Epi-Two-Dimensional Fluid Flow: A New Topological Paradigm for Dimensionality

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While a variety of fundamental differences are known to separate two-dimensional (2D) and three-dimensional (3D) fluid flows, it is not well understood how they are related. Conventionally, dimensional reduction is justified by an *a priori* geometrical framework; i.e., 2D flows occur under some geometrical constraint such as shallowness. However, deeper inquiry into 3D flow often finds the presence of local 2D-like structures without such a constraint, where 2D-like behavior may be identified by the integrability of vortex lines or vanishing local helicity. Here we propose a new paradigm of flow structure by introducing an intermediate class, termed epi-2-dimensional flow, and thereby build a topological bridge between 2D and 3D flows. The epi-2D property is local, and is preserved in fluid elements obeying ideal (inviscid and barotropic) mechanics; a local epi-2D flow may be regarded as a 'particle' carrying a generalized enstrophy as its charge. A finite viscosity may cause 'fusion' of two epi-2D particles, generating helicity from their charges giving rise to 3D flow.

Phenomenologically, 2-dimensional (2D) fluid flow is very different from 3-dimensional (3D) flow in that the former is less-turbulent and more capable of generating and sustaining large-scale vortical structures [1]. This is because the dynamics of vortices in 2D systems is constrained, resulting in the suppression of some essential mechanisms of turbulence. Here we generalize by replacing the usual *geometrical* constraint by a *topological* constraint that extracts the essential property of 2D flow. We call such constrained flow *epi-2D*.

Conventionally, the 2D geometrical constraint is believed appropriate for a fluid with limited depth and slow variation of physical quantities in the vertical direction compared to those in the horizontal directions. However, 2D-like behavior, our epi-2D flow, may occur in flows without being strictly 2D. For example, sometimes in 3D systems, flow may not be totally 3D by having subdomains in which the flow is 2D-like [2].

In this work we precisely formulate the concept of epi-2D flow. The difference in the invariants of 3D and 2D systems serves as a guide: as is well-known, the *helicity* is a constant of motion in an ideal 3D flow [3, 4], while in 2D geometry the helicity degenerates to zero, being compensated by the *enstrophy* (or its generalization as described below). Usually, these two different invariants are regarded as attributes of different dimensionality [5]; but, we switch the viewpoints and use the invariants as discriminants of dimensionality. Interestingly, the class of flows that conserve (appropriately generalized) enstrophy is much larger than the geometrical 2D class. This extended class is what we refer to as epi-2D. We will see that the 'fusion' of two epi-2D flows may yield a 3D flow, transmuting the corresponding enstrophies into helicity. In fact, the epi-2D behavior is a local property, so we can formulate a 'particle picture' of transmutation.

Consider first the basic equations and conservation laws of fluid mechanics. Let M be a 3D domain containing an ideal (invicid) barotropic fluid. We assume $M = \mathbb{T}^3$, the 3-torus, and ignore boundary effects [6]. Denoting by ρ the mass density, $\boldsymbol{V}$ the fluid velocity, P the pressure, the governing equations are

$$\partial_t \rho = -\nabla \cdot (\boldsymbol{V}\rho), \tag{1}$$

$$\partial_t \boldsymbol{V} = -(\boldsymbol{V} \cdot \nabla)\boldsymbol{V} - \nabla h, \tag{2}$$

where $\rho^{-1}\nabla P = \nabla h$ with an enthalpy $h = h(\rho)$, The energy of the system is

$$H = \int_M \left[\frac{1}{2}|\boldsymbol{V}|^2 + \varepsilon(\rho) \right] \rho \, \mathrm{d}^3 x, \tag{3}$$

where $\varepsilon(\rho)$ is the internal energy per unit mass and $\partial(\rho\epsilon)/\partial\rho = h$. It follows by direct calculation that the energy H, the helicity,

$$C = \int_M \boldsymbol{V} \cdot \boldsymbol{\omega} \, \mathrm{d}^3 x, \tag{4}$$

with $\boldsymbol{\omega} = \nabla \times \boldsymbol{V}$ being the vorticity, and the total mass, $N = \int_M \rho \, \mathrm{d}^3 x$, are conserved.

