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We investigate the statistical arrow of time for a quantum system being monitored by a sequence of measurements. For a continuous qubit measurement example, we demonstrate that time-reversed evolution is always physically possible, provided that the measurement record is also negated. Despite this restoration of dynamical reversibility, a statistical arrow of time emerges, and may be quantified by the log-likelihood difference between forward and backward propagation hypotheses. We then show that such reversibility is a universal feature of non-projective measurements, with forward or backward Janus measurement sequences that are time-reversed inverses of each other.

The classical dynamics of a conservative system is time-reversible. If we watch a movie backwards in the absence of friction, it will show dynamics perfectly consistent with the laws of motion, so we may not distinguish whether we watch the movie forward or backwards in time from the dynamics alone. However, when the system has more than a few degrees of freedom—such as during the starting break in a game of pool—then the likelihood that the evolution is either forward or backward in time may differ, so it becomes possible to distinguish an arrow of time statistically. The existence of such an arrow of time is a fundamental question, and has been of interest in many areas of physics [1–3].

The quantum dynamics of a conservative and unmeasured system is similarly time-reversible. For example, the Schrödinger equation becomes invariant under time-inversion if the position-space wavefunction is complex-conjugated. This is a special case of a general anti-unitary time-reversal operation [4], and is sufficient to restore time symmetry for a closed quantum system.

The introduction of a sequence of measurements seems to break such dynamical symmetry, however, for two distinct reasons. First, obtaining definite measurement results traditionally collapses the wavefunction, which produces non-unitary evolution that is distinct from the Schrödinger equation and not reversed by the same anti-unitary operation. Second, the randomness of each measurement creates an intrinsic asymmetry between an unknown future and a definite past. These reasons have contributed to the view that quantum mechanics is fundamentally asymmetric in time [5, 6].

We seek to clarify this apparent discrepancy between classical and quantum reversibility. In the past, such efforts to restore reversibility have tried reformulating quantum mechanics in a more symmetric way [7]. For example, the "two-time" formalism of Aharonov, Bergmann, and Lebowitz [8] removes the indefiniteness of the future by introducing a second boundary condition (or postselection) that brackets a time interval, and avoids non-unitary state collapse by considering infinitesimally weak measurements that do not affect the state within that interval [9]. Physical measurements have nonzero strength, however, so will (at least partially) collapse the state and seemingly spoil the reversibility of such a scheme [10]. Nevertheless, partial collapses of the state may still be fully restored probabilistically ("wavefunction uncollapse"), even if the initial state is unknown [11, 12]. This uncollapsing phenomenon has been confirmed experimentally in superconducting and optical systems [13–15], which raises the question once more whether the time symmetry of a sequence of several such measurements could be similarly restored.

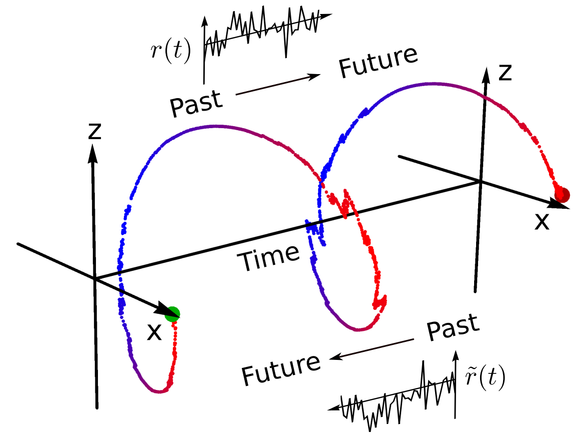

FIG. 1. A single quantum trajectory of a continuously monitored qubit. The x and z Bloch sphere coordinates of a qubit change due to both unitary and measurement dynamics. The red and blue colors denote positive and negative values of the x coordinate. The boundary states are shown as green and red dots. Is time running forward with measurement record $r(t)$ or backward with flipped record $\tilde{r}(t)$?

