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Multi-parameter estimation in networked quantum sensors

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We introduce a general model for a network of quantum sensors, and we use this model to consider the question: When can entanglement between the sensors, and/or global measurements, enhance the precision with which the network can measure a set of unknown parameters? We rigorously answer this question by presenting precise theorems proving that for a broad class of problems there is, at most, a very limited intrinsic advantage to using entangled states or global measurements. Moreover, for many estimation problems separable states and local measurements are optimal, and can achieve the ultimate quantum limit on the estimation uncertainty. This immediately implies that there are broad conditions under which simultaneous estimation of multiple parameters cannot outperform individual, independent estimations. Our results apply to any situation in which spatially localized sensors are unitarily encoded with independent parameters, such as when estimating multiple linear or non-linear optical phase shifts in quantum imaging, or when mapping out the spatial profile of an unknown magnetic field. We conclude by showing that entangling the sensors *can* enhance the estimation precision when the parameters of interest are global properties of the entire network.

Quantum networks are central to a growing number of quantum information technologies, including quantum computation [1, 2] and cryptography [3, 4]. Many important metrology problems can be framed in terms of networks, including mapping magnetic fields [5–9], phase imaging [10–16] and global frequency standards [17]. However, there is no general consensus on whether entanglement within a network of sensors can enhance the precision to which the network can measure a set of unknown parameters: entanglement provides significant enhancements in some cases [17, 18] but not others [14, 19]. Given the immense challenges faced in the creation and manipulation of entangled states, developing a complete understanding of when such resources are advantageous for multi-parameter estimation is of paramount importance.

In this letter we introduce and analyze a general model that encompasses a wide range of those quantum multi-parameter estimation (MPE) problems that might naturally be termed a "quantum sensing network" (QSN). Our QSN model (Fig. 1) includes any situation in which spatially or temporally localized sensors are encoded with independent parameters. Hence, our results have direct implications for multi-mode linear [10–16] or non-linear [11] optical phase shift estimation for quantum imaging, mapping unknown spatially or temporally changing fields [5–9], estimating many-qubit Hamiltonians [18], and networks comprised of clocks [17], BECs [20], interferometers [14], or hybrid elements [21]. Beyond these examples, any situation in which independent parameters are unitarily imprinted on different quantum subsystems fits into our model.

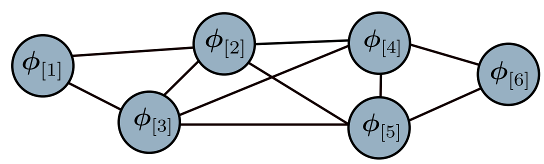

FIG. 1. A network of quantum sensors. The k^{th} node represents a "sensor" into which the vector parameter $\boldsymbol{\phi}_{[k]}$ is encoded via a local unitary evolution. The connections between the nodes denote that, in general, the sensors can be entangled, and/or global measurements can be performed.

Using our model we show that, if the generators of all of the unknown parameters commute, no fundamental precision enhancement can be achieved by entangling the sensors or by performing global measurements. In this case, states that are separable between the sensors – which are often easier to prepare experimentally – can achieve the ultimate quantum limit. We then look at the case of non-commuting parameter generators; here we demonstrate that entanglement between sensors can *at most* enhance the estimation precision by a factor of two. We conclude by showing that entangling the sensors *can* significantly enhance the precision when estimating global parameters, such as the average of all the unknown parameters in the network [17].

Whenever a protocol employs entangled resources it is fundamentally indivisible into separate, independent estimations at each location: it is intrinsically a *simultaneous* [9–16, 22] estimation method. As such, our results directly imply that there are broad conditions under which simultaneous estimation cannot outperform a strategy that estimates each parameter individually, conclusively proving that enhancements from simultaneous estimation [9–13, 16, 22] are not generic.