The 2D geometrical reduction can be obtained as follows. Let z be a 'perpendicular' coordinate in the Cartesian (x, y, z) system and let $\boldsymbol{e}_z = \nabla z$. The reduction with $\boldsymbol{e}_z \cdot \boldsymbol{V} = 0$ and $\partial_z = 0$ yields the 2D system on the x-y plane (a flat torus $\mathbb{T}^2$). Using $\boldsymbol{V} = (v_x, v_y, 0)^{\mathrm{T}}$ and $\boldsymbol{v} = (v_x, v_y)^{\mathrm{T}}$, the vorticity becomes $\boldsymbol{\omega} = \nabla \times \boldsymbol{V} = \omega\boldsymbol{e}_z$, where $\omega = \partial_x v_y - \partial_y v_x$. Because $\boldsymbol{V} \cdot \boldsymbol{\omega} = 0$ for this 2D reduction, helicity conservation is now trivial: $C \equiv 0$. Interestingly, however, a different invariant emerges: the *generalized enstrophy*

$$Q = \int_M f(\vartheta)\rho \, \mathrm{d}^2 x, \tag{5}$$

with the *potential vorticity* $\vartheta = \omega/\rho$, is now a constant of motion (f being an arbitrary smooth function). It is also

easy to show the constancy of the 'local' enstrophy that is defined by replacing the domain M of the integral (5) by an arbitrary co-moving (i.e. transported by the flow $\boldsymbol{v}$) sub-domain $\Sigma(t)$. For the simple choice $f(\vartheta) = \vartheta$, the local enstrophy reads $Q = \int_{\Sigma(t)} \omega \, \mathrm{d}^2 x = \oint_{\partial\Sigma(t)} \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{x}$, and the constancy of this Q is known as Kelvin's circulation theorem. For an incompressible flow ($\nabla \cdot \boldsymbol{v} = 0$), we may assume $\rho = $ constant, and then Q has a special form $\int \omega^2 \, \mathrm{d}^2 x$, which is the usual enstrophy.

Our formulation of epi-2D flow begins by inquiring into the root cause of these invariants, deemed as a reflection of some symmetry. In particular, we replace the geometrical symmetry $\boldsymbol{e}_z \cdot = 0$ and $\partial_z = 0$ that characterizes the 2D system by a *gauge symmetry* that yields an equivalent enstrophy invariant.

The fluid equations (1)–(2) are a Hamiltonian field theory on a phase space $\mathcal{V}$ of fluid variables $\boldsymbol{u} = (\rho, \boldsymbol{V})^{\mathrm{T}}$ [7, 8]. The constancy of the Hamiltonian (energy) H is due to $\partial_t H = 0$. In contrast, the conservation of the total mass N and the helicity C is independent of the choice of Hamiltonian, implying that they are not related to any explicit symmetry of the system. Such constants of motion are called Casimir invariants. A possible mechanism that yields a Casimir invariant is a gauge symmetry in some representation of $\mathcal{V}$. If there is an underlying phase space X of fundamental variables $\boldsymbol{\xi}$, with the physical variables $\boldsymbol{u}$ represented by some specific combinations of $\boldsymbol{\xi}$ where the map $\boldsymbol{\xi} \mapsto \boldsymbol{u}$ is redundant, then a gauge freedom occurs and a Casimir invariant is a Noether charge of the gauge symmetry. One example of this is the relabeling symmetry of the Lagrangian to Eulerian fluid representations [9], but for the purposes here the relevant symmetry is that of the *Clebsch parameterization* [10–14], for which it was shown that N and C are Noether charges [15].

Let X be the phase space of Clebsch parameters

$$\boldsymbol{\xi} = (\varrho, \varphi, p, q, r, s)^{\mathrm{T}} \in X, \qquad (6)$$

where each ξ_j $(j = 1, \cdots, 6)$ is a function on the base space $M = \mathbb{T}^3$ [16]. On the space of *observables* (i.e. smooth functionals on X), we define a canonical Poisson bracket

$$\{F, G\} = \langle \partial_{\boldsymbol{\xi}} F, J \partial_{\boldsymbol{\xi}} G \rangle, \qquad (7)$$

where $\langle \boldsymbol{\eta}, \boldsymbol{\xi} \rangle = \int_M \boldsymbol{\eta} \cdot \boldsymbol{\xi} \, \mathrm{d}^3 x$, $\partial_{\boldsymbol{\xi}} F$ is the gradient of F in X, and J is the symplectic operator

$$J = J_c \oplus J_c \oplus J_c, \quad J_c = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \qquad (8)$$