In this Letter, we demonstrate how to restore time reversal symmetry for a sequence of nonprojective measurements that takes into account the insights from measurement uncollapse. This is a nontrivial problem, since correlation functions of even arbitrarily weak, ostensibly non-invasive, measurements break time-reversal symmetry in general [16, 17]. We solve the general problem by considering two complementary measurement sequences, one pointing into the future, and another into the past, that are time-reversed inverses of each other. We name these complements *Janus sequences*. For qubits, this general solution takes a particularly simple form that can be taken to the limit of time-continuous measurements, producing so-called quantum trajectories [18–23]. Quantum trajectory theory eliminates any remaining separation between Schrödinger equation dynamics and measurement disturbance, and replaces them with a single stochastic process that includes both. In recent years there has been strong experimental evidence in support of this type of conditional quantum dynamics; see e.g. Refs. [24–29] for a sample of current works.

We pose the time-reversibility problem in the following way: Suppose we are given a movie of stochastic quantum state dynamics along with its associated noisy detector output (a sort of "soundtrack" for the movie). We are then asked to determine whether the movie shows the forward evolution of the state, or whether the movie has been reversed, as depicted in Fig. 1. In the simplest case of a monitored qubit, we find that such a movie played backwards obeys time-reversed equations of motion if we also flip the sign of its soundtrack (measurement record). We stress that this is *not* a microscopic time-reversal of the measurement apparatus, nor is it a backward inference (past quantum state) kind of dynamics [30, 31]—our time reversal shows equally valid forward dynamics. After watching the movie for a longer duration while listening to its soundtrack, we can distinguish a forward from a time-reversed movie with increasing certainty in order to probabilistically find the arrow of time.

The usual thermodynamic arrow of time refers to low entropy configurations changing into high entropy under the thermalization process (see e.g. Ref. [32]). Our quantum measurement arrow time is more similar to the situation in non-equilibrium statistical physics. The time arrow can be related to the distinguishability of the distributions of quantities, such as work done between a given process and its time reversed twin [33–35]. In such situations the system entropy can even decrease [36]. However, these treatments typically have a temperature associated with the environment, and the system begins in equilibrium. In contrast, we consider a completely nonequilibrium situation, where all fluctuations are fundamentally quantum in nature arising from the measurement process, with the system starting in an arbitrary state.

Note that to achieve perfect time reversal we must not lose information to the environment, other than to an ideal, quantum-limited, detector. That is, we must consider a system that is being monitored without additional noise or "quantum friction", just as in classical physics. As a physical example, a superconducting qubit like a transmon [37] may be continuously monitored with microwaves using circuit quantum electrodynamics [24, 25], yielding a time-dependent (noisy) homodyne quadrature readout $I(t)$; such monitoring would yield time-reversible evolution if the amplifier were quantum limited with no loss in the readout chain (i.e., no readout inefficiency) [38], and were otherwise decoupled from its environment. Similarly, the current $I(t)$ flowing through a quantum point contact can continuously monitor a double-quantum dot charge qubit with nearly quantum-limited efficiency [21, 39, 40].

Continuous qubit measurement.—We consider such a quantum-limited continuous qubit measurement as an illustrative example, which we will later generalize to an arbitrary sequence of measurements. Specifically, continuously monitoring the σ_z observable produces a noisy informational signal $r(t)$, which is shifted and rescaled so that its average is ± 1 if the quantum system is prepared in eigenstates of σ_z. The stochastic nature of $r(t)$ arises from the intrinsic quantum fluctuations in the detector, e.g., quantum vacuum fluctuations [24], which sets the characteristic measurement time τ for achieving unit signal-to-noise ratio. We also consider a qubit Hamiltonian, given by $H = \hbar\Omega\sigma_y/2$, produced (for example) by a microwave drive, which causes rotation in the x-z plane of the Bloch sphere.