Multi-parameter estimation (MPE) – Consider a quantum system with Hilbert space $\mathcal{H}$, and let $\mathscr{D}(\mathcal{H})$ and $\mathscr{M}(\mathcal{H})$ denote the space of density operators and positive-operator valued measures (POVMs) on $\mathcal{H}$, respectively. We will use the standard framework for a quantum metrology protocol [23–25]: An experimenter picks some $\rho \in \mathscr{D}(\mathcal{H})$ and $\mathcal{M} \in \mathscr{M}(\mathcal{H})$ and implements μ repeats of: i) prepare ρ; ii) let ρ evolve to $\rho_{\boldsymbol{\phi}} = U_{\boldsymbol{\phi}} \rho U_{\boldsymbol{\phi}}^{\dagger}$ where $U_{\boldsymbol{\phi}}$ is a unitary that depends on d unknown parameters $\boldsymbol{\phi} = (\phi_1, \phi_2, \dots, \phi_d)^T$; iii) apply the measurement $\mathcal{M}$ to $\rho_{\boldsymbol{\phi}}$. An estimate of $\boldsymbol{\phi}$ is then calculated from experimental outcomes using an estimator $\boldsymbol{\Phi}$.

A common measure of the estimation uncertainty is the covariance matrix $\mathrm{Cov}(\boldsymbol{\Phi}) = \mathbb{E}[(\boldsymbol{\Phi} - \mathbb{E}[\boldsymbol{\Phi}])\,(\boldsymbol{\Phi} - \mathbb{E}[\boldsymbol{\Phi}])^T]$, where $\mathbb{E}[\cdot]$ is the expected value. For any unbiased estimator, the *quantum Cramér-Rao bound* (QCRB) states that $\mathrm{Cov}(\boldsymbol{\Phi}) \geq (\mathcal{F}\mu)^{-1}$ [26–30], where $\mathcal{F}$ is the *quantum Fisher information matrix* (QFIM) for $\rho_{\boldsymbol{\phi}}$, defined by $\mathcal{F}_{kl} := \mathrm{Tr}[\rho_{\boldsymbol{\phi}} \hat{L}_k \hat{L}_l + \rho_{\boldsymbol{\phi}} \hat{L}_l \hat{L}_k]/2$ with $\hat{L}_k$ solving $\partial \rho_{\boldsymbol{\phi}} / \partial \phi_k = (\rho_{\boldsymbol{\phi}} \hat{L}_k + \hat{L}_k \rho_{\boldsymbol{\phi}})/2$ [26–30]. Note that for matrices A and B, $A \geq B$ denotes that $A - B$ is positive semi-definite. For $d = 1$ and any $\rho_{\boldsymbol{\phi}}$ there is always a measurement and an estimator that saturate the QCRB as $\mu \to \infty$ [29, 31, 32], but for $d > 1$ this is not generally true [26, 28, 33–37]. Some elements of $\boldsymbol{\phi}$ may be of more interest than others, so we introduce a $d \times d$ diagonal *weighting matrix*, W, with $W \geq 0$, and define the scalar quantity $E_{\boldsymbol{\Phi}} := \mathrm{Tr}(W \mathrm{Cov}(\boldsymbol{\Phi}))$ [26, 38, 39]. Throughout this letter, $E_{\boldsymbol{\Phi}}$ is the figure of merit to minimize. The QCRB implies that $E_{\boldsymbol{\Phi}} \geq \frac{1}{\mu} \sum_k W_{kk} [\mathcal{F}^{-1}]_{kk}$.

Quantum sensing networks – In this letter we consider a particular class of quantum MPE problems: *quantum sensing networks* (QSNs). A QSN is, by definition, any estimation problem in which we have s quantum systems, which we will call "quantum sensors", and there are d unknown parameters with each parameter unitarily encoded into *one and only one* of the sensors. It is natural to refer to this model as a QSN because any set of spatially distributed quantum systems that are each "sensing" some locally unitarily encoded parameters is a QSN (although some systems without this spatial structure also fit into this framework).