Given a Hamiltonian H, the adjoint representation of Hamiltonian dynamics is $\mathrm{d}F/\mathrm{d}t = \{F, H\}$, which is equivalent to Hamilton's equation of motion

$$\partial_t \boldsymbol{\xi} = J \partial_{\boldsymbol{\xi}} H. \qquad (9)$$

We relate the physical quantity $\boldsymbol{u} \in \mathcal{V}$ and $\boldsymbol{\xi} \in X$ by $\rho \Leftrightarrow \varrho$ and (denoting $\check{p} = p/\varrho$, and $\check{r} = r/\varrho$)

$$\boldsymbol{V} \Leftrightarrow \boldsymbol{\wp} = \nabla\varphi + \check{p}\nabla q + \check{r}\nabla s. \qquad (10)$$

Writing a vector as (10) is called the *Clebsch parameterization* [10–12]. The five Clebsch parameters $(\varphi, \check{p}, q, \check{r}, s)$ are sufficient to represent every 3-vector (1-form in 3D space) [13]. Inserting (10) into the fluid energy (3), we obtain the Hamiltonian

$$H(\boldsymbol{\xi}) = \int_M \left[\frac{1}{2} \left| \nabla\varphi + \frac{p}{\varrho}\nabla q + \frac{r}{\varrho}\nabla s \right|^2 + \varepsilon(\varrho) \right] \varrho \, \mathrm{d}^3 x. \qquad (11)$$

With this H, the equation of motion (9) reads [6]

$$\begin{cases} \partial_t \varrho + \nabla \cdot (\boldsymbol{V}\varrho) = 0, \\ \partial_t \varphi + \boldsymbol{V} \cdot \nabla\varphi = h - \frac{1}{2}V^2, \\ \partial_t p + \nabla \cdot (\boldsymbol{V}p) = 0, \quad \partial_t q + \boldsymbol{V} \cdot \nabla q = 0, \\ \partial_t r + \nabla \cdot (\boldsymbol{V}r) = 0, \quad \partial_t s + \boldsymbol{V} \cdot \nabla s = 0. \end{cases} \qquad (12)$$

The first equation of (12) is nothing but the mass conservation law (1). Evaluating $\partial_t \boldsymbol{V}$ by inserting (10) and using (12), we obtain (2). Hence, Hamilton's equation (9) with the Hamiltonian (11) describes fluid motion obeying (1) and (2) [11, 17, 18]. The invariance of the helicity C also follows from (12).

We examine how the 2D geometrical reduction works in the Hamiltonian formalism. In a 2D system ($M = \mathbb{T}^2$), we can parameterize a general 2D velocity as

$$\boldsymbol{V} \Leftrightarrow \boldsymbol{\wp} = \nabla\varphi + \check{p}\nabla q, \quad (\check{p} = p/\varrho). \qquad (13)$$

Here, only three Clebsch parameters φ, $\check{p}$, and q are sufficient [13]. The vorticity is $\omega = (\nabla\check{p} \times \nabla q) \cdot \boldsymbol{e}_z$. The helicity cannot be defined in the 2D space. The potential vorticity is a scalar $\vartheta = \omega/\varrho$, and the generalized enstrophy reads $Q = \int_{\Sigma(t)} f(\omega/\varrho)\varrho \, \mathrm{d}^2 x$, where a sub-domain $\Sigma(t)$ is moved by the group-action of $\mathrm{e}^{t\boldsymbol{v}}$. We can easily verify $\mathrm{d}Q/\mathrm{d}t = 0$ by (12).

Epi-2D flow is obtained in the 3D setting with the phase space X on the base space $M = \mathbb{T}^3$ by setting $r = 0$ [19]. The corresponding physical fields are $\rho \Leftrightarrow \varrho$ and

$$\boldsymbol{V} \Leftrightarrow \boldsymbol{\wp} = \nabla\varphi + \check{p}\nabla q, \quad (\check{p} = p/\varrho). \qquad (14)$$

This yields a 2D-like representation, but there is a difference between (13) for 2D flow and (14) for epi-2D flow, for the latter resides in the 3D domain $\mathbb{T}^3$.

