Spectrum Estimation of Density Operators with Alkaline-Earth Atoms

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Spectrum estimation of density operators with alkaline-earth atoms

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We show that Ramsey spectroscopy of fermionic alkaline-earth atoms in a square-well trap provides an efficient and accurate estimate for the eigenspectrum of a density matrix whose n copies are stored in the nuclear spins of n such atoms. This spectrum estimation is enabled by the high symmetry of the interaction Hamiltonian, dictated, in turn, by the decoupling of the nuclear spin from the electrons and by the shape of the square-well trap. Practical performance of this procedure and its potential applications to quantum computing and time-keeping with alkaline-earth atoms are discussed.

The eigenspectrum of a d-dimensional density matrix $\hat{\rho}$ of a system characterizes the entanglement of the system with its environment [1]. As it gives access to quantities such as purity, entanglement entropy, and more generally Renyi entropies, the eigenspectrum is an indispensable tool for studying many-body quantum states and processes in general and quantum information processors in particular [2, 3]. A strategy to estimate the spectrum specifies the measurements to be performed on n copies of $\hat{\rho}$, along with a rule that specifies the estimated spectrum given measurement outcomes. It is natural that an optimal measurement should be invariant under arbitrary permutations [symmetry group S_n] and arbitrary simultaneous rotations [symmetry group $SU(d)$] of all n copies. The well-known empirical Young diagram (EYD) algorithm involves a single joint entangled measurement on all n copies which satisfies these symmetries, by projecting onto irreducible representations of $S_n \times SU(d)$[4–9]. In this Letter, we show that Ramsey spectroscopy on n fermionic alkaline-earth atoms stored together in a square trap can be used for spectrum estimation. We require each atom to have a copy of $\hat{\rho}$ stored in the d-dimensional nuclear spin. Then spatially uniform Ramsey pulses between electronic states result in a joint measurement with $S_n \times SU(d)$ symmetry, reminiscent of the EYD measurement.

Two unique features of fermionic alkaline-earth atoms are the metastability of the optically excited state $|e\rangle = {}^3\mathrm{P}_0$ and the decoupling of the nuclear spin from the $(J = 0)$ electrons in both the ground state $|g\rangle = {}^1\mathrm{S}_0$ and in $|e\rangle$. Thanks to these two features, alkaline-earth atoms have given rise to the world's best atomic clocks [10, 11] and hold great promise for quantum information processing with nuclear and optical electronic qubits [12–17] and for quantum simulation of two-orbital, high-symmetry magnetism [18–23]. Spectrum estimation of $\hat{\rho}$, using a copy of $\hat{\rho}$ stored in the nuclear spin of each of n $|g\rangle$ atoms, would be of great value in all of these applications. First, it can determine whether $\hat{\rho}$ describes a pure state, in which case the fermions would be identical and s-wave scattering would not interfere with clock operation. Second, it can be used to assess how faithfully the nucleus stores quantum information as one manipulates the electron [12, 13, 16]. Finally, this procedure can be used to characterize the entanglement of a given nuclear spin with others in a many-atom state obtained via evolution under a spin Hamiltonian [18–24]; this would require n copies of the many-atom state.

As illustrated in Fig. 1(a), to estimate the spectrum of $\hat{\rho}$, whose n copies are stored in the nuclear spins of n $|g\rangle$ atoms, we transfer all n atoms into a single square well, with at most one atom per single-particle orbital. For sufficiently weak interactions, due to energy conservation and the anharmonicity of the trap, the n occupied orbitals of the well remain unchanged throughout the experiment and play the role of individual sites. Thanks to the decoupling of the d-dimensional nuclear spin from the electrons, s-wave interactions give rise to a spin Hamiltonian with nuclear-spin-rotation $SU(d)$ symmetry [18, 19].

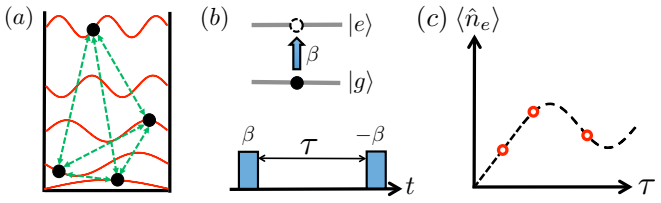

FIG. 1. Spectrum estimation with alkaline-earth atoms. (a) n copies of a d-dimensional density matrix $\hat{\rho}$ are stored in the nuclear spin of n fermionic alkaline-earth atoms trapped in a single square-well trap and prepared in their ground electronic state $|g\rangle$. (b) A Ramsey sequence is applied consisting of two pulses of area β and $-\beta$, respectively, coupling $|g\rangle$ to the first excited electronic state $|e\rangle$. (c) The number $\langle\hat{n}_e\rangle$ of e atoms is measured for different dark times τ (red circles) between the pulses, allowing one to extract the spectrum of $\hat{\rho}$.