Our model, illustrated in Fig. 1, encompasses many metrology problems in the literature [5–18, 20, 21] (see examples later). More formally, a QSN is any MPE problem in which the total Hilbert space $\mathcal{H}$ may be decomposed as $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_s$ for some $\{\mathcal{H}_k\}$, and the unitary evolution may be decomposed as $U_{\boldsymbol{\phi}} = U_1(\boldsymbol{\phi}_{[1]}) \otimes U_2(\boldsymbol{\phi}_{[2]}) \otimes \cdots \otimes U_s(\boldsymbol{\phi}_{[s]})$, where $\boldsymbol{\phi}_{[k]}$ denotes the d_k-dimensional sub-vector of $\boldsymbol{\phi}$ encoded onto the k^{th} sensor by the unitary U_k, with $\sum_k d_k = d$. Let $\boldsymbol{\phi}_{[1]} = (\phi_1, \dots, \phi_{d_1})^T$, $\boldsymbol{\phi}_{[2]} = (\phi_{d_1+1}, \dots, \phi_{d_1+d_2})^T$, etc.

Often we wish to compare probe states ρ that contain the same quantity of "resources" $R(\rho)$, for some $R : \mathscr{D}(\mathcal{H}) \to \mathbb{R}_{\geq 0}$. There is no universally applicable definition for the resources within a state; we will consider functions of the form $R(\rho) = \mathrm{Tr}[(\hat{R}_1 + \hat{R}_2 + \cdots + \hat{R}_s)(\rho)]$, where $\hat{R}_k$ is any Hermitian operator acting non-trivially only on sensor k and satisfying $R(\rho_{\boldsymbol{\phi}}) = R(\rho)$ (so resources are conserved under the evolution). This includes the resource counting in most standard metrology problems. E.g., in optical metrology with s modes the total average number of photons is the standard resource [10–15], given by $\hat{R}_k = \hat{n}_k$, where $\hat{n}_k$ is the number operator on mode k (which commutes with the standard parameter generator, $\hat{n}_k$). In atomic sensing, the resource is normally the total number of atoms [40–43]. This is obtained by taking the Hilbert space of each sensor to be the direct sum of the n-atoms Hilbert space for $n = 0, 1, 2, \dots$, and $\hat{R}_k$ to be the atom-counting operator, which commutes with all atom-number conserving Hamiltonians.

QSNs with commuting parameter generators – The *generator* of ϕ_k is defined by $\hat{H}_k := -i(\partial U_{\boldsymbol{\phi}}^{\dagger} / \partial_{\phi_k}) U_{\boldsymbol{\phi}}$ [44, 45]. Our main results are separated into two cases: when the generators all commute, and when they do not. First, consider any QSN in which the generators all commute. Informally, our first result is that for *any* such estimation problem sensor-separable states can enable an estimation uncertainty that is at least as small as can be achieved with sensor-entangled states. This also implies that, in this setting, simultaneous estimation provides no intrinsic advantage over individual estimation; the latter can achieve the ultimate quantum limit. We now state this precisely:

Theorem 1. *Consider any QSN in which* $[\hat{H}_k, \hat{H}_l] = 0$ *for all* k, l *and where we wish to minimize* $E_{\boldsymbol{\Phi}}$ *where* $E_{\boldsymbol{\Phi}} = \mathrm{Tr}(\mathrm{W}\mathrm{Cov}(\boldsymbol{\Phi}))$ *for some specified* W*. For any estimator, probe* ρ *and measurement* $\mathcal{M}_\rho$*, there exists an estimator, a probe* φ *and a measurement* $\mathcal{M}_\varphi$ *for which*

1. φ *is separable between sensors.*
2. $R(\varphi) \leq R(\rho)$.
3. $\mathcal{M}_\varphi$ *is implementable by independent measurements of each sensor.*
4. $E_{\boldsymbol{\Phi}}(\varphi, \mathcal{M}_\varphi) \leq E_{\boldsymbol{\Phi}}(\rho, \mathcal{M}_\rho)$ *in the asymptotic* μ *limit.*

