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Logarithmically slow relaxation in quasi-periodically driven random spin chains

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We simulate the dynamics of a disordered interacting spin-chain subject to a quasi-periodic time-dependent drive, corresponding to a stroboscopic Fibonacci sequence of two distinct Hamiltonians. Exploiting the recursive drive structure, we can efficiently simulate exponentially long times. After an initial transient, the system exhibits a long-lived glassy regime characterized by a logarithmically slow growth of entanglement and decay of correlations analogous to the dynamics at the many-body delocalization transition. Ultimately, at long time-scales, which diverge exponentially for weak or rapid drives, the system thermalizes to infinite temperature. The slow relaxation enables metastable dynamical phases, exemplified by a "time quasi-crystal" in which spins exhibit persistent oscillations with a distinct quasi-periodic pattern from that of the drive. We show that in contrast with Floquet systems, a high-frequency expansion strictly breaks down above fourth order, and fails to produce an effective static Hamiltonian that would capture the pre-thermal glassy relaxation.

Introduction – Interacting quantum many-body systems often exhibit chaotic dynamics that rapidly scramble quantum information and lead to highly entangled states whose local properties are thermal and classical [1, 2]. A dramatic exception occurs in isolated and disordered systems where many-body localization (MBL) can arrest thermalization, resulting in quantum coherent dynamics at arbitrarily high energy density [3–5]. This dichotomy naturally raises fundamental questions about when and how a system thermalizes. What are the universal features governing the dynamical approach to the final – thermal or non-thermal – state? More practically, what classes of protocols allow one to manipulate a many-body system without rapidly scrambling its stored quantum information?

Given their large bandwidth and dense spectrum, one might naively expect that any persistent dynamical manipulation of an isolated, interacting quantum many-body system leads to runaway heating to a featureless infinite-temperature state. Indeed, random time-dependent manipulations have recently been shown to cause rapid growth of entanglement, accompanied by universal hydrodynamic features [6–8]. However, this expectation is violated in time-periodically driven (Floquet) systems with strong disorder, in which sufficiently rapid driving maintains MBL and indefinitely avoids heating [9–11]. Even in the absence of disorder, rapid periodic driving leads to long-lived pre-thermal phenomena [12–21]. Floquet-MBL systems have been shown to exhibit remarkable dynamic phenomena from spontaneous time-translation symmetry breaking [22–27] and dynamical topological phases with no equilibrium analog [22, 28–37].

The stark contrast between the behaviors under random and periodic driving can be understood by a simple argument: local time-dependent Hamiltonians can only make local re-arrangements. In strongly disordered systems, such rearrangements have a non-zero energy cost and are generically non-resonant with harmonics of the driving frequency. This heuristic forms the basis for more sophisticated considerations for the stability of Floquet-MBL systems [11], which are supported by numerical simulations [9, 10], and cold-atom experiments [38]. Using similar arguments, one can rule out the stability of MBL to random time-dependent drives, which have continuous frequency spectra capable of resonantly inducing arbitrary local transitions leading to thermalization.

In this paper, we consider an intermediate case between periodic and random driving by subjecting a strongly disordered quantum many-body system to a drive with quasi-periodic time-dependence. The quasi-periodic drive has a dense, but sharply discontinuous frequency spectrum that occupies a set of measure zero. A priori, it is not clear whether the density of spectral content will drive heating and thermalization or whether its sparsity will preserve MBL. We find that quasi-periodic driving does eventually lead to thermalization to a featureless infinite temperature state, but only after a long time $t_{\rm th}$ that grows exponentially in the inverse driving strength and the rate of driving. While reminiscent of pre-thermalization in delocalized Floquet systems [12–16], the dynamics before $t_{\rm th}$ are not described by an effective finite temperature equilibrium. Instead, this regime shows a logarithmically slow relaxation of correlations and growth of entanglement, which we will call glassy dynamics. This glassy behavior is analogous to the critical dynamics at the transition between MBL and thermal systems in non-driven settings [40–42]. We explore to what extent the quasi-periodic evolution can be reduced to an effective static Hamiltonian, connecting our study to the question of reducibility of differential equations with quasi-periodic coefficients [43, 44]. The glassy relaxation regime can host new metastable dynamical phases, which we illus-