Furthermore, the interaction strength between square-well orbitals labeled by positive integers $p \neq q$ is proportional to $\int_0^\pi \mathrm{d}x \sin^2(px)\sin^2(qx) = \pi/4$ and is thus independent of p and q, giving rise to the site-permutation symmetry S_n [25]. Critically, the resulting Hamiltonian has $S_n \times SU(d)$ symmetry.

Remarkably, the independence of the interaction strength on p and q also makes the motional temperature of the atoms irrelevant.

Our Ramsey protocol begins with the initial state of the n-atom system $|G\rangle\langle G| \otimes \rho^{\otimes n}$, where $|G\rangle = |g\ldots g\rangle$ and each nuclear spin is in the same state $\hat{\rho}$. The first Ramsey pulse of area β between $|g\rangle$ and $|e\rangle$ [Fig. 1(b)] is implemented over short time $t_P = \beta/\Omega$ (so that interactions can be ignored), using Hamiltonian $\hat{H}_P = \frac{\Omega}{2}\sum_{k=1}^n \left(\hat{\sigma}^k_{eg} + \hat{\sigma}^k_{ge}\right)$ with Rabi frequency Ω and $\hat{\sigma}^k_{\mu\nu} = |\mu\rangle_k\langle\nu|_k$. Since s-wave e-e interactions are lossy [21], we assume that the trapping of $|e\rangle$ atoms is temporarily loosened during the dark time τ [15], so that only g-g interactions contribute via the spin Hamiltonian

$$\hat{H}_{\mathrm{D}} = U\sum_{j<k}\hat{\sigma}^j_{gg}\hat{\sigma}^k_{gg}(1-\hat{s}_{jk}) - \delta\sum_k \hat{\sigma}^k_{ee}. \qquad (1)$$

In the supplement we discuss the approach with a more general Hamiltonian [26]. Here $\hat{s}_{jk} = \sum_{r,r'=1}^d |r\rangle_j|r'\rangle_k\langle r'|_j\langle r|_k$ exchanges nuclear spins on sites j and k (so two identical fermions indeed do not s-wave interact), δ is the detuning of the Ramsey-pulse laser from the g-e transition, $U = 4\pi\hbar a_{gg}\omega_\perp/L$, a_{gg} is the s-wave g-g scattering length, L is the length of the square well, and $\omega_\perp$ is the frequency of the potential that freezes out transverse motion of the atoms [25]. After the second Ramsey pulse of area $-\beta$, the state is $\hat{\rho}' = \hat{W}^\dagger\hat{V}\hat{W}|G\rangle\langle G|\hat{\rho}^{\otimes n}(\hat{W}^\dagger\hat{V}\hat{W})^\dagger$, where $\hat{W} := \exp[-it_P\hat{H}_P]$ and $\hat{V} := \exp[-i\tau\hat{H}_D]$. Finally, the number of $|e\rangle$ atoms $\langle\hat{n}_e\rangle = \mathrm{Tr}\,[\hat{n}_e\hat{\rho}']$ is measured, where $\hat{n}_e = \sum_j \hat{\sigma}^j_{ee}$.

We envisage starting with $m \times R$ sets of n atoms, each with nuclear spin state $\hat{\rho}$. We denote the eigenspectrum of $\hat{\rho}$ as $\vec{p} = (p_1, p_2, \ldots, p_d)$, ordered for future convenience as $p_1 \geq p_2 \geq \cdots \geq p_d$. For each dark time $\tau_1, \tau_2, \ldots \tau_R$, we repeat the Ramsey protocol m times and compute the average [Fig. 1(c)] to yield estimates of $\langle\hat{n}_e(\tau_1,\vec{p})\rangle, \langle\hat{n}_e(\tau_2,\vec{p})\rangle, \ldots \langle\hat{n}_e(\tau_R,\vec{p})\rangle$. Our key finding is that $\vec{p}$ can be inferred by fitting the measured values to a pre-calculated expression of the mean number of e atoms $\langle\hat{n}_e(\tau,\vec{p})\rangle$.

