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Exact Boson-Fermion Duality on a 3D Euclidean Lattice

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Abstract

The idea of statistical transmutation plays a crucial role in descriptions of the fractional quantum Hall effect. However, a recently conjectured duality between a critical boson and a massless 2-component Dirac fermion extends this notion to gapless systems. This duality sheds light on highly non-trivial problems such as the half-filled Landau level, the superconductor-insulator transition, and surface states of strongly coupled topological insulators. Although this boson-fermion duality has undergone many consistency checks, it has remained unproven. We describe the duality in a non-perturbative fashion using an exact UV mapping of partition functions on a 3D Euclidean lattice.

Introduction - The idea of duality underlies some of the most fascinating aspects of quantum statistical mechanics. A system that appears strongly coupled and nearly intractable may be dual to a weakly interacting system. Duality thus enables us to solve some highly non-trivial problems in theoretical physics. In $D = 2 + 1$ dimensions, a class of duality mappings known as *particle-vortex duality* follow from the fact that conserved currents can be expressed in terms of dual electric and magnetic fluxes. The simplest such example is the *boson-vortex duality* that maps a boson such as a Cooper pair to a dual vortex degree of freedom [1–3], and plays a key role in our understanding of superfluid-insulator quantum phase transitions. In this letter, we study another particle-vortex duality which transmutes bosonic and fermionic statistics. We demonstrate an exact mapping of partition functions of a strongly coupled boson, and its free fermion vortex.

It is well-known that in $D = 2 + 1$ dimensions, there can be exotic quantum effects such as fractional statistics and statistical transmutation. In a system with a gap, these features can be simply understood as Berry phase effects arising in adiabatic transport [4]. Statistical transmutation in a long-wavelength description is implemented via coupling matter to Chern-Simons (CS) gauge fields [5–7]. These effective field theories motivate mean-field treatments of the fractional quantum Hall effect based on “flux attachment”, and have enjoyed much success [8–10].

A more non-trivial question is whether similar ideas survive in a gapless context, say near a critical point between two quantum Hall phases (see e.g. [11, 12] for related discussions). Since such systems possess gapless excitations and are in general strongly coupled, the demonstration of statistical transmutation becomes a much more challenging problem. Recently, significant progress towards such a description has been made in the form of a conjectured duality between critical bosons coupled to a level-1 Chern-Simons gauge field and a massless two component Dirac fermion [13–15] (see details below). More interestingly, this conjectured boson-fermion duality combined with the familiar boson-vortex duality generate an entire web of dualities [14, 15]. Members of this web have important potential applications. Most notably, the fermion-fermion duality [16] from this web has been proposed to describe the half-filled Landau level [17] and the surface states of strongly coupled topological insulators [18–20]. On the other hand, from a more formal perspective, along this line people have been exploring non-abelian dualities [13, 21–24] and many more extensions [25–29]. It is seen the recent progress on boson-fermion duality has unified many different

branches of theoretical physics.

Explicitly, in 3D Euclidean spacetime, the boson-fermion duality maps between the Lagrangian densities

$$\begin{aligned}
-\mathcal{L}_{E, \text{boson}} &= -|(\partial_\mu - ib_\mu)\phi|^2 - r|\phi|^2 - \lambda|\phi|^4 + i\frac{\epsilon^{\mu\nu\lambda}}{4\pi}(A-b)_\mu\partial_\nu(A-b)_\lambda \\
&\quad \updownarrow \\
-\mathcal{L}_{E, \text{fermion}} &= \bar{\psi}\sigma^\mu(\partial_\mu - iA_\mu)\psi + m\bar{\psi}\psi + i\frac{\epsilon^{\mu\nu\lambda}}{8\pi}A_\mu\partial_\nu A_\lambda.
\end{aligned} \tag{.1}$$

Here ϕ is a complex boson, b a dynamical $U(1)$ gauge field, ψ a 2-component Dirac fermion with σ^μ the Pauli matrices, and A a background $U(1)$ gauge field (more precisely, A is a $Spin_c$ gauge field, but this difference does not alter our discussion below) [30]. This boson-fermion duality holds in the gapped case [6] with $\text{sgn}(r) = \text{sgn}(m)$, and, as conjectured, can be extended to the gapless case $r = m = 0$. In the gapless case, λ is at the Wilson-Fisher fixed point.