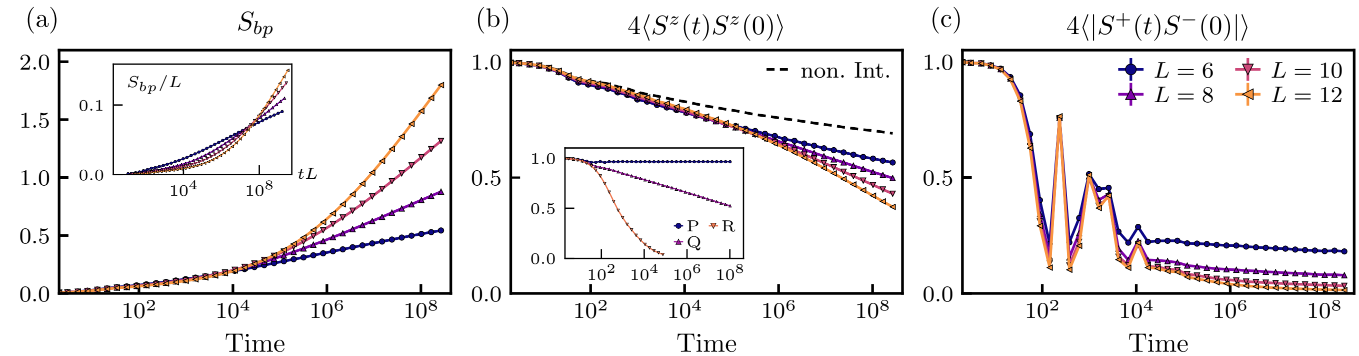

FIG. 1. **Quasi-periodically Driven Spin Chain.** – Time evolution under the quasi-periodic driving sequence, with $J_0 = 0, \delta J = \pi/30, \lambda = 1$ and varying L (markers defined in (c)). All quantities are averaged over states in the global $S^z = 0$ sector of the spin chain and averaged over at least 3000 disorder realizations. (a) The bi-partite entanglement $S_{bp}(t)$. Inset: The normalized entanglement S_{bp}/L. (b) Onsite correlation function $C^{zz}(t)$ on site $i = L/2$. This plot additionally shows (dashed line) the case of driving an $L = 150$ chain in the non-interacting limit of (2); see [39]. Inset: Comparison of driving with periodic (P), quasi-periodic (Q) and random (R) sequences of the elementary unitaries, with $L = 8$. The random case is averaged over 20 different random sequences, each with 100 disorder realizations. (c) Correlation function $C^{+-}(t)$ on site $i = L/2$.

trate with a quasi-periodic analog of time-translation symmetry breaking – a "time quasi-crystal".

Model – To address the fate of a quantum many-body system under quasi-periodic driving, we numerically simulate spin-1/2 chains, subjected to a stroboscopic drive consisting of a Fibonacci sequence of unitary evolutions:

$$U_n = U_{n-2}U_{n-1}, \tag{1}$$

for $n \geq 2$. The sequence is initialized by two elementary unitaries formed from two different static Hamiltonian evolutions: $U_0 = \exp(-i\lambda H_+)$ and $U_1 = \exp(-i\lambda H_-)$, where

$$H_\pm = \sum_{i=1}^{L} h_i S_i^z + \sum_{i=1}^{L-1} (J_0 \pm \delta J)\, \mathbf{S}_i \cdot \mathbf{S}_{i+1}. \tag{2}$$

The h_i are random fields drawn independently for each site from a uniform distribution $h \in [-2\pi, 2\pi)$, J_0 is a static interaction, δJ represents the strength of the quasi-periodic driving and $\lambda \in [0, 1]$ is the characteristic driving time-scale. We will focus on the regime $|J_0 \pm \delta J| \lesssim 1.7$, where $H_\pm$ as static Hamiltonians would be MBL [45]. As such, they are separately described by emergent local integrals of motion (LIOM) with definite S^z value [46]. Unless otherwise noted, we will take $J_0 = 0$. An appealing feature of the recursive nature of the drive is that it enables simulation of exponentially long Fibonacci times $t_n = F_{n+1} \sim \varphi^{n+1}$ with only n unitary multiplications; here $\varphi = (1 + \sqrt{5})/2$ is the golden ratio. This enables us to simulate the long-time physics, limited only by machine precision.