Although our approach is valid for all n, as n increases, the distribution of measurement outcomes $\hat{n}_e/n$ becomes tightly peaked about its expectation value $\langle\hat{n}_e\rangle/n$ given by the following expression in the large n limit:

$$\frac{\langle\hat{n}_e(\tau,\vec{p})\rangle}{n} = \frac{\sin^2\beta}{2}\left[1 - \sum_{r=1}^d p_r\cos(\omega_r\tau)\right] + \tilde{\mathcal{O}}\left(\frac{1}{\sqrt{n}}\right), \qquad (2)$$

where $\omega_r = U(n-1)(1-p_r)\cos^2\frac{\beta}{2} + \delta$. We use the notation that a tilde over the $\mathcal{O}$ indicates that we ignore logarithmic factors. Therefore the number of required repetitions m decreases with n, making our approach particularly appealing in the regime of large n [see Fig. 2(a)].

The limiting cases of Eq. (2) are easily understood. Indeed, Rabi π-pulses ($\beta = \pi$) give zero since $\hat{H}_D \to -n\delta$, so $\hat{W}^\dagger\hat{V}\hat{W} = \exp[in\delta\tau]$. Similarly, $\langle\hat{n}_e\rangle = 0$ in the absence of Rabi pulses ($\beta = 0$) since no $|e\rangle$ atoms are ever created. If $\hat{\rho}$ describes a pure state, in which case one of the p_r is unity while the rest vanish, the interaction U drops out (as it should for identical fermions) and we recover the familiar non-interacting expression.

EYD spectrum estimation.—Before presenting the derivation of the number of e atoms, $\langle\hat{n}_e(\tau,\vec{p})\rangle$, it is useful to review the original EYD spectrum estimation algorithm. For the familiar case of qubits ($d = 2$, or, equivalently, spin-1/2), the EYD algorithm can be stated as:

Letting $(p, 1-p)$ with $p \geq 1/2$ be the spectrum of $\hat{\rho}$, in the limit $n \to \infty$, a single measurement on $\hat{\rho}^{\otimes n}$ of the total spin $\hat{S}^2$ [with possible outcomes $S(S+1)$ with nonnegative $S = n/2, n/2-1, \ldots$] gives an outcome satisfying $p = 1/2 + S/n + \mathcal{O}(1/\sqrt{n})$.

This result follows from the fact that for large n the measurement outcome distribution $\Pr(S|n,p)$ becomes peaked with mean and standard deviation $(p-1/2)n$ and $\sqrt{p(1-p)n}$ to leading order in n, as shown in Fig. 2(b)

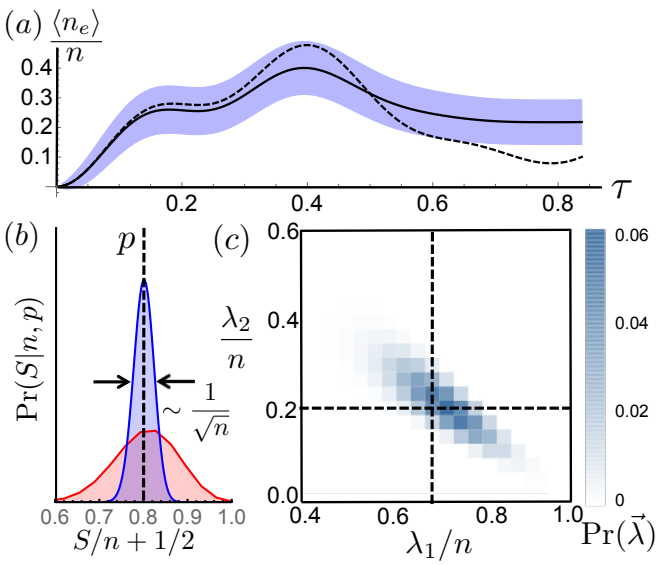

FIG. 2. (a) For spectrum $\vec{p} = (0.7, 0.2, 0.1)$ and $n = 30$, we compare the true expectation value $\langle\hat{n}_e(\tau,\vec{p})\rangle/n$ (solid line) with that estimated using mean-field theory (dashed line). The blue region indicates outcomes that are within one standard deviation of $\langle\hat{n}_e(\tau,\vec{p})\rangle/n$, where the standard deviation is estimated using the mean field result Eq. (5). (b) The normalized probability distribution $\Pr(S|n,p)$ for measurement outcome S (and the estimate $S/n + 1/2$ for p) for $n = 30$ and $n = 300$ copies of $\hat{\rho}$ with spectrum $(p, 1-p)$ with $p = 0.8$. (c) For $n = 30$, the probability distribution is shown for different outcomes $(\lambda_1, \lambda_2, n-\lambda_1-\lambda_2)$ given spectrum $(0.7, 0.2, 0.1)$.

