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Stability landscape of shell buckling

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We measure the response of cylindrical shells to poking and identify a stability landscape, which fully characterizes the stability of perfect shells and imperfect ones in the case where a single defect dominates. We show that the landscape of stability is independent of the loading protocol and the poker geometry. Our results suggest that the complex stability of shells reduces to a low dimensional description. Tracking ridges and valleys of this landscape defines natural phase-space coordinates for describing the stability of shells.

From soda cans to aerospace engineering, the need of high-fidelity estimates of the buckling loads of shell structures is of critical importance for reliability or to increase payload capability [1–3]. Past laboratory testings with cylindrical shells have suggested that defects strongly reduce the buckling resistance of thin-walled structures [4–6]; thus, predicting the critical buckling loads is challenging and requires a priori knowledge of the defect specifics. However, those defects are in general unknown and in many cases difficult to identify. An attractive question is to investigate if there is a more general framework for characterizing the stability of shells and the classification of defects. Promising paths have been recently proposed for estimating the resistance of shells to buckling by probing them via poking from the side [7–9]. Additionally, a new approach has recently been proposed where instead of considering the linear instability of a shell with defects, a finite, nonlinear destabilizing perturbation is imposed on an otherwise perfect shell [10]. In this framework, buckling dynamics are governed by fully nonlinear edge-states located on the border of the basin of attraction of a stable unbuckled fixed point. These strategies lay a foundation for the more general description of the stability of thin-walled structures; opening a new approach for testing, predicting and controlling the stability of shell structure. However, these concepts have never been explored experimentally.

Here we experimentally uncloak the underlying stability structure of thin cylindrical shells by analyzing the buckling of commercial aluminum cans. We investigate the response and stability of cans when subjected to finite size lateral perturbations by laterally poking them with a point probe. The force-displacement curves of the probe at different axial loads unveil a rich landscape in the three-dimensional phase space spanned by axial load, probe displacement and probe force. Distinct regions of stability, independent of the loading protocol and independent on the poker geometry emerge. We specifically identify a ridge characterizing the finite work required to force-trigger buckling, and we identify a valley leading to a minimally buckled state and representing the easiest route to buckle. We suggest that this stability landscape is archetypal to the buckling of perfect shells as well as imperfect ones; specifically in the case where one localized geometric defect dominates. Importantly, the salient topographical features are not aligned parallel to either the axial or poking force axis, indicating that more suitable coordinates for analyzing the stability of thin shells are defined by tracking ridges and valleys.

We investigate the stability of variety of real indus-
is simultaneously recorded. Initially, \( F \) shell at a constant speed 10 mm/min. The poker force stored while the horizontal poker is advanced towards the \( F = 1 \) mm/min to a pre-set axial load of the poker tip. The range results do not appear to depend on the geometry shown in Fig. 1(b) and tabulated in Table 1. Within this range, the results do not change significantly. Eventually, when \( F \) decreases, buckling is also indicated by the sharp drop of \( F \). When buckling, the point on the shell's surface that was in contact with the aluminum surface. Concomitantly with these dynamics, buckling is also indicated by the sharp decrease in \( F \), as shown in the insets of Fig. 2(a).

Up to this point we have focused on several typical examples, introducing three main possible scenarios for the dynamics. Combining all measured force-displacement data for a range of axial loads, \( F_p(D_p, F_A) \) reveals a well defined surface in the three-dimensional phase space \((F_A, D_p, F_p)\), as shown in Fig. 2(b). It is important to note that the response curves of the poker are reversible throughout the majority of the axial load and poker force ranges. Fig. 2(c) presents these ranges, showing reversibility even in the non-monotonic phases of the response. This indicates the absence of irreversible plastic deformation. The goal of this letter is to introduce and highlight the significant topographical features of this stability landscape and discuss their physical significance. Although, the stability of each individual shell is uniquely determined by its specific defects, we hypothesize that the stability landscape is a generic representation of the stability of cylindrical shells. The landscape is similar for different shells of identical geometry as long as – without poking – the axial load remains far enough from the threshold where the shell spontaneously buckles. We speculate that closer to the threshold for spontaneous buckling, which depends on defects, the landscape will be smoothly distorted.

Even with a side poker it is not always possible to trigger buckling. At low axial loads \( F_A \) the cylindrical shell will not buckle regardless of the poking displacement and \( F_p(D_p) \) monotonically increases up to large poker penetrations of several millimeters. Under large axial loads,
characterizes the stability of the thin shell. The color indicates of independent poker force measurements (1 in Table 1). The surface generated by the curves defines a landscape, which characterizes the stability of the thin shell. The color indicates \( F_p \) of Fig. 3(a): the points of maximal force \( (D_p^{\text{max}}, F_p^{\text{max}}) \), and the buckling points for poker displacements \( (D_p^c, F_p(D_p^c)) = 0 \).

