Inner Nonlinear Waves and Inelastic Light Scattering of Fractional Quantum Hall States as Evidence of the Gravitational Anomaly

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Inner nonlinear waves and inelastic light scattering of fractional quantum Hall states as an evidence of the gravitational anomaly

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We developed quantum hydrodynamics of inner waves in the bulk of fractional quantum Hall states. We show that the inelastic light scattering by inner waves is a sole effect of the gravitational anomaly. We obtained the formula for the 'oscillator-strength', or 'mean energy' of optical absorption expressed solely in terms of independently measurable static structure factor. The formula does not explicitly depend on a model interaction potential.

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Introduction Excitations in the bulk of FQH state are neutral modes of density modulations. These modes are generally gapped. Evidence of collective modes were seen in inelastic light scattering [1, 2]. The numerically obtained spectrum of small systems [3, 4], also shows a dispersive branch of a collective excitation.

Experimental accessibility of dispersion of neutral modes of FQH states calls for a better understanding of inner waves. There is a renewed interest to the subject. Some recent papers are [7].

Here we show that inelastic light scattering by inner FQH waves is a sole effect of the gravitational anomaly. This observation gives a geometric interpretation to inner waves, and also a new formula for the 'oscillation strength' of optical absorption Δ_k.

The gravitational anomaly only recently entered the QHE literature (e.g., [10]). It is an elusive phenomena which appeared as a higher order gradient correction to bulk transport coefficients [23]. What would be clean, experimentally accessible bulk effects of the gravitational anomaly? We argue that the gravitational anomaly governs one of the major observable in FQH, the *inelastic light scattering*.

A natural approach to studying inner waves is hydrodynamics. It goes back to the seminal paper [5] Girvin, McDonald and Platzman (GMP). Our analysis is based on more recent development of FQH hydrodynamics [8] (see, also [9]). As the GMP theory, the recent hydrodynamics approach has roots in a similarity between FQH states and a superfluid, but with the essential addition: the superfluid is *rotating and incompressible*.

We briefly describe the central point of the paper. The correspondence [8] between FQHE and a superfluid identifies electrons with quantized vortices in a fast rotating incompressible superfluid. Such hydrodynamics can be reformulated as Helmholtz law (see, e.g., [12]): *vortices of an incompressible flow are frozen (or passively drugged by) the flow.* Since vortices represent electrons they could be probed by light. Then, the Helmholtz law forbids inelastic light scattering. Being perturbed by light, vortices instantaneously change the flow and remain frozen into a new flow. They can not accelerate against the flow.

Our main observation is that the quantization subtly corrects the Helmholtz law through the gravitational anomaly. The inelastic light scattering is the effect of this correction.

The gravitational anomaly comes to the stage to prevent a quantization schemes to violate diffeomorphism invariance, the relabeling symmetry of the fluid. It is quite remarkable, that optical probes directly test this fundamental symmetry.

Hydrodynamic description of inner FQHE waves faces a long standing problem of quantizing of incompressible hydrodynamics, specifically the flows with an extensive vorticity, the chiral flows. Accounting the gravitational anomaly described below, represents, perhaps the first consistent quantization of incompressible flows, whose applications go beyond QHE.

Before we proceed, an important comment about spectrum of incompressible waves is in order. The GMP theory [5] adopted variational approach initially developed by Feynman for the superfluid helium [6]. The GMP's approach assumes that a certain two-body Hamiltonian $H = \sum_q V_q \rho_q \rho_{-q}$, where ρ_q is the electronic density mode, indeed, delivers FQH state. Then it assumes that excitations include a single mode density modulation $|k\rangle = \rho_k|0\rangle$, and interprets the diagonal matrix element of the Hamiltonian $\Delta_k = \frac{\langle k|H|k\rangle}{\langle k|k\rangle}$ as a variational approximation to the excitation spectrum. The net result is expressed in terms of a model potential V_q.

