Color Superconductivity and Charge Neutrality in Yukawa Theory

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Color superconductivity and charge neutrality in Yukawa theory

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It is generally believed that when Cooper pairing occurs between two different species of fermions, their Fermi surfaces become locked together so that the resultant state remains "neutral", with equal number densities of the two species, even when subjected to a chemical potential that couples to the difference in number densities. This belief is based on mean-field calculations in models with a zero-range interaction, where the anomalous self-energy is independent of energy and momentum. Following up on an early report of a deviation from neutrality in a Dyson-Schwinger calculation of color-flavor-locked quark matter, we investigate the neutrality of a two-species condensate using a Yukawa model which has a finite-range interaction. In a mean field calculation we obtain the full energy-momentum dependence of the self energy and find that the energy dependence leads to a population imbalance in the Cooper-paired phase when it is stressed by a species-dependent chemical potential. This gives some support to the suggestion that the color-flavor-locked phase of quark matter might not be an insulator.

1. Introduction. In a system containing a high density of fermions of two different species, there may be Cooper pairing between the two species. This situation arises generically in quark matter, where Cooper pairing of quarks is most energetically favorable in the flavor-antisymmetric channel [1–3], and can also occur in cold atomic gases [4]. It is generally believed that in a BCS condensate with cross-species pairing, the Fermi momenta of the two species are locked to a common value, so that even in the presence of a chemical potential μ_δ that would favor one species over the other, the number densities remain equal. The charge imbalance $N_\delta \equiv N_1 - N_2$ remains zero for a range of μ_δ, so, for example, the charge imbalance susceptibility, $dN_\delta/d\mu_\delta$ at $\mu_\delta = 0$, is zero.

One example where this arises is the color-flavor-locked (CFL) phase of dense quark matter [5]. The CFL phase is believed to be an insulator, with no electrons present, because BCS pairing between quarks of different flavors locks the Fermi momenta of the three flavors to the same value, ensuring that the quark population remains electrically neutral even in the presence of an electrostatic potential that would favor up quarks over down and strange quarks. Thus the electron density in CFL quark matter remains zero.

This rigid locking of the Fermi momenta has been demonstrated for NJL-type models [6], where there is a contact interaction between the fermions, so the fermion self energy, including the fermion-number-violating ("anomalous") component that arises from pairing, is independent of the energy and momentum. However, a study of the CFL phase using the Dyson-Schwinger approach [7], where the anomalous self-energy is energy and momentum dependent, found that there were electrons in the CFL phase. The authors of Ref. [7] suggest that this arises from the energy dependence of the anomalous self-energy.

In this paper we show explicitly, in a mean-field treatment of a model where the fermions interact via a Yukawa boson, that the energy dependence of the anomalous self-energy Δ leads to a non-zero charge imbalance susceptibility via a factor of $\partial\Delta/\partial k_4$ in the relevant integral. The form factor of our interaction is the free boson propagator, not including any in-medium effects on the boson self-energy. However, this simple model is sufficient to give an energy and momentum dependent anomalous self-energy.

The single-flavor Yukawa model was studied previously by Pisarski and Rischke [8], but they neglected the energy-momentum dependence of the anomalous self energy, arguing that this was valid at strong coupling.

In very recent work, Sedrakian et al. [9] studied a two-flavor model interacting via σ and π mesons. They calculated the energy dependence (though not momentum dependence) of the anomalous self energy, and also included in-medium corrections to the boson self energy. They did not calculate the charge imbalance.