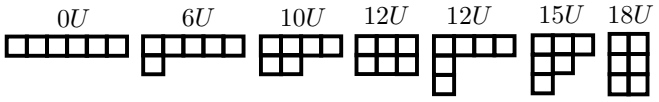

FIG. 3. The Young diagrams $\vec{\lambda} = (\lambda_1, \lambda_2, ..., \lambda_d)$ for $n = 6$, $d = 3$. With all atoms in $|g\rangle$, the interaction Hamiltonian $\langle G|\,\hat{H}_D\,|G\rangle = U\sum_{j<k}(1-\hat{s}_{jk})$ has $S_n \times SU(d)$ symmetry and is therefore diagonal in $\vec{\lambda}$-subspaces. The energy in $\langle G|\,\hat{H}_D\,|G\rangle$ is displayed above each Young diagram. Notice two of the Young diagrams correspond to the same energy.

[26]. Note that the measurement operator $\hat{S}^2$ has symmetry group $S_n \times SU(2)$. The action of this symmetry group within each eigenspace of $\hat{S}^2$ corresponds one-to-one to a distinct irreducible representation of $S_n \times SU(2)$.

This generalizes to arbitrary d. Thanks to Schur-Weyl duality [27], the irreducible representations (irreps) of $S_n \times SU(d)$ in the d^n-dimensional nuclear-spin Hilbert space $\mathcal{H}$ of n atoms are in one-to-one correspondence with d-row Young diagrams $\vec{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_d)$ whose row lengths satisfy $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d$ and $\sum \lambda_i = n$ [see Fig. 3]. We write $\mathcal{H} = \bigoplus_{\vec{\lambda}} \mathcal{H}_{\vec{\lambda}}$, where the $\vec{\lambda}$-subspace $\mathcal{H}_{\vec{\lambda}} \subset \mathcal{H}$ supports the $\vec{\lambda}$-irrep. Any operator on $\mathcal{H}$ with $S_n \times SU(d)$ symmetry has $\mathcal{H}_{\vec{\lambda}}$ as eigenspaces.

In the EYD algorithm, one measures the Young diagram on $\hat{\rho}^{\otimes n}$. The distribution of outcomes $\Pr(\vec{\lambda}|n, \vec{p})$ has a single peak near $n\vec{p}$ [see Fig. 2(c)] with a typical deviation $\sum_i |\frac{\lambda_i}{n} - p_i|$ of $\mathcal{O}(n^{-1/2})$ (for fixed d) [8].

The experimental complexity associated with changing from the $S_n \times SU(d)$ irrep basis $\mathcal{H}_{\vec{\lambda}}$ to the (generally easier to measure) computational basis makes implementing the EYD algorithm [28] seem like a daunting task in practice. The main result of this Letter is that the standard tool of Ramsey spectroscopy applied to fermionic alkaline-earth atoms in a square-well trap naturally accomplishes essentially the same task, allowing for efficient spectrum estimation.

A hint at why our proposal achieves this goal is that the Hamiltonian restricted to the ground electronic state, $\langle G|\,\hat{H}_D\,|G\rangle = U\sum_{j<k}(1-\hat{s}_{jk})$, is an operator on $\mathcal{H}$ with $S_n \times SU(d)$ symmetry. Therefore $\langle G|\,\hat{H}_D\,|G\rangle$ has subspaces $\mathcal{H}_{\vec{\lambda}}$ as energy eigenspaces, which can be probed by Ramsey spectroscopy. However the energies $E(\vec{\lambda}) = \frac{U}{2}n(n-1) - \frac{U}{2}\sum_{i=1}^{d}\lambda_i(\lambda_i - 2i + 1)$ are not in one-to-one correspondence with subspaces $\mathcal{H}_{\vec{\lambda}}$ for $d > 2$ [see Fig. 3 for an example]. Therefore, even if it were possible experimentally, direct measurement of the energy associated with $\langle G|\,\hat{H}_D\,|G\rangle$ would not be sufficient to perform the EYD algorithm. We will see that, remarkably, by accessing restrictions of $\hat{H}_D$ to different electronic states, Ramsey spectroscopy is powerful enough to uniquely identify $\vec{\lambda}$, thus enabling spectrum estimation.