Such approach is justified for compressible fluids, like helium, where atomic density modulation, is a linear wave. In this case a single-mode $|k\rangle = \rho_k|0\rangle$ is a long lived state. Contrary to GMP's major assumption, a single-mode state *does not* approximate a long-lived excitation of incompressible fluids, such as of FQHE. A reason for it is that incompressible waves are essentially nonlinear. A single-mode state decays into multiple modes, does not have a spectrum, and Δ_k has no direct relation to true excitations, as it seems commonly accepted in the literature.

Still, we argue that Δ_k could be measured in optical absorption and give a new formula for Δ_k in terms of the structure factor. It refines the GMP formula which expresses Δ_k in terms of model potential V_q.

Correspondence between FQH-states and fast rotating superfluid The analogy between Laughlin's states and a superfluid was suggested in [5, 9], and developed to a correspondence in [8]. In short, a drift of vortices in a fast rotating superfluid and a motion of electrons in FQH regime are governed by same equations.

Fast rotating superfluid is a dense media of same sense vortices with a quantized circulation, which we denote by $2\pi\Gamma$. The total vorticity of the fluid is compensated by a solid rotation with a frequency Ω, such that the mean density of vortices is $\rho_0 = \Omega/(\pi\Gamma)$. We assume that vortices are in a liquid phase (do not crystallize). The frequency of rotation Ω corresponds to the Larmor frequency $\Omega = eB/2m_*$ with an effective mass m_*. The 'mass' is the only phenomenological parameter of the theory determined by the spectral gap. Its energy scale is the Coulomb interaction $\hbar\Omega \sim e^2/\ell$, where $\ell = \sqrt{\hbar/eB}$ is the magnetic length. Then vortices correspond to electrons if the vortex circulation in units of $\hbar/m_*$ is the inverse of the filling fraction and the gap in the spectrum is of the order of $\hbar\Omega$

$$\Gamma = \left(\frac{\hbar}{m_*}\right)\nu^{-1}, \quad \Omega = \frac{eB}{2m_*}. \tag{1}$$

This correspondence differs from that of GMP [5]. The authors of [5] referred to the work of Feynman [6], who considered atomic density modes of a compressible superfluid at rest. Rather, we discuss the modes of vorticity of rotating incompressible superfluid [24].

We will measure distance in units of magnetic length and the energy (the bulk gap) in units of the $2\hbar\Omega$, setting $\ell = \hbar = m_* = 1$. In these units, the mean density $\rho_0 = 1/(2\pi\Gamma) = \nu/2\pi$.

Helmholtz law Hydrodynamics of a 2D incompressible flow can be cast in the Helmholtz form: *material derivative of vorticity vanishes.* If $\boldsymbol{u} = (u_x, u_y)$ is velocity of a flow, $\omega = \boldsymbol{\nabla}\times\boldsymbol{u}$ is the vorticity, $D_t = \partial_t + \boldsymbol{u}\cdot\boldsymbol{\nabla}$ is the material derivative, and the fluid is incompressible $\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$, then the Euler equation in the Helmholtz form reads

$$D_t\omega = 0. \tag{2}$$

In the context of FQH vorticity is identified with the electronic density. In a rotating frame with no net vorticity the correspondence reads

$$\rho(\boldsymbol{r}) = \rho_0 + \frac{1}{2\pi\Gamma}\omega(\boldsymbol{r}). \tag{3}$$

The velocity of the flow $\boldsymbol{u}$, does not have a measurable analog in FQHE. It could be thought as a transversal part of the fictitious gauge field attaching a flux of magnetic field to electrons.

It is quite remarkable that essential features of Laughlin's states are encapsulated in the quantum Helmholtz equation. We will see some of it now.