2. Yukawa Theory. Our version of the Yukawa model contains a massless fermion ψ that comes in two flavors and, to allow pairing in the $J^P = 0^+$ channel, two "tastes" which would become colors in a full QCD treatment. The fermions have a Yukawa coupling of strength g with a Yukawa boson of mass m. The Lagrangian density is

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu + \gamma^0\mu)\psi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - g\bar{\psi}\psi\phi \ . \quad (1)$$

The coupling to the Yukawa boson breaks chiral symmetry, so the internal symmetry group is

$$\frac{SU(2)_{\rm flavor}}{Z_2} \times \frac{SU(2)_{\rm taste}}{Z_2} \times U(1)_{\rm B} \ . \quad (2)$$

To probe the charge balance of the system we couple the two flavors to separate chemical potentials $\mu_1 = \mu + \mu_\delta$ and $\mu_2 = \mu - \mu_\delta$. We study Cooper pairing in the channel that is a singlet (and therefore antisymmetric) in flavor, taste, and spin, since this channel is known to dominate

Cooper pairing in NJL models and in weakly coupled QCD [3]. This leaves the flavor and taste symmetries unbroken, and breaks $U(1)_{\rm B} \to Z_2$. The free energy of this model in the mean field approximation is (see Chapter 5 in Ref. [10])

$$\Omega = \frac{-T}{2V}\sum_k \ln\det\mathcal{S}^{-1}(k) + \frac{8g^2T^2}{V^2}\sum_{q,k} D(k-q)f(q)f(k)\ , \tag{3}$$

where $\mathcal{S}^{-1}(k)$ is the inverse fermion propagator, the boson propagator is $D(k) = 1/(k^2+m^2)$, and $f(q)$ is related to the anomalous self-energy in momentum space

$$\Delta(k) = \frac{g^2T}{V}\sum_q D(k-q)f(q)\ . \tag{4}$$

The inverse fermion propagator in Nambu-Gor'kov space is

$$\mathcal{S}^{-1}(k) = \begin{pmatrix} S_+^{-1}(k) & T_-^{-1}(k) \\ T_+^{-1}(k) & S_-^{-1}(k)\end{pmatrix}\ , \tag{5}$$

where the terms on the diagonal are given by

$$S_\pm^{-1}(k) = (\gamma^0(ik_4 \pm \mu) - \gamma^i k_i)\otimes \mathbb{1}_f \otimes \mathbb{1}_t \pm \gamma^0\mu_\delta \otimes \sigma_{3_f}\otimes \mathbb{1}_t \tag{6}$$

and the off-diagonal terms by

$$T_\pm^{-1}(k) = \pm\Delta(k)\otimes\sigma_{2_f}\otimes\sigma_{2_t}\ . \tag{7}$$

In each entry, the first factor in the tensor product lives in Dirac space, the second factor lives in two dimensional flavor space and the third factor lives in two-dimensional taste space. The possibility of Cooper pairing is incorporated by the off-diagonal ("anomalous") terms, which represent the violation of quark number symmetry via the condensate, allowing a quark to evolve into an antiquark. The inverse propagator has eight distinct eigenvalues, each of which is 4-fold degenerate. Since the determinant is the product of the eigenvalues, we can use this in Eq. (3) to obtain the free energy

$$\Omega = -\frac{2T}{V}\sum_k \log(X) + \frac{8g^2T^2}{V^2}\sum_{q,k} D(k-q)f(q)f(k), \tag{8}$$

$$X \equiv \prod_{s_i}\left(ik_4 + s_1\sqrt{\Delta(k)^2 + (|\vec{k}| + s_2\mu)^2} + s_3\mu_\delta\right)\ ,$$

where each of the three s_i varies over ± 1 in the product, yielding 8 factors altogether. Minimizing the free energy (8) with respect to $f(k)$ gives the gap equation

$$\Delta(k) = \frac{g^2T}{V}\sum_q D(k-q)W(q)\ , \tag{9}$$

where we have defined

$$W(q) \equiv \frac{1}{4}\sum_{t_1,t_2}\frac{\Delta(q)}{\Delta(q)^2 + (|\vec{q}| + t_1\mu)^2 + (q_4 + t_2 i\mu_\delta)^2}\ , \tag{10}$$

where each of the two t_i varies over ± 1 in the sum. Comparing this with (4), we find that at the minimum of the free energy $f(k) = W(k)$. Using this in (8), we find the free energy of the mean-field ground state,

$$\Omega = -\frac{2T}{V}\sum_k \log(X) + \frac{8T}{V}\sum_k \Delta(k)W(k)\ . \tag{11}$$

We emphasize that this expression is only valid when $\Delta(k)$ is a solution of the gap equation.