Results – We focus on three observables: the z-component of spin $C^{zz}(t) = 4\langle S_i^z(t)S_i^z(0)\rangle$, whose total value is conserved by the evolution, and whose local dynamics are related to spin-transport, the transverse spin-fluctuations $C^{+-}(t) = 4\langle|S_i^+(t)S_i^-(0)|\rangle$, which encodes the dephasing of quantum superpositions of up and down spins, and the bi-partite (half-system) entanglement entropy $S_{\mathrm{bp}}(t)$.

Before discussing the results, we summarize the behavior of these quantities in static MBL, periodically driven (Floquet) MBL, and thermalizing systems. In a static or Floquet-MBL system, $C^{zz}(t)$ tends to a non-zero constant at long times, indicating the absence of spin-transport and emergent conservation laws that produce infinite memory of the initial spin configuration [9, 10, 46, 47]. The transverse fluctuations, $C^{+-}(t)$ decay as a power law in time from dephasing due to classical interactions among the local conserved quantities [48]. This dephasing also produces a logarithmically slow growth of entanglement $S_{\mathrm{bp}}(t) \sim \log t$ [49–51]. On the other hand, in strongly thermal or randomly driven systems, the non-zero spin conductivity and chaotic scrambling leads to an exponential decay of correlation functions $C_{zz}, C_{+-} \sim e^{-t/t_{\mathrm{th}}}$ and a linear growth in $S_{\mathrm{bp}}(t) \sim t$ [52, 53]. Finally, a clean delocalized system subject to rapid periodic driving exhibits a pre-thermalization regime, in which the system initially equilibrates with respect to an effective Hamiltonian at finite temperature. Pre-thermalization persist up to a time exponentially long in the driving frequency [12–15], after which the system heats to a featureless infinite temperature state.

Figure 1 shows C^{zz}, C^{+-}, and S_{bp} for quasi-periodic driving, in a quench from an initial product state. These observables are averaged over initial states and disorder realizations. We observe three distinct regimes: First, there is a short-time transient regime in which there is no distinction between periodic, quasi-periodic and random driving (Fig. 1b inset). Next, there is a long-lived glassy relaxation regime where S_{bp} grows and C^{zz} decays logarithmically slowly. Finally, after a time-scale t_{th} that is exponentially long for weak or rapid driving, the system ultimately heats up to infinite temperature with a non-

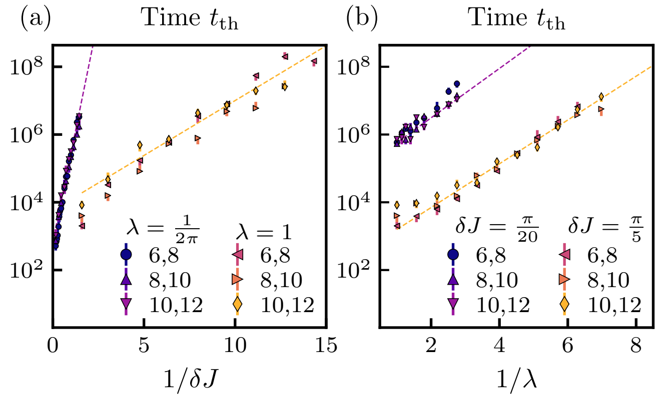

FIG. 2. **Thermalization time t_{th}.** – Thermalization time extracted from the crossing of S_{bp}/L, between pairs of L $(6,8),(8,10),(10,12)$. (a) As a function of $1/\delta J$ for $\lambda = 1/2\pi, 1$ and (b) as a function of λ for $\delta J = \pi/20, \pi/5$. Error-bars are linear estimates in Fibonacci time; dashed lines are fits of form $\log t_n \sim 1/\lambda, 1/\delta J$ to the $(10,12)$ crossing.