While this boson-fermion duality is extremely useful and has undergone many non-trivial consistency checks, it has remained unproven (however, see the demonstrations [25, 27] and [29]). Recall the familiar boson-vortex duality was originally presented as an essentially exact mapping between the lattice gauge theories of an XY model and an abelian Higgs model [1, 2]. A proof of the boson-fermion duality to such rigor is in need.

In this letter, we demonstrate an exact mapping between two 3D Euclidean lattice gauge theories. The first theory is an XY model minimally coupled to a $U(1)$ Chern-Simons gauge field. We implement the Chern-Simons theory on the lattice using gapped Wilson's lattice fermion, thereby manifestly preserving gauge invariance and the compactness of the gauge group. The second theory is a free, massless Dirac fermion, also realized by Wilson's lattice fermion. We show there is an exact mapping relating these two theories which holds even at criticality. Our method is so simple that it can be generalized to other members of the web of dualities [14, 15], and beyond [13, 21–29]. We will develop these ideas elsewhere.

Basic Ingredients - We consider bosons and fermions on a 3D Euclidean spacetime lattice, taken to be a cubic lattice for simplicity. There are two basic ingredients in the model. The

first is a 3D XY model, which has a partition function

$$Z_{XY}[B] = \int D\theta e^{-H_{XY}[B]/T}, \quad \int D\theta = \prod_n \int_{-\pi}^{\pi} \frac{d\theta_n}{2\pi},$$

$$-H_{XY}[B] = \sum_{n\mu} \cos(\theta_{n+\hat{\mu}} - \theta_n - B_{n\mu}), \quad (.2)$$

where n labels lattice sites, $\hat{\mu} = \hat{1}, \hat{2}, \hat{3}$ are elementary displacement vectors, and $n\mu$ labels the link between the sites n and $n + \hat{\mu}$. The XY model consists of angles $\theta_n \in (-\pi, \pi]$ on each site n , and is coupled to a $U(1)$ gauge field $B_{n\mu} \in (-\pi, \pi]$ on each link $n\mu$. Note that although the lattice XY “spins” $\vec{s}_n = e^{i\theta_n}$ have unit length, upon coarse graining, the amplitude will no longer be constrained. Thus, at distances large compared to the lattice spacing, the XY model describes a complex self-interacting scalar close to its Wilson-Fisher fixed point [31].

The second ingredient will be two sets of Grassmann fields $\bar{\chi}_n, \chi_n$ describing 2-component Dirac fermions on the lattice sites. We will use Wilson’s lattice fermions, which have the following partition function:

$$Z_W[A] = \int D\bar{\chi} D\chi e^{-H_W[A](M) - H_{\text{int}}(U)}, \quad \int D\bar{\chi} D\chi = \prod_n \int d^2\bar{\chi}_n d^2\chi_n,$$

$$-H_W[A](M) = \sum_{n\mu} \left(\bar{\chi}_n \frac{\sigma^\mu - R}{2} e^{-iA_{n\mu}} \chi_{n+\hat{\mu}} + \bar{\chi}_{n+\hat{\mu}} \frac{-\sigma^\mu - R}{2} e^{iA_{n\mu}} \chi_n \right) + \sum_n M \bar{\chi}_n \chi_n. \quad (.3)$$

H_W is the bare theory of massive Dirac fermions on a Euclidean lattice. The first two terms of the second line of Eq. (.3) are associated with the lattice links whereas the third term, the Dirac mass term, is associated with lattice sites. We will analyze H_W below. $H_{\text{int}}(U)$ is some lattice scale interaction of strength U that will be chosen later (not necessarily physically motivated) for the purpose of constructing a simple exact lattice duality.

Suppose we turn off H_{int} for now. Consistent with the doubling theorem [32], there are $2^D = 8$ 2-component Dirac fermion modes in the IR (see supplementary material). The parameter R was introduced by Wilson as a way to assign different Dirac masses to these Dirac fermion modes [33, 34]. It represents a momentum-dependent correction to fermion masses and is otherwise invisible in the continuum limit. In 3D, even though R gaps out the doubled fermions, these modes never decouple. Instead, they produce Chern-Simons terms. As is well-known, a massive 2-component Dirac fermion with mass m produces a CS term

of coefficient $-\text{sgn}(m)/2$. Thus, when all 8 fermion modes are gapped, integrating them out results in the following CS term in the long wavelength limit:

$$-\mathcal{L}_{E,\text{eff}}[A] = i \frac{C}{4\pi} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda + (\text{terms with higher derivatives}). \quad (.4)$$

The coefficient C depends on M, R as follows [35, 36] (see supplementary material):

$$C = \begin{cases} 0, & 3|R| < |M| \\ \text{sgn}(R), & |R| < |M| < 3|R| \\ -2\text{sgn}(R), & |M| < |R|. \end{cases} \quad (.5)$$

We will address the effects of H_{int} later.