From now on, we work in the zero-temperature limit.

3. Charge imbalance susceptibility. We now briefly review the argument that the charge imbalance susceptibility is zero in NJL models, and describe why it is nonzero in a Yukawa model. The NJL model is the limit of the Yukawa model where $D(k-q) \to 1/m^2$. A momentum cutoff Λ is then required, but the k_4 integral can be left unregulated. The anomalous self-energy Δ is then independent of energy and momentum, and the gap equation (9) becomes

$$\Delta = \frac{g^2}{2m^2}\int\frac{d^4q}{(2\pi)^4}\left(\frac{\Delta}{\Delta^2 + (|\vec{q}| - \mu)^2 + (q_4 + i\mu_\delta)^2} + \frac{\Delta}{\Delta^2 + (|\vec{q}| + \mu)^2 + (q_4 + i\mu_\delta)^2}\right)\ . \tag{12}$$

where the q_4 integration contour can be closed in the upper or lower half-plane, since the poles come in complex conjugate pairs with opposite sign residues. Since the residues of the poles are independent of μ_δ, the gap equation is independent of μ_δ as long as one can eliminate μ_δ by performing a change of integration variable $q_4' = q_4 - i\mu_\delta$ without moving any poles from the upper half plane to the lower half plane and vice versa.

The poles of the integrand in the gap equation (12) are

$$q_4 = i\left(-\mu_\delta \pm \sqrt{\Delta^2 + (|\vec{q}| \pm \mu)^2}\right)\ . \tag{13}$$

We can see that the poles with a "+" sign in front of the square-root move toward the lower half plane from the upper half plane as we increase μ_δ, and they first cross the real axis when $\mu_\delta = \Delta$ (with $|\vec{q}| = \mu$). Therefore, as long as $\mu_\delta < \Delta$, the gap equation and hence the anomalous self-energy Δ and the free energy of the paired state $\Omega_{\rm BCS}$ are independent of μ_δ and the charge imbalance N_δ and all its derivatives vanish [6].

For a Yukawa interaction the gap equation (9,10) is

$$\Delta(k) = g^2\int\frac{d^4q}{(2\pi)^4}\frac{1}{(k_4 - q_4)^2 + (\vec{k} - \vec{q})^2 + m^2}W(q)\ . \tag{14}$$

Unlike the NJL case, the presence of a scalar propagator in the gap equation results in an energy- and momentum dependent anomalous self-energy. Because the anomalous self-energy and the scalar propagator have q_4

dependence, a shift in q_4 by $q_4' = q_4 - i\mu_\delta$ does not result in a μ_δ independent gap equation. This raises the possibility that N_δ and its derivatives may no longer be zero. Below we will describe explicit calculations that find that this is indeed the case.

4. Number Density and Susceptibility. The charge imbalance $N_\delta = -d\Omega/d\mu_\delta$ only receives a contribution from the first term in the free energy (8), since the second term only depends on μ_δ via $\Delta(k)$, and in the ground state $\delta\Omega/\delta\Delta(k) = 0$. Thus

$$N_\delta = 2\int \frac{d^4k}{(2\pi)^4}\frac{1}{X}\frac{\partial X}{\partial \mu_\delta}\ . \tag{15}$$