Proof. This may be proven by constructing such a φ and $\mathcal{M}_\varphi$, for arbitrary ρ and $\mathcal{M}_\rho$. First consider pure ρ, i.e., $\rho = \psi = |\psi\rangle\langle\psi|$. We now find a mapping from ψ to a state φ that satisfies conditions 1 and 2, and that has an equal or smaller QCRB on $E_{\boldsymbol{\Phi}}$. Consider the state $|\varphi\rangle = \bigotimes_{k=1}^{s} (\sum_{\lambda_k} \| \langle \psi | \lambda_k \rangle \| |\lambda_k\rangle)$, where $\{|\lambda_k\rangle\}$ is a set of orthonormal mutual eigenstates of the generators for all of the parameters encoded into sensor k. By construction, ψ and φ have the same statistics for any operator that is diagonal in the eigenbasis of the generators, and

φ is separable between sensors. As the resource operator commutes with $U_{\boldsymbol{\phi}}$, it commutes with the parameter generators, implying φ satisfies conditions 1 and 2.

For a pure state and commuting generators $\mathcal{F}_{kl} = 4(\langle \hat{H}_k \hat{H}_l \rangle - \langle \hat{H}_k \rangle \langle \hat{H}_l \rangle)$ [9, 12, 14]. Using this we find that ψ and φ have the same block-diagonal QFIM elements, where the block diagonals are the sub-QFIMs for each $\boldsymbol{\phi}_{[k]}$, denoted $\mathcal{F}_{[kk]}$, and φ has a block-diagonal QFIM (ψ in general does not). Now for any QFIM $[\mathcal{F}^{-1}]_{[kk]} \geq [\mathcal{F}_{[kk]}]^{-1}$, with saturation only for a block-diagonal QFIM (see the Supplemental Material [46]), and hence the diagonal elements of the inverse QFIM of φ are all smaller than or equal to those of ψ. Using $E_{\boldsymbol{\Phi}} \geq \frac{1}{\mu} \sum_l W_{ll} [\mathcal{F}^{-1}]_{ll}$, and noting that when the generators commute there always exists a measurement and estimator that asymptotically saturate the QCRB [27], we see that condition 4 is satisfied by some measurement and estimator. It only remains to show that for one such measurement condition 3 holds, and for every mixed state ρ, there exists a pure state with equal or lower $E_{\boldsymbol{\Phi}}$ and the same resources. We prove this in the Supplemental Material [46]. □

Theorem 1 has practical implications for a range of important estimation problems. For example, consider estimating a set of d optical phases encoded into d modes (defined with respect to a classical phase reference [47]). Theorem 1 implies that, for any mode-entangled state and measurement, there is a mode-separable state and measurement (acting on only that mode and a local phase reference) that provides an equal or lower estimation uncertainty, for the same average number of photons through the d phase shifts. So, although highly mode-entangled states can provide high estimation precision [10–12, 16], this entanglement is not necessary. This supersedes the results of Ref. [14], which apply only to mode-symmetric states.

Importantly, Theorem 1 is only directly applicable when the set of states, from which we wish to find the best ρ, is the set of all density operators on $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_s$. Hence, if we restrict the allowed ρ to $\mathbb{S} \subset \mathscr{D}(\mathcal{H})$, Theorem 1 is only applicable if $\mathbb{S}$ contains all ρ on some smaller Hilbert space $\mathcal{H}'$ that still factorizes. This is not the case for some global constraints on the state. This reconciles our theorem with Humphreys *et al.* [10], who show that highly-entangled "generalized NOON states" provide a precision enhancement over individual estimation strategies, for the d-optical-phases problem, when only states with definite total photon number are considered.

Interestingly, Theorem 1 may be extended to further classes of $\mathbb{S}$. This includes any $\mathbb{S}$ containing pure states whereby every state in $\mathbb{S}$ can be mapped to a sensor-separable state in $\mathbb{S}$ with the same measurement statistics for operators diagonal in the eigenbasis of the generators (the proof is a trivial adaption of that given above). This implies that, if considering only Gaussian optical states in the d-phases problem, entanglement cannot reduce the estimation uncertainty. As such, our theorem strengthens and complements the results of Ref. [15].

Theorem 1 may also be applied to other important metrology scenarios: It implies that the estimation of non-linear optical phase shifts on many modes [11] does not benefit from mode-entanglement, and in a network of clocks [17], if each clock is used for local timekeeping then entangling the clocks will not enhance the precision. A magnetic field sensing problem is considered later.