Mean-field solution.—To infer the spectrum, we need to calculate the Ramsey measurement expectation value,

$$\frac{\langle \hat{n}_e(\tau, \vec{p})\rangle}{n} = \frac{\mathrm{Tr}\left(\hat{\rho}^{\otimes n}\ \hat{n}_e(\tau)\right)}{n}, \tag{3}$$

defining $\hat{n}_e(\tau) := \langle G|W^\dagger V^\dagger W \hat{n}_e W^\dagger V W|G\rangle$, which is an operator on $\mathcal{H}$ with $S_n \times SU(d)$ symmetry. We now show that, within the mean-field approximation, Eq. (3) can be evaluated using the expression in Eq. (2).

Without loss of generality, we choose the eigenbasis of the initial nuclear-spin density matrix $\hat{\rho}$ as the nuclear spin basis. At the mean-field level, time evolution under $\hat{H}_P$ and $\hat{H}_D$ does not create coherence between different nuclear spin states. Let $\rho^{rr}_{\mu\nu}$ be the entry $\langle \mu r|\,\hat{\rho}(\tau)\,|\nu r\rangle$ of the single-atom density-matrix $\hat{\rho}(\tau)$, where μ, ν denote the electronic state (g or e), while r denotes nuclear spin. Then the dark-time evolution keeps ρ^{rr}_{gg} and ρ^{rr}_{ee} unchanged, while

$$\frac{\partial \rho^{rr}_{eg}}{\partial \tau} = i\left[\delta - U(n-1)\left(\rho^{rr}_{gg} - \sum_{r'}\rho^{r'r'}_{gg}\right)\right]\rho^{rr}_{eg}. \tag{4}$$

Putting this together with the two Ramsey pulses, we recover Eq. (2) without the $1/\sqrt{n}$ correction. Since there is at most one e atom in every site (spatial mode), the variance of $\hat{n}_e/n$ within the mean-field approximation is

$$\langle(\hat{n}_e/n)^2\rangle - \langle\hat{n}_e/n\rangle^2 = \frac{\langle\hat{n}_e/n\rangle - \langle\hat{n}_e/n\rangle^2}{n}. \tag{5}$$

This $1/\sqrt{n}$ standard deviation scaling is the same as that of the deviation of the mean-field value of $\langle\hat{n}_e(\tau, \vec{p})\rangle/n$ from its exact value [26]. However the exact expression is still important for small n which would occur when technical limitations prevent us from putting all available atoms into the same trap or when atoms are produced in small batches. In that case, we would need to repeat the experiment many times and will be sensitive to the deviation of the meanfield value from the exact result. Therefore, we now evaluate Eq. (3) exactly.

Exact solution.—To avoid clutter, we drop hats on operators and arrows on vectors and introduce abbreviations: $\mathfrak{c} := \cos\frac{\beta}{2}$, $\mathfrak{s} := \sin\frac{\beta}{2}$. We define a basis $|E\rangle$ of binary vectors, $E = (E_1, E_2, ..., E_n) \in \{0,1\}^n$, where the kth atom is in electronic state $|g\rangle$ ($|e\rangle$) when $E_k = 0$ ($E_k = 1$). We also denote by $|E|$ the number of 1's in E. Expanding $W|G\rangle$ in the $|E\rangle$ basis,

$$n_e(\tau) = \sum_{E', E\in\{0,1\}^n} i^{|E'|-|E|}\mathfrak{c}^{2n-|E|-|E'|}\mathfrak{s}^{|E|+|E'|} \times \langle E'|\,V^\dagger W n_e W^\dagger V\,|E\rangle\,. \tag{6}$$

Since $W n_e W^\dagger$ is a sum of single-atom operators, terms in which strings E and E' differ on more than one site vanish. When $E' = E$,

$$\langle E|\,V^\dagger W n_e W^\dagger V\,|E\rangle = \langle E|\,W n_e W^\dagger\,|E\rangle = (n - |E|)\mathfrak{s}^2 + |E|\mathfrak{c}^2, \tag{7}$$

since $V|E\rangle = e^{i\delta|E|\tau}\exp\left[-i\alpha\sum_{j<k\notin E}(1-\hat{s}_{jk})\right]|E\rangle$. Here $\alpha = U\tau$, $j<k\notin E$ is a sum over all pairs $j<k$ such that $E_j=0$ and $E_k=0$. Terms with $E'=E$ thereby sum to $2n\mathfrak{c}^2\mathfrak{s}^2 = \frac{n}{2}\sin^2\beta$ in Eq. (6).