The charge imbalance susceptibility is $\chi_\delta \equiv dN_\delta/d\mu_\delta$ evaluated at $\mu_\delta = 0$. In principle N_δ depends on μ_δ both explicitly and via the dependence of $\Delta(k)$ on μ_δ, but we can take $\chi_\delta = \partial N_\delta/\partial\mu_\delta|_\Delta$ because we evaluate the derivative at $\mu_\delta = 0$ and $\Delta(k)$ can only depend on μ_δ at $\mathcal{O}(\mu_\delta^2)$ or higher. This follows from the flavor symmetry which ensures that the free energy Ω is an even function of μ_δ, and $\Delta(k)$ being defined by requiring $\delta\Omega/\delta\Delta(k) = 0$. From (15) and (8) we find

$$\begin{aligned}\chi_\delta &= \frac{1}{\pi^4}\int dk_4\ d^3\vec{k}\,\left(2k_4^2(U_+^2+U_-^2) - U_+ - U_-\right),\\ U_\pm &\equiv \frac{1}{\Delta(k)^2 + (|\vec{k}|\pm\mu)^2 + k_4^2}.\end{aligned} \tag{16}$$

Integrating by parts, we find

$$\chi_\delta = \frac{4}{\pi^3}\int dk_4\ dk k^2\left(\frac{\partial U_+}{\partial\Delta(k)} + \frac{\partial U_-}{\partial\Delta(k)}\right)k_4\frac{\partial\Delta(k)}{\partial k_4}. \tag{17}$$

The factor involving $U_\pm$ is negative for all $(k_4, \vec{k}\,)$. In NJL models the factor $\partial\Delta/\partial k_4$ is zero, but in models with more realistic interactions, such as the Yukawa model, we expect that $k_4\partial\Delta/\partial k_4$ will be negative since the interaction between fermions effectively weakens at high energy or momentum, so the anomalous self energy should decrease at high energy. Thus we expect χ_δ to be positive, meaning that the charge imbalance grows with the relevant chemical potential μ_δ.

Our numerical results, presented below, confirm these expectations.

5. Numerical Results. To study examples of the result obtained above, we numerically solved the gap equation (9), including all energy and momentum dependence, by an iterative procedure. We discretized the function $\Delta(k_4, |\vec{k}|)$ on a grid in energy-momentum space with cutoffs Λ_{k4} in the range $\sim 10^6$ to 10^{10} MeV and $\Lambda_{\vec{k}}$ in the range $\sim 10^4$ to 10^6 MeV. Since the Yukawa model is renormalizable, we obtained cutoff-insensitive results as long as $(\mu, m) \ll \Lambda_{\vec{k}} \ll \Lambda_{k4}$. The typical $(k_4, \vec{k})$ grid sizes were (512×256). The grid points were generated by using a Gauss-Legendre quadrature rule. The points and weights were then remapped to achieve better resolution of regions where the integrand shows strong variation. The integration was performed on a graphics processing unit.

	m [MeV]	25	50	75
$\Delta_{\text{spectral}} \approx 50$ MeV	$g =$	4.787348	4.871645	4.969172
	$\chi_\delta =$	2472.1	2174.6	1919.7
$\Delta_{\text{spectral}} \approx 75$ MeV	$g =$	5.322668	5.388405	5.466438
	$\chi_\delta =$	3028.9	2791.1	2559.8
$\Delta_{\text{spectral}} \approx 100$ MeV	$g =$	5.726067	5.778862	5.841499
	$\chi_\delta =$	3470.5	3279.9	3077.2

TABLE I: The charge imbalance susceptibility $\chi_\delta = dN_\delta/d\mu_\delta|_{\mu_\delta=0}$ in MeV^2 for nine parameter sets, with different Yukawa boson masses and spectral gaps, evaluated at fermion chemical potential $\mu = 350$ MeV. We find $\chi_\delta > 0$ in all cases.