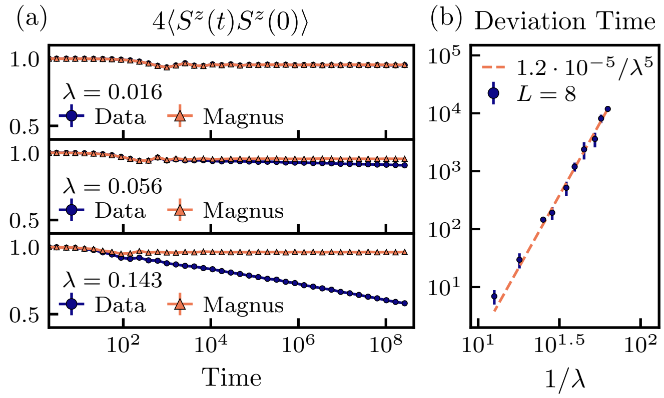

FIG. 3. **Magnus Expansion.** – (a) Onsite correlation function $C^{zz}(t)$ for $L = 8, J_0 = 0, \delta J = \pi/5$ and different λ compared to that obtained by the Magnus expansion Hamiltonian at third-order. (b) Initial time at which $C^{zz}(t)$ of the Magnus expansion deviates by more than 10^{-4} from the data.

zero rate, signaled by linear growth of entanglement and rapid decay of correlations. Ultimately, S_{bp} will saturate to its thermal value and C^{zz}, C^{+-} decay to zero.

The behavior of this quasi-periodic system is markedly distinct from the other scenarios mentioned above, as contrasted in the inset of Fig. 1b. Similar to and MBL system, C^{+-} shows aperiodic oscillations that decay slowly. Unlike an MBL system, however, C^{zz} does not saturate to a non-zero value. Taken together, these imply that the glassy relaxation regime does not possess LIOM. Nonetheless, it does not exhibit the rapid decay characteristic of a thermal system.

There are two ways we can identify the thermalization time t_{th}: as the time where C^{zz} curves of different L separate from each other after the logarithmic decay or as the time where the normalized entanglement S_{bp}/L cross at a single point as a function of Lt (Fig. 1a inset). These two ways of extracting t_{th} follow each other closely and allow us to extract the parametric dependence of t_{th} on δJ and λ (Fig. 2) [54]. At small λ and δJ we find an asymptotic dependance which is consistent with $t_{\mathrm{th}} \sim e^{1/\lambda}, t_{\mathrm{th}} \sim e^{1/\delta J}$, implying an anomalously slow dephasing and decay over an extremely long time-scales. At larger $\lambda, \delta J$ there may deviations from this form. In this respect, the logarithmic decay is reminiscent of the long-lived pre-thermal regime of non-MBL Floquet systems [12–16]. However, the entanglement growth in this region is slower than linear and consistent with logarithmic growth, which would not be the case of a system equilibrating to an effective finite temperature and pre-thermal Hamiltonian. We note that such logarithmic decay is observed at the phase transition between MBL and thermal phases [40–42]; here, we see this critical-like behavior without fine-tuning.

It is interesting to compare these results to those of a non-interacting analog of (2) (dashed line in Fig. 1b, for detailed comparison see [39]). The non-interacting system also exhibits a slow decay regime, but in this case there is no cross-over to fast thermalization ($t_{\mathrm{th}} = +\infty$). This suggests that, despite the absence of local conserved quantities, the long lived glassy relaxation regime in the interacting case is nonetheless governed by the dynamics of emergent single-particle-like degrees of freedom.

(Ir)reducibility of the quasi-periodic drive – High-frequency expansions provide a useful tool for understanding pre-thermalization behavior in Floquet systems. They enable the computation of an effective static pre-thermal Hamiltonian and the expansion breakdown at long times indicates the onset of thermalization. Here, we attempt to develop a generic expansion of the many-body time-evolution operator organized in powers of λ – effectively a Magnus expansion – taking advantage of the special self-similar structure of the Fibonacci drive. Technical details are given in the Supplemental Material [39].

We can analytically construct a recursive Magnus expansion for $\Omega_n = \log U_n$, using the local deflation rule structure of quasiperiodic sequences [55, 56]. We can generate U_{n+1} from U_n by replacing $U_0 \to U_1$ and $U_1 \to U_0 U_1$ in the product defining U_n. We expand Ω_n onto a basis of nested commutators and construct and solve difference equations for the coefficients in this expansion, order-by-order in the degree k of the commutator basis. Up to degree two:

$$\Omega_n = F_{n-1}\Omega_0 + F_n\Omega_1 + \tfrac{1}{2}\left\{(-1)^n + F_{n-2}\right\}[\Omega_0, \Omega_1]\,.$$

Explicit expressions for degrees $k = 3, 4$ are given in the Supplemental Material [39]. In order to assign an effective static Hamiltonian interpretation, the asymptotic form for all coefficients need to be $\sim \varphi^n$, as above. However, for $k \geq 4$, the asymptotic behavior is $\sim \varphi^{(k-2)n}$. Therefore, the time where the non-Hamiltonian evolution dominates becomes increasingly short $t_n \sim \lambda^{-(k-1)/(k-3)}$. We note that this breakdown is fundamentally different from the breakdown of thermalization in the Floquet-Magnus case for periodic driving, which is due to a lack of convergence of the expansion.