The Helmholtz law reflects a geometric meaning of hydrodynamics: incompressible flows are generated by a successive action of volume preserving diffeomorphisms. In QHE this concept has been suggested in [14]. Therefore, FQH inner waves, and the equivalent problem of a quantum hydrodynamics, both, seen as a problem of the quantization of the group of volume preserving diffeomorphisms. This group is generated by density modes operators $\bar\rho_k = \int e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\bar\rho(\boldsymbol{r})\,d^2\boldsymbol{r}$, with the algebra

$$[\bar\rho_k,\ \bar\rho_{k'}] = \mathrm{i}e_{kk'}\bar\rho_{k+k'}, \tag{4}$$

with the structure constants $e_{kk'} = \boldsymbol{k}\times\boldsymbol{k'}$. On the torus they are $e_{kk'} = 2e^{\frac{1}{2}(\boldsymbol{k}\cdot\boldsymbol{k'})}\sin(\frac{1}{2}\boldsymbol{k}\times\boldsymbol{k'})$. Here we used *bar* to emphasize quantization as in [5]. The classical limit of (4) is the Poisson brackets of hydrodynamics [15].

Nonlinear waves Few important properties already follow from (2). A well-known fact is that the 2D incompressible hydrodynamics does not assume linear waves. In the language of quantum theory, this means that single density modes are not long-lived states.

However, the Euler equation can be linearized about an inhomogeneous background. If we impose a periodic density modulation $|k_0\rangle$, then on top of it, there are linear waves $\rho_{q-k_0}|k_0\rangle = \rho_{q-k_0}\rho_{k_0}|0\rangle$. This suggests that in contrast to a single-mode, the 2-modes states do have a spectrum. This assertion agrees with the interpretation of inelastic light scattering experiments of Pinczuk et all [1] as Raman type 2-modes process by He and Platzman [4]. We address the spectrum of inner waves elsewhere.

Another consequence mentioned already is: Helmholtz law prohibits absorption of light. We show how this problem is resolved by the quantization.

Quantization of Euler equation Quantization of Euler equation meets essential difficulties. The advection term $\boldsymbol{u}\cdot\nabla\omega = \nabla(\boldsymbol{u}\cdot\omega)$ where two operators sit at the same point requires a regularization. The problem in a general setting has a long history of failures and commonly considered nearly impossible. A scheme of regularization where points are split $\boldsymbol{u}(\boldsymbol{r}+\frac{\epsilon}{2})\cdot\omega(\boldsymbol{r}-\frac{\epsilon}{2})$ leads to inconsistencies. The difficulty is that the point splitting distance, itself depends on the flow $\epsilon[\boldsymbol{u}]$. Hence, a regularization scheme is specific to the flow and can not be practical to all varieties of flows at once. However, if the flow consists of a dense media of vortices, the chiral flow, the quantization could be achieved. In this case, a variable short distance cut off is the distance between vortices $\epsilon \sim 1/\sqrt{\rho}$.

We will use the complex notations. We denote the complex velocity by $\mathrm{u}_z = u_x - \mathrm{i}u_y$ and use the stream function ψ and the traceless part of the fluid momentum flux tensor $\Pi_{ij} = u_i u_j - \frac{1}{2}\delta_{ij}\boldsymbol{u}^2$. In complex coordinates $\mathrm{u}_z = 2\mathrm{i}\partial_z\psi$ and $\Pi_{zz} = \mathrm{u}_z^2$. We will write the advection term as

$$\boldsymbol{u}\cdot\boldsymbol{\nabla}\omega = \mathrm{i}[\partial_z^2\Pi_{\bar{z}\bar{z}} - \partial_{\bar{z}}^2\Pi_{zz}]. \quad (5)$$

Hence, we have to give a quantum meaning to u_z^2. For that we recall a notion of the projected density operator.