QSNs with non-commuting parameter generators – There are a variety of important estimation problems for which the generators do *not* commute [9, 48, 49], such as estimating the three spatial components of a magnetic field [9], or estimating completely unknown unitaries [49]. We now adapt Theorem 1 to the case of non-commuting parameter generators.

Consider an arbitrary QSN with some non-commuting parameter generators. In our model, the generators of parameters imprinted on different sensors always commute, so only the generators of parameters encoded into the same sensor can be non-commuting. When estimating parameters with non-commuting generators, it is known that the optimal estimation protocol will generally require a probe that is entangled with an ancilla [48, 49]. In a QSN, other sensors in the network can potentially play a similar role to ancillas, and so sensor-entanglement might reduce estimation uncertainty. However, any enhancement in the estimation precision gained from entanglement between sensors can instead be obtained by entangling each sensor with a local ancilla. The cost of this is that resources can be consumed by the ancillary system; twice the resources might be required to obtain the same estimation precision without sensor-entanglement. We can state this precisely in the following theorem:

Theorem 2. *Consider any QSN in which we wish to minimize* $E_{\boldsymbol{\Phi}}$*. For any estimator, probe state* $\rho \in \mathscr{D}(\mathcal{H})$ *and measurement* $\mathcal{M}_\rho \in \mathscr{M}(\mathcal{H})$*, there exists an estimator, probe* $\varphi \in \mathscr{D}(\mathcal{H} \otimes \mathcal{H})$ *and measurement* $\mathcal{M}_\varphi \in \mathscr{M}(\mathcal{H} \otimes \mathcal{H})$ *for which*

1. φ *is separable between sensors, but each sensors can be entangled with a local ancilla.*
2. $R(\varphi) \leq 2R(\rho)$.
3. $\mathcal{M}_\varphi$ *is implementable by independent measurements of each sensor.*
4. $E_{\boldsymbol{\Phi}}(\varphi, \mathcal{M}_\varphi) \leq E_{\boldsymbol{\Phi}}(\rho, \mathcal{M}_\rho)$ *in the asymptotic* μ *limit.*

A complete proof is provided in the Supplemental Material [46] (it closely follows the proof of Theorem 1). Note that condition 2 in this theorem depends on how resources used in ancillary sensors are counted, and here we have counted resources in the ancillas and sensors equally. If ancillas are considered cost-free then condition 2 improves to $R(\varphi) \leq R(\rho)$. Whether entanglement with a local ancilla is practically plausible is application dependent. Theorem 2 can be applied to a range of practical QSN problems. For example, if we wish to characterize a multi-dimensional field at multiple locations, then entanglement between atomic sensors at these locations can provide no improvement in precision compared to entangling these atoms with some local ancillary system (which may contribute to total resources used). This complements the results of Ref. [9], which provides strategies for single-site estimation of multi-dimensional fields.

Estimating global functions of $\boldsymbol{\phi}$ – In some sensing problems it may not be necessary to estimate $\boldsymbol{\phi}$. Instead, the parameter(s) of interest could be some function(s) of $\boldsymbol{\phi}$, e.g., $\sum_k \phi_k$. In this case, the aim is to optimize the QSN for estimating these functions, and this encompasses many important problems, including measuring: phase differences in one [50] or more [14] interferometers; the average or sum of many parameters [17]; a linear gradient [51, 52]. A *global* property of the network is some vector (or scalar) with elements that are functions of $\{\phi_k\}$ depending non-trivially on many or all of the ϕ_k, which includes the examples given above. We now show that the optimal protocol for estimating global properties of a QSN often requires sensor-entangled states.