Epi-2D flow is generated by the reduced Hamiltonian

$$H(\boldsymbol{\xi}) = \int_M \left[\frac{1}{2} \left| \nabla\varphi + \frac{p}{\varrho}\nabla q \right|^2 + \varepsilon(\varrho) \right] \varrho \, \mathrm{d}^3 x, \qquad (15)$$

giving the 3D equations (1) and (2). While s does not appear in (15), it obeys the same equation (12) but with

$\boldsymbol{V}$ independent of s. Such a field, co-moving with the epi-2D flow, is called a *phantom* field [20]. Or, s is a gauge field with the observables blind to its initial value.

As previously remarked, (14) cannot represent an arbitrary 3D flow: epi-2D flow may have a finite vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{\wp} = \nabla \check{p} \times \nabla q$, but its helicity density $\boldsymbol{\wp} \cdot (\nabla \times \boldsymbol{\wp}) = \nabla \varphi \cdot (\nabla \check{p} \times \nabla q) = \nabla \cdot (\varphi \nabla \check{p} \times \nabla q)$ is an exact differential, implying zero helicity, $C = \int_M \boldsymbol{\wp} \cdot (\nabla \times \boldsymbol{\wp}) \, \mathrm{d}^3 x = 0$.

As for 2D, this degeneracy of the helicity is compensated by a different invariant obtained by extending the generalized enstrophy to 3D. With the phantom s,

$$Q := \int_{\Omega(t)} f(\vartheta) \, \varrho \, \mathrm{d}^3 x, \quad \vartheta = \frac{\boldsymbol{\omega} \cdot \nabla s}{\varrho}, \qquad (16)$$

with arbitrary f and an arbitrary co-moving 3D volume element $\Omega(t) \subset M$, is seen to be conserved upon making use of (12). If we choose $s = z$, then (16) reduces to the 2D form $Q = \int_{\Sigma(t)} f(\omega/\varrho) \varrho \, \mathrm{d}^2 x$. In what follows, we choose the simplest case $f(\vartheta) = \vartheta$.

Local epi-2D regions within a 3D flow can be exploited to define particle-like behavior. With the general 3D parameterization $\boldsymbol{V} \Leftrightarrow \boldsymbol{\wp} = \nabla \varphi + \check{p} \nabla q + \check{r} \nabla s$, a region in which $\check{r} = 0$ may be called an *epi-2D domain*. Since $\check{r}$ co-moves with the fluid, every infinitesimal volume element, say $\Omega_j(t)$ with elements indexed by j, included in an *epi-2D domain* may be viewed as a quasiparticle, which we call an *epi-2D particle*. The generalized enstrophy evaluated for the vorticity $\boldsymbol{\omega}_+ = \nabla \check{p} \times \nabla q$ in $\Omega_j(t)$, denoted by $Q_+(\Omega_j)$, is a constant of motion. Here the index '+' is used to distinguish from counterpart domains where $\check{p} = 0$, for which $Q_-(\Omega_j) = \int_{\Omega_j(t)} \vartheta_- \varrho \, \mathrm{d}^3 x$, where $\vartheta_- = (\boldsymbol{\omega}_- \cdot \nabla q)/\varrho$ with the vorticity $\boldsymbol{\omega}_- = \nabla \check{r} \times \nabla s$.

We call $Q_\pm(\Omega_j)$ the *charge* of the epi-2D particle Ω_j. It is remarkable that both Q_+ and Q_- are invariant even in 3D flows. Both charges essentially measure the 'circulations' of the decomposed components of the flow (cf. the comment following (5)). In fact, Kelvin's circulation theorem applies in general 3D flows. The merit of the use of the charges $Q_\pm$, in comparison with the conventional circulation, is in that they can delineate the local flow structure. As far as a particle (volume element) carries only Q_+ or Q_-, it is epi-2D. However, when the vorticity is in a *mixed state*, i.e., $\boldsymbol{\omega} = \boldsymbol{\omega}_+ + \boldsymbol{\omega}_- = \nabla \check{p} \times \nabla q + \nabla \check{r} \times \nabla s$, the particle becomes 3D, and then, helicity is created from the charges Q_+ and Q_-. Indeed, the integrands of Q_+ and Q_- (the charge densities) are, respectively,

$$\mathcal{Q}_+ = (\nabla \check{p} \times \nabla q) \cdot \nabla s, \quad \mathcal{Q}_- = (\nabla \check{r} \times \nabla s) \cdot \nabla q,$$