Normal ordering and quantization States on the lowest Landau level (LLL), and also flows of rotating superfluid, are realized as Bargmann space [5, 18]. It is a space of holomorphic functions with the inner product $\langle g|f\rangle = \int e^{-\frac{1}{2}|z|^2}\, g^*(\bar{z})f(z)dzd\bar{z}$. The density operators acting in the Bargmann space obeying the algebra (4) are realized by the normally ordered operator $\bar{\rho}_k = \sum_i e^{-\frac{\mathrm{i}}{2}\mathrm{k}z_i^\dagger}e^{-\frac{\mathrm{i}}{2}\bar{\mathrm{k}}z_i}$, where k is a complex wave vector, and $z_i^\dagger = 2\partial_{z_i}$. GMP called it projected (onto LLL) density operator. It is organized such a state $|k\rangle = \bar{\rho}_k|0\rangle$ is holomorphic, hence belongs to LLL. It is also chiral $\overline{\rho}_k^\dagger = \bar{\rho}_{-k}$. Similarly, the 2-modes operator entered the momentum flux tensor on the Bargmann space is represented by normal ordered string

$$\overline{\rho_k\rho_{k'}} = \sum_{i,j} e^{-\frac{\mathrm{i}}{2}\mathrm{k}z_i^\dagger}e^{-\frac{\mathrm{i}}{2}\mathrm{k}'z_j^\dagger}e^{-\frac{\mathrm{i}}{2}\mathrm{k}^*z_i}e^{-\frac{\mathrm{i}}{2}\mathrm{k}'^*z_j}. \quad (6)$$

The projected density modes generate coherent states of LLL, and also states of rotating superfluid if z_i is a coordinate of a vortex.

The next step is to express momentum flux tensor on the Bargmann space by the generators $\overline{\rho}_k$. We get an insight by computing it for the ground state where the density is uniform.

We denote the Wick contraction $\overbrace{AB} = \overline{AB} - \bar{A}\;\bar{B}$ and compute $\overbrace{\mathrm{uu}}$. The contraction of 2-density modes follows from (6)

$$\overbrace{\rho_k\rho_{k'}} = N\delta_{k+k',0}(1 - e^{-\frac{1}{2}k^2}). \quad (7)$$

Equivalently the contraction of two stream functions is

$$\overbrace{\psi(\boldsymbol{r})\psi(\boldsymbol{r}')} = \frac{2\pi}{\nu}\int e^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')}\Big[\frac{1-e^{-\frac{1}{2}k^2}}{k^4}\Big]\frac{d^2k}{(2\pi)^2}. \quad (8)$$

Now we can compute $\overbrace{\mathrm{u}_z(\boldsymbol{r})\mathrm{u}_z(\boldsymbol{r}')} = -4\partial_z\partial_{z'}\overbrace{\psi(\boldsymbol{r})\psi(\boldsymbol{r}')}$. In the hydrodynamic limit ($|r-r'|\gg\ell$) $\overbrace{\mathrm{u}_z(\boldsymbol{r})\mathrm{u}_z(\boldsymbol{r}')} \sim (z-z')^{-2}$. As $r\to r'$ the net result is zero due to the rotational symmetry. Effect of short distance regularization does not show up.

Gravitational anomaly in hydrodynamics Now we extend these calculations when u_z^2 is sandwiched between two flow states with a non-uniform density. In this case the cut-off as $r\to r'$ is not uniform. The result follows from the geometric interpretation of the fluid flow. In this picture the distance between particles (vortices) is interpreted as a metric $ds^2 = \rho|dz|^2$ of an auxiliary evolving Riemann surface. The scalar curvature of this surface is

$$\mathcal{R} = -4\rho^{-1}\partial_z\partial_{\bar{z}}\log\rho. \quad (9)$$

The distance between particles is invariant under a change of coordinates, or by relabeling particles. In hydrodynamics, this fictitious symmetry typically applied to fluid atoms. In our approach, it is a relabeling symmetry of vortices. We want to keep it in quantization.