After buckling the dimple snaps inwards to a position \( (D_p^a, 0) \) [14]. \( D_p^{\text{max}} \) decreases linearly as a function of \( F_A^0 \), as shown in Fig. 3(b). The trajectory of the maximum \( F_p^{\text{max}} \) is a monotonically decreasing function of \( F_A^0 \) and does not depend on the poker size (1 and 3 in Table 1), as shown in Fig. 3(c). Importantly, the stability is not altered if the compression is load-controlled instead of displacement-controlled (4 in Table 1), as shown in Fig. 3(b) and Fig. 3(c) [15]. The described characteristics of the load-displacement curves can also be obtained, with less noise, by non-destructively probing a single can at different loads (2 in Table 1), as shown in the insets of Fig. 3(b) and Fig. 3(c).

Buckling occurs for a sufficiently high axial load and beyond a critical displacement of the poker \( D_p^c \). In all our experiments, \( D_p^c \) is at least two times larger than the shell thickness, indicating that buckling in this case is a nonlinear instability triggered by finite amplitude perturbations. The critical poker displacement \( D_p^c \) decreases linearly as a function of \( F_A^0 \), as shown in Fig. 3(d). In a range of \( F_A^0 \) between 400 N and 600 N, \( D_p^c \) is comparable to \( D_p^a \), indicating that buckling cannot occur below such loading. Moreover, if we first fix the position of the poker, and then increase the axial load until buckling, we find the same values of \( D_p^c \) (5 in Table 1), suggesting that the stability landscape is independent of the loading and poking history, as shown in the inset of Fig. 3(c).

In an intermediate range of axial loads centered around \( F_A^0 = 600 \) N, \( F_p(D_p) \) has a local minimum, \( (D_p^{\text{min}}, F_p^{\text{min}}) \), as sketched in Fig. 3(a). Following the minima of the stability landscape down to zero poker force leads to a minimally buckled state. Buckling does not occur for loads smaller than this minimal buckling load.

Traditionally, the linear instability of cylindrical shells against catastrophic buckling is primarily considered. At a critical axial load any infinitesimal perturbation will destabilize the system causing it to buckle. Real shells have defects and are thus weaker; it is common to only consider the linear stability of the now defected shell base state [16]. Here we are examining the stability of shells under conditions where they are linearly stable and a finite amount of work, \( E_p = \int_{0}^{D_p^c} F_p(x)dx \), is required to destabilize them. When the axial load is larger, less work is required to collapse the shell and as the axial load approaches the critical load for the linear instability, \( E_p \) approaches zero (Fig. 3(e)) and in qualitative agreement with recent computations of the stability of pressurized spherical shells [9]. This consistency with recent computations also suggests that our concept of landscape of stability could be extended to any shell. In the linearly stable system one may think of \( E_p \) as an energy barrier to buckling that is decreased by defects.

Our results suggest that the stability of shells can be coherently described by a two-dimensional surface in the three-dimensional phase space \( (F_A, D_p, F_p) \), schematically described in Fig. 4. It is an important conclusion that the complex stability of commercial soda cans reduces to such a low dimensional description, as naively such systems should be fully dominated by stochastic defects rendering buckling thresholds unpredictable. The stability landscape presents a number of important features. The gap between the unstable fixed point \( D_p^c \) and the stable one \( D_p^a \), shown qualitatively in Fig. 2(b), is a flat lake representing the unstable region surrounded by a basin and protected by an energy barrier, \( E_p \). In the landscape, the energy barrier takes the form of a

**FIG. 2.** a. Force-displacement curves for seven cans. Insets: axial load measurements. b. Three-dimensional representation of independent poker force measurements (1 in Table 1). The surface generated by the curves defines a landscape, which characterizes the stability of the thin shell. The color indicates \( F_p \). c. Three-dimensional representation of consecutive non-destructive poker force measurements on the same can, probing reversible elastic deformations without reaching buckling (2 in Table 1).
FIG. 3. All individual cans share universal stability features. a. Definitions of the main features. b. $D_{p}^{\text{max}}$ at $F_{p}^{\text{max}}$ vs $F_{A}$. Small dots, large dots and circles correspond to the protocols 1, 3 and 4 respectively, indicated in Table 1. Inset: single can tests. c. $F_{p}^{\text{min}}$ vs $F_{A}$. d. Critical poker displacement $D_{p}^{\text{c}}$ required to trigger buckling (magenta) and displacement $D_{s}^{p}$ to catch up with the post-buckled surface (green) vs $F_{A}$. Inset, hollow squares: the shell is first poked and then loaded (5 in Table 1). The poker displacement $D_{p}^{\text{min}}$ at the minimum of poking force is plotted with blue stars, and the minimum poker force $F_{p}^{\text{min}}$ is reported in the second inset. e. The elastic energy barrier, $E_{p}$ vs $F_{A}$.