while the integrand of C (the helicity density) is

$$\mathscr{C} = \boldsymbol{\wp} \cdot (\nabla \times \boldsymbol{\wp}) = \check{r} \mathcal{Q}_+ + \check{p} \mathcal{Q}_- + \mathscr{C}_{\mathrm{ex}}, \qquad (17)$$

where $\mathscr{C}_{\mathrm{ex}} = \nabla \varphi \cdot (\nabla \check{p} \times \nabla q + \nabla \check{r} \times \nabla s)$ is the exact part of the helicity density. The residual helicity density $\mathscr{C}_{\mathrm{r}} = \mathscr{C} - \mathscr{C}_{\mathrm{ex}} = \check{r} \mathcal{Q}_+ + \check{p} \mathcal{Q}_-$ describes the coupling of
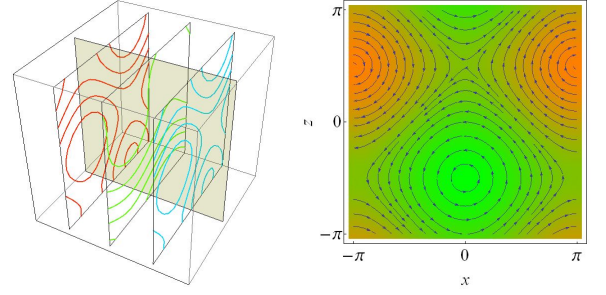

FIG. 1: An epi-2D flow given by $\boldsymbol{V}_+$. (Left) The integrable vortex lines. (Right) Contours of $\mathcal{Q}_+$ and the flow vector on the surface $y = 0$ indicated by the gray cross-section in the left figure (color code ranges from orange= 2 to green= -2).

epi-2D particles; evidently, either $\check{r} = 0$ or $\check{p} = 0$ causes $\mathscr{C}_{\mathrm{r}}$ to vanish. Conversely, the combination of two charges $\mathcal{Q}_+$ and $\mathcal{Q}_-$ yields $\mathscr{C}_{\mathrm{r}}$. Therefore, $\mathscr{C}_{\mathrm{r}} = 0$ can be used as an alternative definition of epi-2D flow.
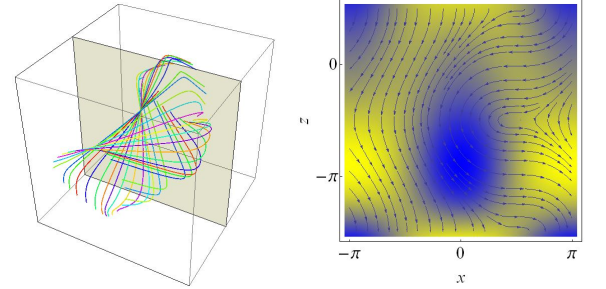
FIG. 2: The combination $\boldsymbol{V} = \boldsymbol{V}_+ + \boldsymbol{V}_-$ yields a 3D flow. (Left) The vortex lines become chaotic (non-integrable). (Right) Contours of $\mathscr{C}_{\mathrm{r}}$ together with the surface-aligned component of the flow vector on the surface $y = 0$ indicated by the gray cross-section in the left figure (color code ranges from blue= 5 to yellow= -1).

We give an illustrative example to visualize epi-2D flow and its transition to 3D. Let (x, y, z) be Cartesian coordinates, and $\boldsymbol{V} = \alpha \boldsymbol{V}_+ + \beta \boldsymbol{V}_-$ with

$$\boldsymbol{V}_+ = \begin{pmatrix} b \sin y - c \cos z \\ 0 \\ a \sin x \end{pmatrix}, \quad \boldsymbol{V}_- = \begin{pmatrix} 0 \\ c \sin z - a \cos x \\ -b \cos y \end{pmatrix},$$

where α, β, a, b and c are arbitrary real constants, and the vorticities satisfy $\boldsymbol{\omega}_\pm = \nabla \times \boldsymbol{V}_\pm = \boldsymbol{V}_\mp$. When $\alpha = \beta = 1$, $\boldsymbol{V}$ is the famous ABC flow [21], satisfying $\nabla \times \boldsymbol{V} = \boldsymbol{V}$. We may cast $\boldsymbol{V}$ into the Clebsch form (10) with

$$\begin{cases} \varphi = (\alpha a \sin x - \beta b \cos y) z, \\ \check{p} = \alpha [b \sin y - c \cos z - (a \cos x) z], \quad q = x, \qquad (18) \\ \check{r} = \beta [c \sin z - a \cos x - (b \sin y) z], \quad s = y. \end{cases}$$