The partition function for a boson coupled to a CS term can now be written:

$$Z[A] = \int Db \, Z_{XY}[b] \, Z_W[A - b], \quad \int Db = \prod_{n\mu} \int_{-\pi}^{\pi} \frac{db_{n\mu}}{2\pi} \quad (.6)$$

where $b_{n\mu}$ is a fluctuating $U(1)$ gauge field that lives on the links. We have not imposed a gauge fixing condition for $\int Db$, since the gauge redundancy is just an overall factor. Provided that $H_W + H_{\text{int}}$ satisfies certain UV boundary conditions to be described below, integrating out the fermions above will lead to a level-1 CS term for the difference $(A - b)_\mu$. Since the XY boson couples to b_μ , we expect this will impart statistical transmutation and convert the boson into a fermion.

Exact UV Mapping - We next present an exact mapping from the partition function $Z[A]$ above to that of a Wilson fermion with a different mass M' . To do this, we work in the representation of boson currents $j_{n\mu}$ defined on the links, obtained by Fourier series [37]:

$$e^{\frac{1}{T} \cos(\theta_n - \theta_{n+\hat{\mu}} - b_{n\mu})} = \sum_{j_{n\mu}=-\infty}^{\infty} I_{j_{n\mu}}(1/T) e^{i(\theta_n - \theta_{n+\hat{\mu}} - b_{n\mu})j_{n\mu}} \quad (.7)$$

where $I_j(x) = I_{-j}(x)$ is the j th modified Bessel function. It represents a sum over all possible boson tunneling terms, each term resulting in moving an integer $j_{n\mu}$ number of bosons across the link, with a tunneling amplitude $I_{j_{n\mu}}(1/T)$. Integrating out θ_n leads to a ‘‘Gauss’s Law’’ constraint on each site:

$$Z_{XY}[b] = \sum_{\{j_{n\mu}\}} \left(\prod_{n\mu} I_{j_{n\mu}}(1/T) e^{-ib_{n\mu}j_{n\mu}} \right) \left(\prod_n \delta_{\Delta_\mu j_{n\mu}} \right), \quad \sum_{\{j_{n\mu}\}} = \prod_{n\mu} \sum_{j_{n\mu}=-\infty}^{\infty}. \quad (.8)$$

The divergence-free condition is of course just a restatement of the $U(1)$ conservation of the original XY model.

We then consider the Wilson fermions. Following Wilson, we set $R = +1$. The choice of the magnitude 1 does not affect the IR physics, but it simplifies the lattice model as $\pm\sigma^\mu - 1$ project out one linear combination of the 2-component fermion. For reasons to be made apparent below, for $|R| = 1$ we choose H_{int} to be a hopping-hopping interaction:

$$\begin{aligned} -H_{\text{int}}(U) &= \sum_{n\mu} \frac{U}{2} \left(\bar{\chi}_n \frac{\sigma^\mu - R}{2} e^{-iA_{n\mu}} \chi_{n+\hat{\mu}} + \bar{\chi}_{n+\hat{\mu}} \frac{-\sigma^\mu - R}{2} e^{iA_{n\mu}} \chi_n \right)^2 \\ &= \sum_{n\mu} U \left(\bar{\chi}_n \frac{\sigma^\mu - R}{2} \chi_{n+\hat{\mu}} \right) \left(\bar{\chi}_{n+\hat{\mu}} \frac{-\sigma^\mu - R}{2} \chi_n \right). \end{aligned} \quad (.9)$$