cliff to the left of the lake that is decorated by a ridge $R = (D_{p}^{\text{max}}, F_{p}^{\text{max}}(F_{A}))$. Although buckling may also suddenly occur in our experiments [17] by a violent drop of $F_{p}$ to 0, in all our tests this never occurs for $D_{p} < D_{p}^{\text{max}}$, thus, the ridge may indicate the separating line between a globally stable (i.e. defect-insensitive) region to the left of the ridge and a defect-sensitive region to its right. It is thus potentially useful to envision ridge-tracking as a new non-intrusive method for probing the stability of shells as close as possible to their linearly unstable limit.

Another useful feature of the stability landscape is the minimal buckling point: for $F_{A}$ smaller than a critical value, the shells never buckle, consistent with the phenomenological design rule set by NASA [4] and hinted to by calculations of edge states [10]. Although in this regime there is no buckling ($F_{p} \neq 0$), the landscape is not featureless. Instead the curve $V = (D_{p}^{\text{min}}(F_{A}), F_{p}^{\text{min}}(F_{A}))$ highlights a valley with a very steep slope leading to the tip of the lake where $D_{p}^{c} = D_{s}^{p}$. This tip is the point of minimal buckling, where buckling does not result in any measurable deformation of the shell.

Our results suggest an appealing and generic framework for studying the stability of thin-walled structures. Shell buckling is characterized by the topographical features of the stability landscape, namely an unstable flat lake surrounded by a ridge and valley of stability. It is tempting to consider that the ridge and valley indicate the most natural coordinate system for probing the stability of such structures instead of the somewhat arbitrary coordinates $F_{A}^{0}$ and $D_{p}$. Thus, systematic tracking of the ridge and the valley of the stability landscape is likely to have a significant impact in the development of new non-destructive testing protocols in structural engineering.

We have shown that buckling of real defected cylindrical shells is underscored by a universal stability landscape. By probing away from the linearly unstable state we are exposing the underlying stability characteristics of minimal buckling point: for $F_{A}$ smaller than a critical value, the shells never buckle, consistent with the phenomenological design rule set by NASA [4] and hinted to by calculations of edge states [10]. Although in this regime there is no buckling ($F_{p} \neq 0$), the landscape is not featureless. Instead the curve $V = (D_{p}^{\text{min}}(F_{A}), F_{p}^{\text{min}}(F_{A}))$ highlights a valley with a very steep slope leading to the tip of the lake where $D_{p}^{c} = D_{s}^{p}$. This tip is the point of minimal buckling, where buckling does not result in any measurable deformation of the shell.

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We have shown that buckling of real defected cylindrical shells is underscored by a universal stability landscape. By probing away from the linearly unstable state we are exposing the underlying stability characteristics of
the near-perfect shell. The probe induces a localized shell deformation; thus, illuminating how a dominant localized defect modifies the shell stability. The local geometric defect induced by the poker, interact with intrinsic defects in the shell material. Consequently, we speculate that at high axial loads a smoothly distorted stability landscape encodes the influence of localized intrinsic defects.

Here, we present the landscape for a single poker measurement revealing the influence of a dominant localized defect. Likewise, a system poked from several directions simultaneously with numerous pokers will show a different stability landscape. In fact, the full stability landscape is hyper-dimensional and any probing geometry defines its own section of the full stability landscape; a section spanned by the amplitude of that specific probing mode, the probe force and axial load [10]. While the full stability landscape lies in the hyperspace spanned by all possible probe geometries and associated force amplitudes, it is reasonable to expect that the single localized defect case studied in this letter, is of practical importance and may inform the design rules of thin-walled structures, such as rockets, airplanes, and beer cans.

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[14] Note that in the situations where during the buckling process several dimples are formed (more common at high axial loads), $D_{sp}$ only indicates the depth of the dimple in the direction of the poker and the other dimples are not taken into account.
[15] For load-controlled compression, the shell is totally crushed when buckling is induced; thus, $D_{sp}$ cannot be measured.
[17] A violent drop in the poking force is observed for axial loads of $F_{0A} = 859$ N or $F_{0A} = 1084$ N in Fig. 2(a). In these two cases the force drops to zero significantly faster than the axial loading rate. We did not observe any rate dependence of the mechanical response in the loading rate range that we tested (0.1 to 10 mm/min).