When $\alpha = 1$ and $\beta = 0$, we obtain an epi-2D flow $\boldsymbol{V}_+$ with $\mathcal{Q}_+ = c \sin z - a \cos x$. The co-presence of $\boldsymbol{V}_+$ and $\boldsymbol{V}_-$

TABLE I: Topological classification of flows. $\boldsymbol{V}$: 3D flow, $\boldsymbol{V}_{\mathrm{r}} = \boldsymbol{V} - \nabla\varphi$: solenoidal component, $\mathscr{Q}_{\pm}$: generalized enstrophy density, $Q_{\pm} = \int_{\Omega(t)} \mathscr{Q}_{\pm}\,\mathrm{d}^3 x$: generalized enstrophy, $\mathscr{C}$: helicity density, $\mathscr{C}_{\mathrm{r}}$: residual helicity density, $C = \int_M \mathscr{C}_{\mathrm{r}}\,\mathrm{d}^3 x$: helicity. Notice that the local integral $\int_{\Omega(t)} \mathscr{C}_{\mathrm{r}}\,\mathrm{d}^3 x$ is not a constant of motion.

classification	representation	invariants
vorticity-free $(\nabla \times \boldsymbol{V} = 0)$	$\boldsymbol{V} = \nabla\varphi$	$\mathscr{C} = 0$ $\mathscr{Q} = 0$
helicity-free $(\boldsymbol{V} \cdot \nabla \times \boldsymbol{V} = 0)$	$\boldsymbol{V} = \breve{p}\nabla q$	$\mathscr{C} = 0$ $Q_{\pm}$
epi-2D $(\boldsymbol{V}_{\mathrm{r}} \cdot \nabla \times \boldsymbol{V}_{\mathrm{r}} = 0)$	$\boldsymbol{V} = \nabla\varphi + \breve{p}\nabla q$	$\mathscr{C}_{\mathrm{r}} = 0$ $Q_{\pm}$
general	$\boldsymbol{V} = \nabla\varphi + \breve{p}\nabla q + \breve{r}\nabla s$	C $Q_{\pm}$

creates a 3D flow; when $\alpha = \beta = 1$, $\boldsymbol{V}$ has the residual helicity density $\mathscr{C}_{\mathrm{r}} = r\mathscr{Q}_{+} + p\mathscr{Q}_{-}$. In Fig. 1, we show the structure of $\boldsymbol{V}_{+}$ and the distribution of $\mathscr{Q}_{+}$ on a surface $y = 0$. Figure 2 depicts the 3D flow $\boldsymbol{V} = \boldsymbol{V}_{+} + \boldsymbol{V}_{-}$ and $\mathscr{C}_{\mathrm{r}}$ on a cross-section of $y = 0$.

Table I summarizes our newly proposed classification of flows. The smallest set hosts vorticity-free, potential flow (or *lamellar* field [22]). The next of the hierarchy includes 'weighted' potential flows (or *complex lamellar* field) that have vorticity but still zero helicity density. A further generalization yields the epi-2D flows that have only exact helicity density (hence, $\mathscr{C}_{\mathrm{r}} = 0$). The epi-2D class subsumes conventional 2D systems where we may take $s = z$ (the perpendicular coordinate); this is possible since $\boldsymbol{\omega}$ is aligned to the fixed vector $\boldsymbol{e}_z$ [23]. As the generalization of the *a priori* base space of a 2D system, an epi-2D flow has intrinsic vortex surfaces (cf. [24]). While the direction of $\boldsymbol{\omega}$ changes dynamically, the vortex lines remain integrable, keeping the similarity to 2D flows (cf. [2]). Contrary to 2D flow, however, epi-2D flow allows for vortex stretching, which may make the epi-2D particle thinner (thus, the possibility of singularity generation is not precluded). A general 3D flow may be viewed as a mixed state of epi-2D particles, with each particle carrying a charge of Q_{+} or Q_{-} [25]. When particles with Q_{+} and Q_{-} occupy a same volume element, they produce a helicity to make the volume 3D (cf. [26] for experimental visualization of knotted vortices). Otherwise, the volume is epi-2D. The epi-2D property is topologically invariant; i.e., an epi-2D volume remains so under ideal fluid motion. If some non-ideal process, such as vortex reconnection, occurs [27–29], however, two epi-2D particles can fuse to generate helicity.