For simplicity, we consider estimating a single linear function of $\boldsymbol{\phi}$; $\theta = \boldsymbol{v}^T\boldsymbol{\phi}$ for some $\boldsymbol{v} \in \mathbb{R}^d$. To fix arbitrary constants, let $\|\boldsymbol{v}\|_2 = 1$ and $v_k \geq 0$ $\forall k$ ($\|\boldsymbol{v}\|_p := [\sum_k |v_k|^p]^{1/p}$). Moreover, consider a QSN consisting of $\leq N$ particles (e.g., atoms or photons) distributed over d sensors, with ϕ_k encoded into sensor k. We take the parameter generators to all be identical (except that they act on different sensors), with the maximal and minimal eigenvalues of the generator for $\leq n$ particles in a sensor, $\lambda_{\max,n}$ and $\lambda_{\min,n}$, satisfying $\lambda_{\max,n} - \lambda_{\min,n} = \kappa n$ for some constant $\kappa > 0$. Denote corresponding orthonormal eigenvectors by $|\lambda_{\max,n}\rangle$ and $|\lambda_{\min,n}\rangle$. Examples that fit into this setting include estimating a function of many linear optical phase shifts, or of a spatially varying 1-dimensional magnetic field with multi-level atoms, or qubits [18].

Although we only wish to estimate θ, there are many unknown parameters. Hence, to bound $\mathrm{Var}(\Theta) = \mathbb{E}[\Theta^2] - \mathbb{E}[\Theta]^2$ (Θ is the estimate of θ) requires the QCRB on $\boldsymbol{\theta} = (\theta, \theta_2, \dots)^T = M\boldsymbol{\phi}$ for some matrix M with $(M\boldsymbol{\phi})_1 = \theta$. We may take M to be orthogonal, as only the first row of M is specified by the problem. The relevant QFIM is then $\mathcal{F}(\boldsymbol{\theta}) = M\mathcal{F}(\boldsymbol{\phi})M^T$ [29].

The optimal n-particle state of sensor k for estimating ϕ_k is $\propto |\lambda_{\min,n}\rangle + |\lambda_{\max,n}\rangle$, so the optimal N-particle QSN sensor-separable state for estimating θ is $\propto (|\lambda_{\min,w_k}\rangle + |\lambda_{\max,w_k}\rangle)^{\otimes d}$ optimized over $\boldsymbol{w} \in \mathbb{N}^d$ with $\|\boldsymbol{w}\|_1 = N$. By calculating the QFIM of this $\boldsymbol{w}$-optimized state, for any pure and sensor-separable state we have $\mathrm{Var}(\Theta) \geq \|\boldsymbol{v}\|_{2/3}^2/(\mu\kappa^2 N^2) \geq \|\boldsymbol{v}\|_1^3/(\mu\kappa^2N^2)$, where μ is the number of experimental repeats. Now, assuming that $v_k/\|\boldsymbol{v}\|_1$ is rational and that N is such that $\tilde{v}_k \equiv N v_k/\|\boldsymbol{v}\|_1$ is an integer $\forall k$, consider the sensor-entangled GHZ-like state

$$|\psi_{\text{GHZ},\boldsymbol{v}}\rangle = \frac{1}{\sqrt{2}}\left(|\lambda_{\max,\tilde{v}_k}\rangle^{\otimes d} + |\lambda_{\min,\tilde{v}_k}\rangle^{\otimes d}\right). \quad (1)$$

The QFIM for this state is $\mathcal{F}(\boldsymbol{\phi}) = \kappa^2N^2\boldsymbol{v}\boldsymbol{v}^T/\|\boldsymbol{v}\|_1^2$, and hence $\mathcal{F}(\boldsymbol{\theta})_{11} = \kappa^2N^2/\|\boldsymbol{v}\|_1^2$ with all other matrix elements zero. This QFIM is singular, but the state depends on θ, so the saturable QCRB for this state is given by $\mathrm{Var}(\Theta) \geq 1/(\mu\mathcal{F}(\boldsymbol{\theta})_{11}) = \|\boldsymbol{v}\|_1^2/(\mu\kappa^2N^2)$.

As $\|\boldsymbol{v}\|_2 = 1$, for *all* non-trivial $\boldsymbol{v}$ (i.e., $\boldsymbol{v}$ with multiple non-zero elements) $\|\boldsymbol{v}\|_1 > 1$. Hence, for all such $\boldsymbol{v}$ entanglement between sensors reduces the estimation uncertainty below what is obtainable with *any* sensor-separable state. Moreover, $\|\boldsymbol{v}\|_1$ is maximal when $\boldsymbol{v} \propto (1, 1, \dots, 1)$, and so the precision enhancement is largest when estimating the average or sum of all d parameters. In this setting, the reduction in the estimation variance is a factor of $1/d$ (as then $\|\boldsymbol{v}\|_1^2/\|\boldsymbol{v}\|_{2/3}^2 = 1/d$).