To proceed, we notice that in the hydrodynamic limit the contraction of stream functions (8) is the Green function of the Laplace operator

$$\overbrace{\psi(\boldsymbol{r})\psi(\boldsymbol{r}')} = \frac{\pi}{\nu}G(\boldsymbol{r},\boldsymbol{r}'). \quad (10)$$

It is natural to assume that in a flow state the contraction is the Green function of the Laplace-Beltrami operator in the metric $\rho|dz|^2$. Then the problem is reduced to a covariant regularization of the Green function as $r\to r'$. Such regularization identifies the short distance cut-off with the geodesic distance $d(\boldsymbol{r},\boldsymbol{r}')$. With this prescription we define the Wick contraction of the momentum flux tensor $\Pi_{zz} = \mathrm{u}_z^2$ as a limit

$$\overbrace{\mathrm{uu}} = \frac{4\pi}{\nu}\lim_{r\to r'}\partial_z\partial_{z'}\Big[G(\boldsymbol{r},\boldsymbol{r}') + \frac{1}{2\pi}\log d(\boldsymbol{r},\boldsymbol{r}')\Big]. \quad (11)$$

The result of this limit is known: it is the Schwarzian of the metric (Supplemented Material (SM))

$$\overbrace{\mathrm{uu}} = \frac{1}{6\nu}\left(\partial_z^2\log\rho - \frac{1}{2}(\partial_z\log\rho)^2\right). \quad (12)$$

Then the contraction of the advection term (5) is expressed through the curvature (9)

$$\overbrace{\boldsymbol{u}\cdot\boldsymbol{\nabla}\omega} = \frac{1}{96\pi}\,\boldsymbol{\nabla}\mathcal{R}\times\boldsymbol{\nabla}\omega. \quad (13)$$

This is the main result of the quantization [25]. We can now treat the hydrodynamics as a field theory, with a constant cut-off, independent on the flow. With the help of (13) we obtain

$$D_t\bar{\rho} = \frac{1}{96\pi}\boldsymbol{\nabla}\mathcal{R}\times\boldsymbol{\nabla}\bar{\rho}. \quad (14)$$

If waves are small and long, $\mathcal{R}\approx -\rho_0^{-2}\Delta\rho$, the correction to the Helmholtz law could be treated in the harmonic and long-wave approximation

$$D_t\bar{\rho}_k = \frac{\pi}{24\nu^2}\sum_q q^2(\boldsymbol{k}\times\boldsymbol{q})\bar{\rho}_q\bar{\rho}_{k-q}. \quad (15)$$

Deviation from Helmholtz law Implication of quantum corrections is that Helmholtz law held for quantum operators does not hold for their matrix elements: material derivative for the projected density mode (14) does not vanish. Acceleration of particles against the flow appears as quantum corrections, but, as we will see, it is the only source for the light scattering. The universal departure from the Helmholtz law is our main result.

Hamiltonian Now we are in a position to determine the Hamiltonian which together with the brackets (4) yields Eq. (14). Here we present the result, leaving calculations to SM.

We write the Hamiltonian separating classical and quantum contributions as $H = \int (\mathcal{H} - \hbar S)\, \rho_0 d^2\boldsymbol{r}$, restoring the units and skipping *bars* in the notations

$$\mathcal{H} = \frac{m_*}{2}\left(\boldsymbol{u}^2 + 2\boldsymbol{u}\cdot\boldsymbol{u}_0 - \pi\Gamma^2\rho\log\rho\right), \tag{16}$$

$$S = -\pi\Gamma\left(\rho\log\rho + \frac{1}{96\pi}(\nabla\log\rho)^2\right). \tag{17}$$

The first two terms in the classical part (16) are the kinetic and centrifugal energies, $\nabla\times\boldsymbol{u}_0 = 2\Omega$. The last term in (16) regularizes the divergency of the kinetic energy at vortex cores. It was known in the theory of superfluid since 1961 paper of Kemoklidze and Khalatnikov [20], see, also, recent [21]. This term is the Casimir invariant, whose Poisson bracket with all local fields vanish. It does not show in equations of motion (14) but enters the current ((3) of SM) as a divergence free term.