Despite this, we find that truncating the expansion at $k = 3$ gives a Hamiltonian evolution which reproduces that data at small λ remarkably well, with the exception of rare anomalous disorder configurations. Indeed, the time where this expansion deviates from the data scales with λ^{-5}, much later than the expected λ^{-3} (Fig. 3). In no case, however, does the Magnus expansion capture the anomalous logarithmic decay of C^{zz} or growth of S_{bp} for $t < t_{\rm th}$, suggesting these are inherently dynamical phenomena not governed by a static Hamiltonian, i.e. not governed by an effective conserved (quasi)-energy.

Fibonacci time quasi-crystal – The existence of an exponentially long lived quasi-MBL regime, with only logarithmically slow decay, raises the prospect of transient phases unique to quasi-periodically driven systems. These are analogous to metastable phases in pre-thermal Floquet settings, but with the important distinction that the quasi-periodically driven system does not require cooling to observe quantum coherent behavior. To illustrate this possibility, we now construct a model that exhibits the quasi-periodic analog of discrete time-translation breaking symmetry [22–27] – a "time quasi-crystal" (TQC). The model uses the Fibonacci sequence of (1), but with elementary unitaries

$$U_0 = e^{-i\theta\sum_i S_i^x}, U_1 = e^{-i\lambda\sum_i\left(J_i S_i^z S_{i+1}^z + h_i^z S_i^z + h_i^x S_i^x\right)}. \quad (3)$$

This model is closely inspired by the periodic version introduced in [22, 24].

Consider the ideal case of (3), where $\theta = \pi, h_i^x = 0$ and random J_i, h_i^z. Then $U_0 \sim \prod_i S_i^x \equiv X$ applies a perfect, global spin-flip, while U_1 is made of only S^z operators. A simple S^z-product state would merely acquire a phase under U_1 and flip under U_0. The time-evolution of a specific spin $\langle S_i^z(t) S_i^z(0)\rangle$ exhibits an oscillating quasi-periodic pattern that is sharply distinct from the driving pattern. An elegant way to capture this difference is to view the quasi-periodic sequence as a projection of a 1d strip cutting through a regular 2d square lattice at an irrational angle (see [39]). The TQC spin response corresponds to a projection from a 2d lattice having a doubled unit cell compared to that for the drive.

Alternatively, we can directly compare the Fourier spectrum of the spin response compared to that of the drive [55, 56]. For this, it is convenient to interpret U_0 in (1) as arising from an instantaneous pulse, so that we can write the evolution in terms of a Hamiltonian with quasi-periodic delta-function "kicking": $H(t) = H_1 + \sum_{m=1}^{M}\delta(t - t_m) H_0$, where $t_m = \lfloor\varphi m\rfloor$ and M is the largest integer such that $t_M \leq t$. In the ideal limit $\theta = \pi, h_i^x = 0$, the correlation function would satisfy $\mathrm{d}C^{zz}(t)/\mathrm{d}t = 2\sum_{m=1}^{M}(-1)^m\delta(t - t_m)$. The spectrum of the spin-response is shifted compared to the drive (see Fig. 4 and Supplemental Material [39]). The distinction