We studied nine different parameter sets, as shown in Table I. Each row shows a set of theories with different masses for the Yukawa boson, but with the coupling tuned to the value shown so that the spectral gap, which is a physically measurable quantity, has the same value in all three cases. To determine the spectral gap we increased μ_δ until we reached the spinodal point at which the condensate and anomalous self-energy vanish [6]. This value of μ_δ is the spectral gap. We find that the charge imbalance susceptibility is non-zero and positive, as our calculations above led us to expect.

In Fig. 1 we show the full energy-momentum dependence of the anomalous self-energy Δ for the Yukawa theory with $g = 4.871645$ and $m = 50$ MeV, evaluated at $\mu = 350$ MeV. This case shows the same qualitative properties that we see for all parameter values that we studied (Table I). The anomalous self energy is symmetric in k_4, and decreases monotonically with $|k_4|$, so $k_4 d\Delta/dk_4$ is always negative, implying via Eq. 17 that $\chi_\delta > 0$. At large k_4, $\Delta \propto 1/k_4^2$. At large 3-momentum $|\vec{k}|$ the self energy drops off roughly as $1/|\vec{k}|^{1.7}$, however the behavior is not monotonic for all $|\vec{k}|$. As $|\vec{k}|$ rises from zero the anomalous self energy grows until at reaches a maximum at $|\vec{k}| \approx \mu$, after which it decreases monotonically, see Fig. 2.

Fig. 3 shows the fractional charge imbalance $y_\delta = 2N_\delta/(N_1+N_2)$, as a function of the charge imbalance chemical potential μ_δ. In an NJL model, where the gap parameter is energy-independent and the Fermi surfaces are locked by pairing, it would remain exactly zero until μ_δ reaches $\Delta_{\text{spectral}}/\sqrt{2}$. This is the Chandrasekhar-Clogston limit [11, 12] where there is a first order transition from a paired state to the unpaired state, so N_δ jumps up to around the value expected for free fermions, $y_\delta{=}2\tilde{\mu}_\delta(3+\tilde{\mu}_\delta^2)/(1+3\tilde{\mu}_\delta^2)$ where $\tilde{\mu}_\delta = \mu_\delta/\mu$.

The solid curve with dots in Fig. 3 shows that in the Yukawa model N_δ does not remain zero, but rises slowly in response to μ_δ. The inset shows how our numerical

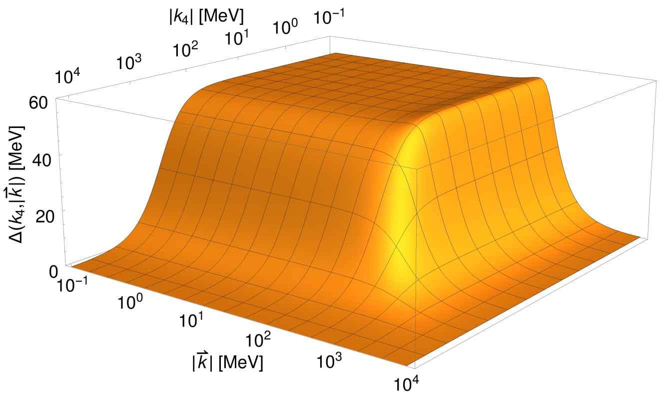

FIG. 1: The full energy and momentum dependence of the anomalous fermion self-energy in Yukawa theory with $m = 50$ MeV and $g = 4.872$.

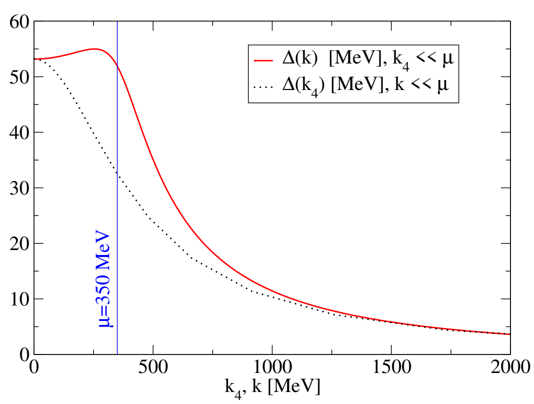

FIG. 2: Slices through the anomalous self-energy $\Delta(k_4, k)$ of Fig. 1, showing that as a function of energy Δ is monotonically decreasing (dotted curve), but as a function of momentum it rises to a maximum around $k = \mu$ and then decreases (solid curve).