Due to the Grassmann algebra of the fermion fields, we may expand the contribution to $Z[A]$ on each link *exactly* (see Fig. 1). Each link $n\mu$ contributes a factor $Z_{n\mu}$ which is

$$\begin{aligned} Z[A] &= \int D\bar{\chi} D\chi \sum_{\{j_{n\mu}\}} \left(\prod_n \delta_{\Delta_\mu j_{n\mu}} e^{M\bar{\chi}_n \chi_n} \right) \left(\prod_{n\mu} Z_{n\mu} \right), \\ Z_{n\mu} &= \int_{-\pi}^{\pi} \frac{db_{n\mu}}{2\pi} I_{j_{n\mu}}(1/T) e^{-ib_{n\mu} j_{n\mu}} \\ &\quad \left[1 + e^{-i(A-b)_{n\mu}} \bar{\chi}_n \frac{\sigma^\mu - R}{2} \chi_{n+\hat{\mu}} + e^{i(A-b)_{n\mu}} \bar{\chi}_{n+\hat{\mu}} \frac{-\sigma^\mu - R}{2} \chi_n \right. \\ &\quad \left. + (1+U) \left(\bar{\chi}_n \frac{\sigma^\mu - R}{2} \chi_{n+\hat{\mu}} \right) \left(\bar{\chi}_{n+\hat{\mu}} \frac{-\sigma^\mu - R}{2} \chi_n \right) \right]. \end{aligned} \quad (.10)$$

The remarkable feature of the expression above is that the integration over the dynamical gauge field $b_{n\mu}$ can be performed *exactly*. Doing so, we find the boson tunneling current $j_{n\mu}$ must be equal to the fermion tunneling current, which can only take values $0, \pm 1$ as seen in the Grassmann expansion above. This shows the bosons and the fermions always form composite particles. Note that the Gauss's Law constraint $\Delta_\mu j_{n\mu} = 0$ for bosons is *automatically* satisfied, because under the Grassmann integral, fermion tunneling currents must form closed loops to have non-vanishing amplitude, as shown in Fig. 1. Thus, after integrating out $b_{n\mu}$,

$$\begin{aligned} \sum_{j_{n\mu}=-\infty}^{\infty} Z_{n\mu} &= I_0(1/T) \left[1 + \frac{I_1(1/T)}{I_0(1/T)} e^{-iA_{n\mu}} \bar{\chi}_n \frac{\sigma^\mu - R}{2} \chi_{n+\hat{\mu}} + \frac{I_1(1/T)}{I_0(1/T)} e^{iA_{n\mu}} \bar{\chi}_{n+\hat{\mu}} \frac{-\sigma^\mu - R}{2} \chi_n \right. \\ &\quad \left. + (1+U) \left(\bar{\chi}_n \frac{\sigma^\mu - R}{2} \chi_{n+\hat{\mu}} \right) \left(\bar{\chi}_{n+\hat{\mu}} \frac{-\sigma^\mu - R}{2} \chi_n \right) \right], \end{aligned} \quad (.11)$$

which describes fermions with renormalized tunneling amplitudes and self-interactions on each link. It is now clear why we have chosen H_{int} to be the hopping-hopping interaction.

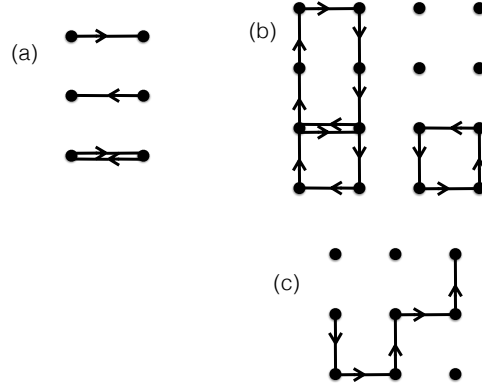


FIG. 1. (a) The various terms that arise on a link $n\mu$ in the exact expansion of Grassmann fields in Z_W . From top to bottom, the contributions are: hopping $\bar{\chi}_{n+\hat{\mu}} \frac{-\sigma^\mu-1}{2} \chi_n$, hopping $\bar{\chi}_n \frac{\sigma^\mu-1}{2} \chi_{n+\hat{\mu}}$, double hopping plus interaction $(1+U) (\bar{\chi}_n \frac{\sigma^\mu-1}{2} \chi_{n+\hat{\mu}}) (\bar{\chi}_{n+\hat{\mu}} \frac{-\sigma^\mu-1}{2} \chi_n)$. (b) In a Grassmann integral, each fermion component must appear exactly once. Consider a conjugate pair of fermion components, say $\chi_{n\uparrow}$ and $\bar{\chi}_{n\uparrow}$. They either appear together in a mass term, or appear separately in two link terms. So the link terms always form closed loops. If this condition is not satisfied as in (c), the contribution vanishes by Grassmann algebra. Thus, all contributions to Z_W manifestly satisfy Gauss's law. (The lattice is 3D. We drew a 2D lattice for clarity.)