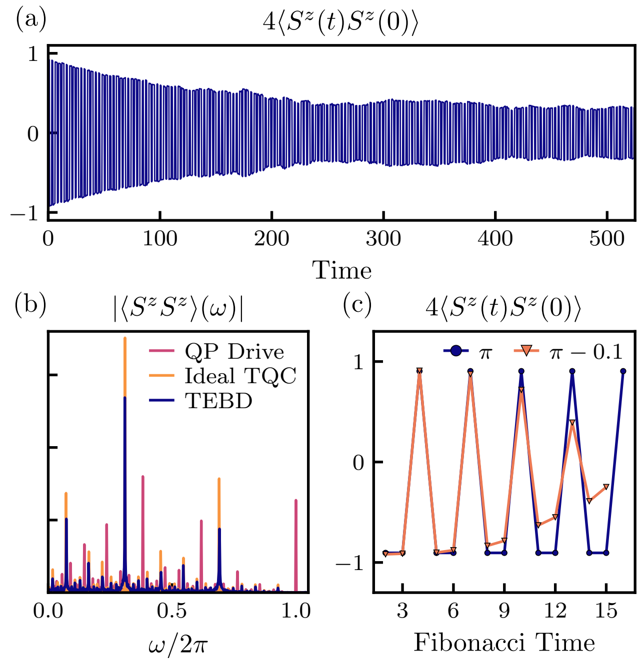

FIG. 4. **Time Quasi-Crystal.** – (a) TEBD data of a single spin in a spin-1/2 chain subjected to drive (3) with U_0 occuring instantaneously. Parameters are $L = 60$, $\lambda = 1$, and $\theta = \pi - 0.1$ as well as random variables drawn from uniform distributions $J \in [2, 8]$, $h_i^z \in [0, 2]$, $h_i^x \in [0, 0.6]$. We show a single disorder realization. (b) Fourier spectra of the quasi-periodic (QP) drive pattern, of the ideal TQC pattern and of the TEBD data. (c) Magnetization at Fibonacci times, for ideal ($\theta = \pi$) and non-ideal ($\theta = \pi - 0.1$) pulse, shows period-3 oscillations characteristic of the TQC.

between the spin-response and drive patterns is even simpler if we consider stroboscopically measuring $C^{zz}(t)$ at Fibonacci times $t_n = F_n$. At these times, the initial spins have been flipped F_{n-1} mod 2 times from their initial state. Since F_k mod 2 form a repeating pattern with period 3; the TQC is characterized by persistent period-3 oscillations in Fibonacci time.

These aspects also generalize straightforwardly to other time quasi-crystal phases. For example, we may replace the Ising spins ($\mathbb{Z}_2$) by N-state clock spins ($\mathbb{Z}_N$) in U_1 and replace S^x by the operator that increments the clock spins in U_0 of (3). In Fibonacci time, the spins would oscillate with the Pisano period $\pi(N)$; for $N = 2, 3, 4, 5$, $\pi(N) = 3, 8, 6, 20$. While the emergence of quasi-periodic correlations that have a different pattern from the drive can occur in ideally driven single spins [57], this is special to fine-tuned drivings. In the many-body set-up (3), the interactions give phase rigidity even away from the ideal limit $\theta = \pi$, as for a Floquet time-crystal [25].

For $\theta \neq \pi$ or $h_x \neq 0$, the model becomes non-integrable and we lose analytic control. Figure 4 shows $C^{zz}(t)$ from time-evolving block decimation (TEBD) [58–60] for system size $L = 60$ starting from a product state. The TEBD calculations were done with Trotter step 0.01λ, keeping the discarded weight below 10^{-7} throughout the time evolution. Away from the ideal limit, the results largely track the ideal oscillations, but we clearly see the overall logarithmic decay in the quasi-periodic oscillations due to the quasi-MBL nature as discussed in the previous sections. In the Heisenberg chain (2) discussed above, the glassy relaxation was smoothly connected to the non-interacting limit. It is intriguing that this behavior is again observed in a system that is unconnected to any free fermion limit due to the longitudinal fields. This again suggests a possible description in terms of an emergent set of effectively single-particle, though non-conserved, degrees of freedom.

Despite that the system eventually thermalizes, for moderately small λ the decay is sufficiently slow to permit many period-3 oscillations in Fibonacci time. This is a fundamentally different type of approximate non-equilibrium order than previously discussed for the cases of pre-thermal order in Floquet systems [12–16], which require cooling to an effective prethermal ground-state.

Beyond this quasi-periodic generalization of a Floquet time-crystal, the slow relaxation in the long-lived regime of glassy relaxation opens the door to more exotic quantum dynamical behavior such as long lived quasi-periodic topological phenomena. Investigating this intriguing possibility, and developing a systematic theoretical framework to characterize such metastable quantum phases will be an important challenge for future work.

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