaly for a conformal field theory with a central charge proportional to $1/G$ [35]. This is our first indication that the symmetry is anomalous.

7 Canonical generators and their algebra

At the horizon, the symmetries of the preceding section obey an algebra

$$\begin{aligned}
&[\delta_{\xi_1}, \delta_{\xi_2}]f \triangleq \delta_{\xi_{12}} f \qquad \text{with } \ \xi_{12} = -(\xi_1 D\xi_2 - \xi_2 D\xi_1)\\
&[\hat{\delta}_{\eta_1}, \hat{\delta}_{\eta_2}]f \triangleq 0\\
&[\delta_{\xi_1}, \hat{\delta}_{\eta_2}]f \triangleq \hat{\delta}_{\eta_{12}} f \qquad \text{with } \ \eta_{12} = -(\xi_1 D\eta_2 - \eta_2 D\xi_1)
\end{aligned} \tag{7.1}$$

This may be recognized as a BMS_3 algebra, or equivalently a Galilean conformal algebra [36]. We must now ask whether these transformations can be realized canonically as in (4.4), that is, whether there exist generators that satisfy

$$\delta L[\xi] = \frac{1}{8\pi G}\int_\Delta [\delta\varphi\, \delta_\xi\kappa - \delta_\xi\varphi\, \delta\kappa]\, n_a = \frac{1}{8\pi G}\int_\Delta [\delta\varphi\, D(D+\kappa)\xi - \xi D\varphi\, \delta\kappa]\, n_a \tag{7.2a}$$

$$\delta M[\eta] = \frac{1}{8\pi G}\int_\Delta \left[\delta\varphi\, \hat{\delta}_\eta\kappa - \hat{\delta}_\eta\varphi\, \delta\kappa\right] n_a = \frac{1}{8\pi G}\int_\Delta \left[-\delta\kappa(D+\kappa)\eta + \frac{1}{2}\frac{D\delta\varphi}{\bar{D}D\varphi}\eta\frac{d^2V}{d\varphi^2}D\varphi\right] \tag{7.2b}$$

where variations of the generators must include the horizon displacement described by (5.3), and the covariant phase space formalism allows us to impose the equations of motion after variation.

It is not at all clear that such generators exist: there is no obvious reason that the near-horizon symmetry (6.2) should have a canonical realization. In fact, though, the quantities

$$L[\xi] = \frac{1}{8\pi G}\int_\Delta \left[\xi D^2\varphi - \kappa\xi D\varphi\right] n_a \tag{7.3a}$$

$$M[\eta] = \frac{1}{8\pi G}\int_\Delta \eta\left(D\kappa - \frac{1}{2}\kappa^2\right) n_a \tag{7.3b}$$

do the job. Using (4.6), we find Poisson brackets[§]

$$\{L[\xi_1], L[\xi_2]\} = L[\xi_{12}] \tag{7.4a}$$

$$\{M[\eta_1], M[\eta_2]\} \triangleq 0 \tag{7.4b}$$

$$\{L[\xi_1], M[\eta_2]\} \triangleq M[\eta_{12}] + \frac{1}{16\pi G}\int_\Delta \left(D\xi_1 D^2\eta_2 - D\eta_2 D^2\xi_1\right) n_a \tag{7.4c}$$

with ξ_{12} and η_{12} as in (7.1). The canonical generators thus give a representation of the symmetry algebra, but with an added off-diagonal central term. The appearance of such central terms is well-understood in classical mechanics [37]; a similar phenomenon in (2+1)-dimensional gravity [38] was critical for the first counting of microstates of the BTZ black hole [6, 7].

[§]The first of these holds even if $D\varphi \neq 0$. The second and third do not—the η transformations are exact symmetries only on a horizon—but the deviations are of order $(D\varphi)^2$.

8 Modes, zero-modes, and entropy

Given the BMS_3 algebra (7.4a)–(7.4c), we may use the results of section 2 to determine an entropy. We must first find a mode decomposition of the generators. For a black hole with constant surface gravity, the appropriate modes are of the form $e^{in\kappa v}$, where v is the advanced time along the horizon, normalized so that $\ell^a\nabla_a v = 1$. Such modes are periodic in imaginary time with period $2\pi/\kappa$, as required for nonsingular Greens functions. For us, κ is not constant, but we can generalize the periodic modes by defining a phase ψ for which

$$D\psi \triangleq \kappa,\ \bar{D}\psi \triangleq 0 \quad \Leftrightarrow \quad d\psi \triangleq -\kappa n_a \quad \Leftrightarrow \quad \psi \triangleq -\int_\Delta \kappa n_a \triangleq -\int \kappa dv \tag{8.1}$$