When E' and E only differ on the kth atom such that $E_k=1$ and $E'_k=0$,

$$\langle E'|V^\dagger W n_e W^\dagger V|E\rangle = -i\mathfrak{c}\mathfrak{s}e^{i\delta\tau}\underbrace{e^{i\alpha\sum_{j\notin E}(1-s_{jk})}}_{\mathcal{A}_E}, \quad (8)$$

as $e^{-i\alpha\sum_{j<l\notin E'}s_{jl}}e^{i\alpha\sum_{j<l\notin E}s_{jl}} = e^{-i\alpha\sum_{j\notin E}s_{jk}}$, which holds since the exponents commute. Defining $\mathcal{A}_E$ as given by the underbrace, the contribution to the sum in Eq. (6) of E and E' that differ on a single atom is

$$-\sum_{k=1}^{n}\sum_{\substack{E\in\{0,1\}^n\\E_k=1}}\mathfrak{c}^{2n-2|E|+2}\mathfrak{s}^{2|E|}\mathrm{Tr}\left[\rho^{\otimes n}(e^{i\delta\tau}\mathcal{A}_E + e^{-i\delta\tau}\mathcal{A}_E^\dagger)\right]. \quad (9)$$

Note that $\mathrm{Tr}(\rho^{\otimes n}\mathcal{A}_E)$ is invariant under site permutation, and therefore depends only on $|E|$. For integer $w=0,1,...,n-1$, define the convenient $|E|=w+1$ representative operator $\mathcal{B}_w := e^{i\alpha\sum_{j=1}^{n-w-1}(1-s_{jn})}$. Then,

$$\frac{\langle\hat{n}_e(\tau,\vec{p})\rangle}{n} = \frac{\sin^2\beta}{2}\left[1-\sum_{w=0}^{n-1}\Pr(w|n,\beta)\ \mathfrak{Re}\left\{e^{i\delta\tau}\mathrm{Tr}(\rho^{\otimes n}\mathcal{B}_w)\right\}\right], \quad (10)$$

where $\Pr(w|n,\beta) := \binom{n-1}{w}\mathfrak{c}^{2(n-w-1)}\mathfrak{s}^{2w}$ is the binomial distribution obtained from expanding $(\mathfrak{s}^2+\mathfrak{c}^2)^{n-1}=1$.

We evaluate $\mathrm{Tr}(\rho^{\otimes n}\mathcal{B}_w)$ in two ways. The first way (presented below) uses group representation theory and illustrates the connection to the EYD algorithm, and yields an expression that can be evaluated conveniently numerically. The second approach (provided in the supplement [26]), is used to prove that the asymptotic result in Eq. (2) deviates from the exact result by $\tilde{\mathcal{O}}(1/\sqrt{n})$.

As $\mathrm{Tr}(\rho^{\otimes n}\mathcal{B}_w)$ is invariant under $S_n\times SU(d)$ actions,

$$\mathrm{Tr}(\rho^{\otimes n}\mathcal{B}_w) = \sum_\lambda \Pr(\vec{\lambda}|n,\vec{p})\mathrm{Tr}_\lambda(\mathcal{B}_w), \quad (11)$$

where $\Pr(\vec{\lambda}|n,\vec{p})$ is the EYD probability distribution, and Tr_λ is a trace over the λ-subspace $\mathcal{H}_\lambda$. Now we show

$$\mathrm{Tr}_\lambda(\mathcal{B}_w) = e^{i\alpha(n-w-1)}\sum_\xi \Pr(\xi|w,\lambda)\sum_{r=1}^{d}\frac{\|\xi^{-r}\|}{\|\xi\|}e^{-i\alpha(\xi_r-r)}, \quad (12)$$

where the sum is over all irreps ξ of S_{n-w}, and $\Pr(\xi|w,\lambda) := \frac{m(\lambda,\xi)\|\xi\|}{\|\lambda\|}$ is a probability distribution defined in terms of the multiplicity $m(\lambda,\xi)$ of irrep ξ of S_{n-w} when regarding λ as a (reducible) representation of the subgroup $S_{n-w}\subset S_n$. For an irrep μ of S_m, its dimension is denoted $\|\mu\|$, the length of the rth row is μ_r, and μ^{-r} is an irrep of S_{m-1} defined by removing a box from the r-th row of μ.