To illustrate these results, we now apply them to a simple – but practically relevant – example: estimating the difference between the magnetic field strength at two locations with N qubits (i.e., gradient estimation). Consider estimating $\theta = (\phi_2 - \phi_1)/\sqrt{2}$ with ϕ_k for $k = 1, 2$ generated by $\hat{J}_{z,k} = \frac{1}{2}\sum_j \sigma_{z,k,j}$ on sensor k, which consists of n_k qubits for $n_1 + n_2 = N$, where $\sigma_{z,k,j}$ is the σ_z operator on qubit j in sensor k. Our results imply that a global GHZ-like state $\propto |{\downarrow}\rangle^{n_1}|{\uparrow}\rangle^{n_2} + |{\uparrow}\rangle^{n_1}|{\downarrow}\rangle^{n_2}$ with $n_1 = n_2 = N/2$ has an uncertainty reduction of $1/2$ compared to any sensor-separable state. However, if we instead wish to estimate ϕ_1 and ϕ_2 (or $\phi_2 - \phi_1$ and $\phi_2 + \phi_1$), then the above state is not appropriate, as it is sensitive only to $\phi_2 - \phi_1$. In this case, Theorem 1 implies that the optimal probe state is separable between the atoms at the two sites (the optimal state is then a local GHZ-like state at each site). Importantly, note that these conclusions do not necessarily hold if ϕ_1 and ϕ_2 have a known dependence: the extreme case is when we know that $\phi_1 = \phi_2$, in which case estimating $\phi \equiv \phi_1 = \phi_2$ is a well-known one-parameter problem, and a global GHZ is optimal [41, 53]. This example can be directly adapted to l-level atoms, > 2 sensors, and more general linear functions.

Recently, Ge *et al.* [54] have applied our results to the estimation of a function of d linear phase shifts, and they have shown how to obtain the $O(d)$ precision enhancement, derived above, by entangling photons using a linear optical network. These interesting results show that the $O(d)$ enhancement proven here is potentially obtainable with current technology.

Conclusions: Quantum metrology is a powerful emerging technology, but while many practical problems unavoidably involve more than one unknown parameter, the critical resources for obtaining the ultimate quantum limit in multi-parameter estimation (MPE) are not yet well-understood. In this setting, simultaneous estimation, entanglement between sensors, and global measurements are possible avenues for improving estimation precision that are not relevant in the single-parameter scenario [9–13, 16, 22].

In this letter we considered a broad class of practically important MPE problems: *quantum sensing networks*, meaning any setting in which the unknowns parameters can be sub-divided into distinct sets each associated with one spatially or temporally localized sensor. We have presented a general model for such estimation problems, and we stated precise theorems that show that simultaneous estimation, entanglement between sensors, and global measurements are broadly *not* fundamentally useful resources for minimizing estimation uncertainty in this setting. The important exception to this is when one or more *global* properties of the network are the parameters of interest, e.g., if only the average of all the parameters is to be estimated. In this case we have shown that entangled states and measurements can, in general, improve estimation precision. In doing so, we have shown that GHZ-like states have a particularly high precision for estimating generic linear functions in a practically relevant class of QSNs, including in optical and atomic sensing networks.

These results provide a rigorous foundation for understanding the role of entanglement and simultaneous estimation in optimal MPE, and they definitively show that these resources are not critical in a broad class of important problems. We anticipate that this letter will prove helpful for guiding the development of sensing technologies for multi-parameter metrology in fields as diverse as optical imaging [10–16], field mapping with atoms [5–9], and sensor networks comprised of BECs [20], clocks [17], or interferometers [14]. Moreover, recently these results been applied to the interesting problem of estimating functions of linear optical phases [54].

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