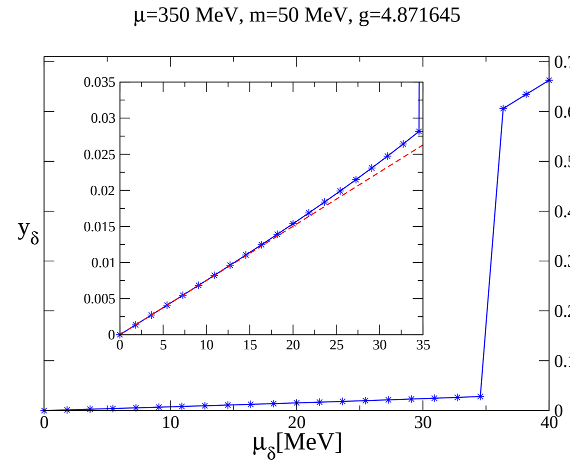

FIG. 3: Fractional difference y_δ in the number densities of the two species as a function of the imbalance potential μ_δ. The dashed line in the inset plot is the linear approximation $N_\delta = \chi_\delta \mu_\delta$ based on the charge imbalance susceptibility χ_δ (Eqs. 16,17).

calculation of N_δ using (15) agrees well with the extrapolation $N_\delta \approx \chi_\delta \mu_\delta$ (dashed red line).

6. Conclusions and Discussion. We have shown that, in a mean field calculation, Cooper pairing between two different species does not guarantee exact equality in the number densities of the two species. This marks a qualitative difference from earlier calculations using NJL models which found that the charge imbalance was zero (the Fermi surfaces were "locked together") as long as the system remained in the paired phase.

Our result confirms and clarifies the suggestive results of Ref. [7], which reported such an imbalance in a Schwinger-Dyson treatment of CFL quark matter. We studied a simpler system where we could clearly identify the energy dependence of the self energy as an essential factor in creating the charge imbalance. The integrand for the charge imbalance susceptibility contains a factor of $k_4 \partial\Delta/\partial k_4$ (Eq. 17). Since the anomalous self-energy Δ drops monotonically as a function of energy, this means that the susceptibility is generically non-zero. This implies that the charge imbalance itself is generically non-zero, although it can be tuned to zero by a specific choice of the chemical potential μ_δ that couples to the difference in number densities. We cannot rule out the possibility that our result is an artefact of the mean field approximation, but that would mean that the most widely used technique for analysis of Cooper pairing is not reliable for calculations of important properties such as susceptibilities.

Our result and that of Ref. [7] have potentially major implications for the phenomenology of quark matter in neutron stars. Previous mean field calculations in NJL models led to the prediction that the CFL phase of quark matter should be an electrical insulator because the pairing between the different quark flavors was thought to ensure that the quark matter contained equal numbers of all three flavors, so there are no electrons present in neutral CFL matter. If this prediction is incorrect, then the phenomenology of the CFL phase, and the signatures by which it might manifest itself in neutron star observations, will be significantly affected. A neutralizing population of electrons would, for example, dominate the specific heat and thermal conductivity and powerfully resist the motion of magnetic field lines [6].

If the charge imbalance responds to an arbitrarily small chemical potential μ_δ then there should be some mode based on the quark degrees of freedom that is massless and carries an $N_1 - N_2$ charge. Further investigation of this mode, along with the study of in-medium corrections to the Yukawa boson propagator, beyond-mean-field effects, an analogous investigation of fermions interacting via gluons, and the phenomenological consequences for quark matter, would be natural extensions of this work.

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