The class of locally epi-2D flows is capable of describing strongly heterogeneous 3D vortex dynamics where the helicity density is localized in narrow subdomain (such local structures often manifest as coherent vortices, and are called *worms*) [30]. We note that vortex stretching

can happen in any of these subdomains.

In conclusion, the newly formulated epi-2D vector fields are useful for delineating between mixed states of order and disorder, which indeed appear as intermittency, coherent vortices, or various local structures in fluid systems. Previously, the framework of 2D geometry was the only one for describing simple (integrable) vortex structures and discussing their moderate (or ordered) dynamics. However, such structures/dynamics can manifest themselves without this *a priori* geometrical constraint; they are more flexible and ubiquitous in general 3D space, as we do observe in actual phenomena. The epi-2D class abstracts the *topological* characteristics of the usual 2D flows; it persists under deformations by ideal fluid motion (including stretching); being a local property, it is suitable for characterizing the mixture of epi-2D and true 3D dynamics; it bridges 2D and 3D by elucidating how 3D flow is created from the epi-2D prototype, or conversely, how epi-2D degenerates to 2D. Here we discussed fluid mechanics, but the paradigm of Table I for 3D vectors applies to a variety of fields, including magnetic fields [31], optical vortices [32], as well as chiral charge-density waves [33].

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[19] By *epi-2D* we mean a class 'over' 2D. It is different from the so-called '2.5D' class of fields that are defined on a 2D base space but have 3 components, or the class of fields that have some continuous symmetry by which the number of independent variables can be reduced to two; see, for example, B. Ettinger and E.S. Titi, SIAM J. Math. Anal. **41**, 269 (2009).

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[22] Historically, Kelvin called a vorticity-free flow ($\nabla \times \boldsymbol{V} = 0$, or $\boldsymbol{V} = \nabla\varphi$) a *lamellar field*, and a helicity-free flow ($\boldsymbol{V} \cdot (\nabla \times \boldsymbol{V}) = 0$, or $\boldsymbol{V} = \check{p}\nabla q$) a *complex lamellar field*. The epi-2D flow is the combination of a lamellar field and a complex lamellar field. The Clebsch parameters $\varphi, \check{p}$ and q may be called *Monge's* potentials; see C. Truesdell, *The kinematics of vorticity*, (Indiana Univ. Press, Bloomington, 1954), Chap. I.

[23] An axisymmetric flow such that $\partial_\theta = 0$ and $\boldsymbol{e}_\theta \cdot \boldsymbol{V} = 0$ is also 2D, in which we may choose $s = \theta$, and then, the enstrophy of $\omega_\theta = r\nabla\theta \cdot \boldsymbol{\omega}$ is constant. A helically symmetric flow such that $\iota\partial_\theta - \partial_z = 0$ and $\boldsymbol{h} \cdot \boldsymbol{V} = 0$ ($\boldsymbol{h} = \iota r\boldsymbol{e}_\theta - \boldsymbol{e}_z$ with a constant ι) is epi-2D, which can be written as $\mathrm{d}\varphi + \check{p}\mathrm{d}\psi$ with φ and $\check{p}$ being functions of r and $\psi = m(\theta + \iota z)$. It is not 2D, because $\boldsymbol{h}$ is not integrable to define a perpendicular 2D manifold, however, it is very close to 2D in that the vortex stretching vanishes; cf. [19].

[24] By Frobenius' theorem, the following two conditions are equivalent for a 1-form on a 3D manifold: (1) u is helicity free ($u \wedge \mathrm{d}u = 0$). (2) u is locally *integrable*, i.e., $u = \alpha\mathrm{d}\beta$ ($\exists\alpha, \beta$). An epi-2D flow consists of a helicity-free flow and a potential flow, which may be viewed as a Pfaffian form (C. Carathéodory, *Calculus of Variations and Partial Differential Equations of the First Order, 3rd Ed.* (AMS Chelsea Publishing, Providence, 2002)). See also Supplemental Material (differential-geometrical formulation).

[25] See Supplemental Material for graphical images of epi-2D particles and their fusion.

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