Redefining the fermion fields $\psi_n = \sqrt{I_1(1/T)/I_0(1/T)} \chi_n$ (and likewise for $\bar{\psi}_n$), we can see a hopping-hopping interaction is generated from the coupling to bosons, so we want to include a bare H_{int} that has the same form as the generated interaction, to act as a “counterterm” and ultimately to produce a free fermion in the IR (see next section). More explicitly, the total partition function of the system in terms of $\psi, \bar{\psi}$ reads

$$Z[A] \propto \int D\bar{\psi} D\psi e^{-H_W[A](M')-H_{\text{int}}(U')}, \quad \frac{M'}{M} = \frac{I_0(1/T)}{I_1(1/T)} = \sqrt{\frac{1+U'}{1+U}} \quad (.12)$$

which describes a Wilson fermion with modified M' and U' ; the function $I_0(1/T)/I_1(1/T)$ increases from 1 to $+\infty$ as T increases from 0 to $+\infty$. This is the exact lattice duality we claimed.

By similar steps, it is also easy to verify the correlation functions satisfy

$$\left\langle e^{i\theta_{n_1}} \chi_{n_1} \cdots e^{i\theta_{n_k}} \chi_{n_k} e^{-i\theta_{n'_1}} \bar{\chi}_{n'_1} \cdots e^{-i\theta_{n'_k}} \bar{\chi}_{n'_k} \right\rangle_{[A]} \propto \langle \psi_{n_1} \cdots \psi_{n_k} \bar{\psi}_{n'_1} \cdots \bar{\psi}_{n'_k} \rangle_{[A]} \quad (.13)$$

under any background $A_{n\mu}$ configuration. This again manifests the point that the fermion ψ is the composite particle formed by the boson $e^{i\theta}$ and the fermion χ due to the mediation

of the gauge field b . Note that $e^{i\theta}\chi$ is invariant under the $U(1)_b$ gauge transformation of b , as ψ should be.

UV Boundary Conditions - In the previous section, we have shown that the 3D XY model coupled to a massive interacting Wilson fermion χ can be exactly mapped in the UV to another (generically massive and interacting) Wilson fermion ψ . In order to realize the conjectured IR duality, we will need to adjust the parameters of the UV theory, so that χ is massive and can be integrated out before the gauge field b to implement a level-1 CS, while ψ has a free massless Dirac fermion mode in the IR. For concreteness, to realize the IR behavior of ψ , we can set $M' = 3, U' = 0$, so that the Dirac mode near lattice momentum $k_\mu = 0$ becomes massless, while the other 7 Dirac modes contribute a level-1/2 CS for the background field A (see supplementary material). From Eq. .12, we see that $M' = 3, U' = 0$ are realized when the UV parameters M, U of χ satisfy

$$1 + U = \frac{M^2}{9} = \left(\frac{I_1(1/T_c)}{I_0(1/T_c)} \right)^2 \quad (.14)$$

where T_c is the value of the “temperature” T of the XY model required to hit the critical point. Such values of M and U lie on a section of a parabola shown by the gray dashed curve in Fig. 2. However, now U *must be* non-zero. We know that when $U = 0$, a Wilson fermion with $1 < M < 3$ can be integrated out to generate a CS term of level $C = 1$. Now we need to show $C = 1$ continues to hold when M, U lie on the said section of the parabola, at least in the vicinity $M \lesssim 3, U \lesssim 0$. This way, the heavy Wilson fermion χ continues to implement the level-1 CS term for $(A - b)$ in the IR.