The quantum part (17) also consists of two terms. The first term is a quantum correction to the Kemoklidze-Khalatnikov term. The second term represents the effect of the gravitational anomaly.

Static structure factor Now we check that the Helmholtz equation (2), and its consequences (14-17), encode independently known long wave expansion of the static structure factor $s_k = \frac{1}{N}\langle 0|\rho_{-k}\rho_k|0\rangle$. This check justifies the hydrodynamic approach.

According to the theory of linear response, the structure factor appears in the harmonic approximation of the Hamiltonian as a rigidity of density modes (see SM)

$$H \approx \frac{1}{2N}\left(\frac{\hbar^2}{m_*\ell^2}\right)\sum_{q\neq 0} s_q^{-1}\rho_{-q}\rho_q. \tag{18}$$

We compute the inverse structure factor by expanding (16,17). The result is

$$s_q^{-1} = \frac{2}{q^2} - \left(\frac{1}{2\nu} - 1\right) + [s_q^{-1}]_+, \tag{19}$$

where $[s_q^{-1}]_+$ is the part of the expansion which consists of positive powers of q. The leading term in $[s_q^{-1}]_+$ followed from the las term in (17) is the effect of the gravitational anomaly.

$$[s_q^{-1}]_+ = \frac{q^2}{24\nu} + \mathcal{O}(q^4). \tag{20}$$

Inverting (19) we obtain

$$s_q = \frac{1}{2}q^2 + \frac{1}{8\nu}(1-2\nu)q^4 + \frac{1}{8\nu^2}\left(\frac{3}{4}-\nu\right)\left(\frac{1}{3}-\nu\right)q^6 + \dots \tag{21}$$

Each of the three terms in (21) is independently known, has a universal meaning, and reflects symmetries of the electronic fluid. The term q^2 corresponds to the kinetic energy $\frac{1}{2}\boldsymbol{u}^2$, q^4 corresponds to the $\rho\log\rho$ term in (16,17) and referred as the 'compressibility' sum rule. Finally, q^6 term represents the gravitational anomaly. It was first obtained in [22] directly from Laughlin's wave function. In equivalent forms, it appeared in [17]. There is no reasons to think that higher terms, but the first three, are universal.

Using (21) we obtain the projected structure factor $\bar{s}_k = \frac{1}{N}\langle 0|\bar{\rho}_{-k}\bar{\rho}_k|0\rangle$. From (7) we have $\bar{s}_q = s_q - (1 - e^{-\frac{1}{2}q^2})$. Hence,

$$\bar{s}_q = (1-\nu)\frac{q^4}{8\nu}\left(1 + \frac{1}{6\nu}(3-10\nu)q^2\right) + \dots \tag{22}$$

Harmonic approximation We can now express correction to the Helmholtz law in terms of the structure factor. Let us compute $[H, \bar{\rho}_k]$ with the Hamiltonian (18). The first term in the expansion of s_q^{-1} (19) gives the material derivative, the second does not contribute. The correction to the Helmholtz law is due to the positive part of the expansion (19), whose leading term is the gravitational anomaly (20). We obtain a refine form of the Eq.(15) valid at all k

$$D_t\bar{\rho}_k = \frac{\pi}{\nu}\sum_q e_{kq}[s_q^{-1}]_+\bar{\rho}_q\bar{\rho}_{k-q}. \tag{23}$$

Optical absorption by nonlinear waves Absorption occurs when light accelerates particles against the flow, i.e., due to departure from the Helmholtz law.