To begin, note $\mathcal{B}_w$ is composed of permutations in the subgroup S_{n-w} of the first $n-w-1$ sites, along with the nth site. From this observation, we regard the representation space λ as a representation of S_{n-w}, to obtain a *reducible* representation $\lambda|^n_{n-w}$ of S_{n-w}. Note that we ignored the $SU(d)$ Hilbert space and considered S_n alone since $\mathcal{A}_E$ is written in terms of elements of S_n, which are each themselves $SU(d)$ symmetric. This decomposes into a direct sum of irreps ξ of S_{n-w} as $\lambda|^n_{n-w}\cong\bigoplus_\xi m(\lambda,\xi)\xi$. The multiplicity $m(\lambda,\xi)$ is the number of distinct paths from λ to ξ, where each step in a path is a Young diagram, with one box removed from the previous step [26]. Since $\mathcal{B}_w$ is invariant under permutation of the first $n-w-1$ sites, we can finally diagonalize $\mathcal{B}_w$ by further restricting each ξ-irrep of S_{n-w} to subgroup $S_{n-w-1}\subset S_{n-w}$; $\mathcal{B}_w$ must have each ξ^{-r}-subspace as an eigenspace. The eigenvalue of the ξ^{-r}-subspace is $e^{i\alpha(n-w-1)}e^{-i\alpha(\xi_r-r)}$ [26], resulting in Eq. (12).

We have introduced three probability distributions $\Pr(\lambda|n,p)$, $\Pr(w|n,\beta)$, and $\Pr(\xi|w,\lambda)$, all of which turn out to be unimodal for large n. In the large n limit, the unimodality together with the fact that $\frac{\|\xi^{-r}\|}{\|\xi\|}\to\frac{\xi_r}{\sum_j\xi_j}$ recovers the mean field result Eq. (2). For n and d which are too large to evaluate $\langle\hat{n}_e(\tau,\vec{p})\rangle/n$ exactly, one can still obtain a more precise estimate with this approach than that given by Eq. (2) by dropping terms associated with negligible contributions to the distributions [26].

Experimental considerations.—In Ref. [25] we suggest an implementation to trap tens of ^{87}Sr atoms in a square well potential by freezing out the x and y directions using a strong red-detuned laser such that $\omega_\perp = 2\pi\times 10$ kHz, and "capping" the ends of the tube of length $L\sim 10\mu$m with a blue-detuned laser. These parameters and the s-wave ^{87}Sr scattering length $a_{gg}=5.1$ nm [29] result in $U=(4\pi a_{gg}\omega_\perp)/L\approx 2\pi\times 10$ Hz, allowing one to trap $\lesssim 20$ atoms.

The relevant timescale for Eq.(2) is $1/(nU)\sim 1$ ms.

One can use a build-up cavity to increase barrier height of the caps and $\omega_\perp$, allowing one to trap more atoms and therefore carry out higher-resolution spectrum estimation.

To avoid losses caused by e-e collisions, we propose temporarily loosening the e trap during the dark time, which is readily doable for our choice of internal states [15]. This should be performed slowly with respect to $\omega_\perp$ and quickly with respect to U.

An experimentally simpler approach is to use β sufficiently small as to make e-e interactions negligible; this will, however, decrease the signal requiring additional repetitions of the experiment. In the supplement, we include e-g collisions in the mean-field treatment [26]. We include analysis of experimental imperfections in the supplemental material [26].

Outlook.—We have shown that alkaline-earth atoms can be used as a special-purpose quantum computer ca-

pable of measuring the spectrum of a density matrix, motivated by EYD. It is possible that many other useful quantum information tasks can be accessed in similar systems with special symmetry properties. In particular, an important extension of our work would be to find an efficient implementation of full-state tomography in current experimental systems. On the other hand, it would also be interesting to know if one can improve on our proposal if one seeks to measure a simpler quantity than the full spectrum [9], such as the purity.

Note.—While finalizing the manuscript, we learned of a proposal [30] to perform spectrum estimation with Rydberg atoms using a sequence of swap operations between two copies of the system, controlled by an ancilla.

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