The modes are then

$$\zeta_n \triangleq \frac{1}{\kappa}e^{in\psi} \qquad \text{(where } \zeta \text{ is either } \xi \text{ or } \eta\text{)} \tag{8.2}$$

where the prefactor of $1/\kappa$ is chosen so that $\{\zeta_m, \zeta_n\} = \zeta_m D\zeta_n - \zeta_n D\zeta_m = -i(m-n)\zeta_{m+n}$. Setting $L_n = L[\xi_n]$ and $M_n = M[\eta_n]$, our BMS_3 algebra reduces to (2.1), with a central term

$$\frac{1}{16\pi G}\int_\Delta \left(D\xi_m D^2\eta_n - D\eta_n D^2\xi_m\right) n_a = \frac{i}{8\pi G}\int_\Delta mn^2 e^{i(m+n)\psi}d\psi \tag{8.3}$$

If we take the integral to be over a single period—essentially mapping the problem to a circle, as is standard in conformal field theory—we obtain a central charge of

$$c_{LM} = \frac{1}{4G} \tag{8.4}$$

We also need the zero-modes of L and M. For M, this is straightforward: from (7.3b),

$$M_0 = M[\eta_0] = -\frac{1}{16\pi G}\int_\Delta \kappa n_a = \frac{1}{16\pi G}\int d\psi = \frac{1}{8G} \tag{8.5}$$

For L, the "bulk" contribution to L_0 vanishes. But L, unlike M, has a boundary contribution. Indeed, the variation leading to (7.2a) involves integration by parts, with a boundary term

$$\delta L[\xi] = \cdots + \frac{1}{8\pi G}\left(\xi D\delta\varphi - (D+\kappa)\xi\,\delta\varphi\right)\Big|_{\partial\Delta} \tag{8.6}$$

As noted in section 5, the covariant phase space approach requires us to set $D\delta\varphi$ to zero at the bifurcation point B. We should certainly not hold φ itself fixed, though, since that would fix φ along the entire horizon, eliminating the η symmetry. Instead, we should fix the conjugate variable κ at B. This requires an added boundary contribution to cancel the variation (8.6),

$$L_0^{\text{bdry}} = \frac{1}{8\pi G}\varphi(D+\kappa)\xi_0\Big|_B = \frac{\varphi_+}{8\pi G} \tag{8.7}$$

where φ_+ is the value of φ at B. Inserting (8.4), (8.5), and (8.7) into (2.3), we finally obtain

$$S = \frac{\varphi_+}{4G} \tag{8.8}$$

precisely the correct Bekenstein-Hawking entropy.

9 Conclusions

We have seen that black hole entropy is indeed governed by horizon symmetries. In contrast to previous efforts, this derivation requires no stretched horizon and no extra angular dependence or other ad hoc ingredients. The main assumptions are merely that dimensional reduction is possible and that the horizon obeys the "boundary conditions" of section 5.

How should we think about the resulting BMS symmetry? It is not a gauge symmetry: our counting arguments imply that states are not invariant, but transform under high-dimensional representations. Nor is it quite a standard asymptotic symmetry: while we can view the horizon as a sort of boundary, it is a boundary that exists only for a restricted class of field configurations. Physically, we are asking a question of conditional probability—*if* a black hole is present, what are its properties?—and the symmetries reflect this condition. In some sense, this is analogous to entanglement entropy, which requires a similar specification of a boundary.

There are obvious directions for generalization. Dimensional reduction picks out the relevant parts of the geometry, but it would be good to explicitly lift the argument to higher dimensions. We should clarify the relationship between the symmetries of this paper and other appearances of BMS symmetry at the horizon [34, 39–42], as well as the related horizon symmetry used by Wall to prove the generalized second law [43]. It should be feasible to significantly relax the boundary conditions of section 5. It may also be possible to make the concept of "approximate symmetry" in section 6 more precise. In this regard, note that the shift parameter η appears in the variation of the action with no transverse derivatives, and can also be rescaled by a constant without changing the algebra, so both its value and its support can be made arbitrarily small.

Finally, if this symmetry is really responsible for the universal properties of black hole entropy, it should be present, if perhaps hidden, in other derivations of entropy. Hints of such a hidden symmetry have been found for loop quantum gravity [44], induced gravity [45], and perhaps near-extremal black holes in string theory [46], but none of these investigations has exploited the full BMS symmetry. Ideally, we might hope to do even more: perhaps our BMS symmetry can be used to couple the black hole to matter and obtain Hawking radiation, as Emparan and Sachs did for the (2+1)-dimensional black hole [47].

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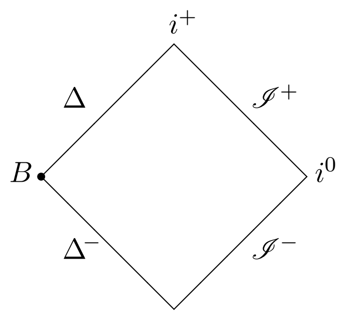
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