Consider an acoustic wave imposed through the Hall bar as in experiment [2]. It creates a state $|k\rangle = \overline{\rho}_k|0\rangle$. In solids the optical absorption measures the differential intensity $S_k(\omega) = \frac{1}{N}\langle k|\delta(H - \hbar\omega)|k\rangle$, and the integrated intensity $\bar{s}_k = \hbar\int S_k(\omega)d\omega = \frac{1}{N}\langle k|k\rangle$, the projected static structure factor. Another objects of interest is the oscillation strength, the first moment of the intensity

$$\bar{f}_k = \int \omega S_k(\omega)d\omega = \frac{1}{N}\langle k|H|k\rangle = \frac{\mathrm{i}}{2N}\langle 0|\dot{\bar{\rho}}_k\ \bar{\rho}_{-k}|0\rangle \tag{24}$$

and the the 'mean energy' $\Delta_k = \bar{f}_k/\bar{s}_k = \langle k|H|k\rangle/\langle k|k\rangle$. In fluids, intensity must be written in a coordinate system moving with the fluid. This means that the time derivative in (24) is the material derivative

$$\bar{f}_k = \frac{1}{2N\mathrm{i}}\langle 0|[D_t\bar{\rho}_k,\ \bar{\rho}_{-k}]|0\rangle. \tag{25}$$

Hence, only the RHS of (23) enters (25).

Typically $S_k(\omega)$ features an asymmetric peak supported by the curve $\hbar\omega = \Delta_k$, rudimentary interpreted as a spectrum of excitations. Such interpretation will be valid, would $\bar\rho_k|0\rangle$ be a long-lived state, as it happens in a compressible fluid. As we commented above, in FQHE, the state $\bar\rho_k|0\rangle$ is short-lived.

Interpretation aside, we compute $\bar f_k$. Eq. (23) reduces (25) to $\langle 0|\bar\rho_{-k}\bar\rho_q\bar\rho_{k-q}|0\rangle$, which we compute with the help of the algebra (4). We express result in terms of $\tilde s_k = (1-\nu)^{-1}e^{k^2/2}\bar s_k$ and in units $\hbar^2/(\pi m_*\ell^2)$, and use the structure constants (4) for the torus

$$\Delta_k = \tilde s_k^{-1}\int \sin^2(\tfrac12 \boldsymbol{k}\times\boldsymbol{q})e^{-\frac{q^2}{2}}[s_q^{-1}]_+(\tilde s_q - \tilde s_{k-q})d^2q \quad (26)$$

Contrary to (4.15) of [5], our formula does not explicitly depend on a model interaction. It is expressed only through independently measured structure factor. We emphasize that beyond terms in (22) the structure factor depends on details of the material, and so as the 'mean energy' (26).

Magneto-roton minimum Both $\bar f_k$ and $\bar s_k$ and their ratio Δ_k feature a broad asymmetric peaks at $k\ell \sim 1$.

At $k\to 0$, $\tilde s_k \sim \frac{k^4}{8\nu}$ and $[s_k^{-1}]_+ \sim \frac{k^2}{24\nu}$. At $k\to\infty$, $\tilde s_k = 1$ and $[s_k^{-1}]_+ = \frac{1}{2\nu}$. Hence, the 'mean-energy' Δ_k smoothly interpolates between

$$\Delta_{k=0} = 4\nu\int q^2[s_q^{-1}]_+\left(\nabla_q^2\tilde s_q\right)e^{-\frac{q^2}{2}}d^2q \quad (27)$$

$$\Delta_{k\to\infty} = \int [s_q^{-1}]_+(\tilde s_q+1)e^{-\frac{q^2}{2}}d^2q. \quad (28)$$

Numerically evaluation of Δ_k from model Hamiltonians [3–5] also shows a minimum. GMP called it magneto-roton minimum. However, it is unclear whether it has a universal meaning. The minimum relies on features of $\bar s_k$ beyond its universal part (19-22).

A sequence of minima in optical absorption reported in [2] for fractions other than Laughlin's. It is not clear whether they are related to the GMP minimum for Laughlin's states.

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