Consider the phase diagram in M and U . Massive fermions with $C = 1$ and $C = 0$ are in different topological phases, so there must be a gapless phase boundary separating the two phase regions, and we know this phase boundary goes through the point $M = 3, U = 0$. We claim the phase boundary looks like the red line in Fig. 2 in the vicinity of the point $M = 3, U = 0$. To justify this claim, it is sufficient to consider $M = 3 + \delta M, U = 0 + \delta U$ and determine the slope $\delta U / \delta M$ such that the Dirac mode near $k_\mu = 0$ remains massless. That is, we equate $\delta M + \Sigma = 0$, where Σ is the self-energy evaluated at $k_\mu = 0, M = 3$ to first order in δU . The slope thus computed is $\delta U / \delta M \approx -8.8$, as drawn in Fig. 2. Therefore, at least in the vicinity $M \lesssim 3, U \lesssim 0$, points satisfying Eq. (.14) lie in the $C = 1$ phase as desired. This completes the verification that the constructed UV duality reproduces the conjectured IR duality. One can also check that when T deviates from T_c , the IR mass of ψ

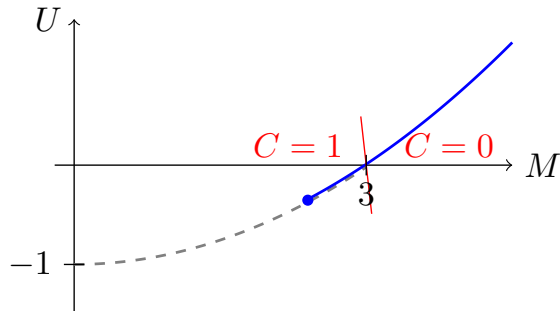


FIG. 2. The gray dashed curve (section of parabola) contains the values of M, U that satisfy Eq. (.14) for some T_c . Suppose we have chosen some M, U on the gray dashed curve, say the blue dot. Then as T increases from 0 to $+\infty$, the corresponding M', U' trace up along the blue section of the parabola. The red line represents the phase boundary between the $C = 1$ and $C = 0$ phase regions in the vicinity of $M = 3, U = 0$. The slope is ≈ -8.8 but we did not compute higher derivatives, so we do not know how the phase boundary looks further away from the point $M = 3, U = 0$. However, this is sufficient to show our choice of blue dot lies within the $C = 1$ phase as desired.

has the expected sign.

The reader may naturally wonder whether our analysis of the UV boundary condition amounts to fine-tuning beyond that which is required to reach a critical point. However, since $H_{\text{int}}(U')$ is irrelevant near the free IR fixed point, its only effect is to renormalize the IR mass. Put differently, M', U' are not independent parameters that need to be tuned (although we tuned to $M' = 3, U' = 0$ for demonstration). [38] The irrelevancy of the interaction in the IR also guarantees that first-order transitions are unlikely for small deviations from the point $M' = 3, U' = 0$. For large deviations from this point, a perturbative analysis is not controlled and other possibilities may occur.

On the other hand, the behavior of the phase boundary(ies) between the $C = -2$ and $C = 1$ phases in the vicinity of $M = 1, U = 0$ remains unclear.

Conclusion -

In this letter, we solved a strongly interacting 3D lattice gauge theory exactly and showed that the solution reproduces a conjectured boson-fermion duality in the long distance limit.

Our idea of realizing ψ as the composite particle $e^{i\theta}\chi$ is not new – it resembles the so-called “parton construction” widely used in theories of fractional quantum Hall systems [39, 40]. In our lattice construction, the following facts are all derived exactly: 1) the boson

$e^{i\theta}$ and the fermion χ form a composite particle ψ , 2) the fermion χ is massive and can be integrated out to produce level-1 Chern-Simons, and 3) the fermionic composite particle ψ becomes massless. In the IR duality, the fermion ψ is viewed as the composite particle $\phi\mathcal{M}$ where \mathcal{M} is the so-called “monopole operator”. Clearly ϕ has been realized as the coarse-grained version of $e^{i\theta}$. Moreover, \mathcal{M} can indeed be realized by the heavy fermion χ , because a heavy fermion excitation moving in a Dirac sea in the $C = 1$ phase attains a 2π flux on itself.

One may ask to what extent our “XY+heavy fermion” lattice construction represents the “Wilson-Fisher+CS” side of continuum duality. In a sense, our construction can be viewed as *giving* an operational definition to the strongly coupled continuum Lagrangian by providing a non-perturbative regularization. Then our claim is that, after integrating out the heavy fermion χ and the fast modes of bosonic fields $e^{i\theta}$ and b , the slow modes of the bosonic fields “look sufficiently like” the continuum Lagrangian of Wilson-Fisher+CS. Claims of such kind, generally not justifiable by analytic means, are implicitly understood in common applications of lattice gauge theory such as lattice QCD [33, 34] and the original presentation of the boson-vortex duality [1, 2]. Our procedure is a derivation of the boson-fermion duality within such criterion.

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