Collecting a particular measurement trace from the detector $r(t)$ allows us to infer the conditional quantum state as a function of time, $\{x(t), y(t), z(t)\}$, from the Bloch equations of motion,

$$\dot{x} = -\Omega z - \frac{xzr}{\tau}, \quad \dot{y} = -\frac{yzr}{\tau}, \quad \dot{z} = \Omega x + \frac{(1-z^2)r}{\tau}, \quad (1)$$

also known as a quantum trajectory. These equations assume ideal conditions, including efficient detection and Markovian evolution, so any residual entanglement between the qubit and detection apparatus is assumed to vanish (e.g., a microwave resonator must operate in the "bad cavity" limit). For non-differentiable $r(t)$, these equations remain valid as stochastic differential equations with time-symmetric (Stratonovich) derivatives [21]. We observe that Eqs. (1) are time reversal invariant under the transformations: $t \mapsto -t$, $\Omega \mapsto -\Omega$, keeping x, y, z invariant, provided that the *record is also flipped*, $r \mapsto -r$. With these changes, a quantum movie for a single measurement run is the same when played backward, as illustrated in Fig. 1 for the special case of $y = 0$, thus restoring time symmetry. Specifically, a forward trajectory with initial state $\rho_i := (x_i, y_i, z_i)$ at time $t = 0$, final state $\rho_f := (x_f, y_f, z_f)$ at time $t = T$, and record $r_F(t)$ is equivalent to a backwards trajectory with initial state ρ_f, final state ρ_i, and reversed record $r_B(t) = -r_F(T-t)$ [41].

Arrow of time.—We will now show that although such continuous qubit measurement dynamics is time reversal invariant, we can nevertheless probabilistically distinguish forward and backward evolution, yielding a statistical arrow of time. The task of distinguishing a past-to-future versus a future-to-past dynamics can be phrased as a hypothesis testing problem: is the movie shown in Fig. 1 of duration T running forward (F) or backward (B)?

To test these hypotheses, let the prior probabilities $P(F)$ and $P(B) = 1 - P(F)$ indicate our initial guess whether the movie is running forward or backward. Let $P_F(r(t)) = P(r(t)|\rho_i)$ be the probability density of obtaining the measurement record $r(t)$, supposing the movie is running forward from an initial state ρ_i; similarly, let $P_B(r(t)) = P(-r(T-t)|\rho_f)$ be the probability density that supposes the movie is running backward from a final state ρ_f. We then use Bayes' rule to compute the likelihood that the movie is running in the forward direction given the movie and its soundtrack,

$$P(F|r(t)) = \frac{P_F(r(t))P(F)}{P_F(r(t))P(F) + P_B(r(t))P(B)}. \quad (2)$$

If we have no *a priori* bias about this question, we set $P(B) = P(F) = 1/2$, to find the likelihood

$$P(F|r(t)) = \frac{\mathcal{R}}{1+\mathcal{R}}, \quad \mathcal{R} = \frac{P_F(r(t))}{P_B(r(t))}. \quad (3)$$

We therefore conclude that we can make no statistical inference *only* if the forward and backward probability densities are identical (i.e., the probability ratio $\mathcal{R} = 1$). The logarithm of this ratio, $\ln\mathcal{R}$, is thus a natural discriminator, with positive values inferring forward motion and negative values inferring backward motion. The mean value $\overline{\ln\mathcal{R}}$ over forward-generated trajectories thus gives an estimate of the statistical arrow of time for continuous quantum measurement, also named the "length of time's arrow" [42]. It is similar to the relative entropy (also known as the Kullback-Leibler divergence) between forward and backwards distributions. Researchers in nonequilibrium statistical physics have used analogous arrow-of-time hypothesis discrimination to quantify the entropy production (or irreversibility) of mesoscopic systems [34, 42–45]. There has been recent cross-pollination of the methodology in these fields [46, 47].

To find the relative probability densities of the trajectories $r(t)$ versus $-r(T-t)$, given a quantum trajectory, we may expand the distribution of results to first order in a small time-step to find $P(r(t)|\rho_i) \propto \exp[-\int_0^T dt'(r(t')^2 - 2r(t')z(t') + 1)/2\tau]$ [48, 49], where the backwards distribution simply time-reverses the integral, and flips the sign of r at every time [50]. The arrow of time ratio $\mathcal{R}$ Eq. (3) is given in terms of the probability densities of the forward trajectories $r(t)$ and the backward trajectories $-r(T-t)$, so [50]

$$\ln\mathcal{R} = \frac{2}{\tau}\int_0^T dt\, r(t)z(t). \quad (4)$$

This relative log-likelihood will then categorize each run of the experiment as being more likely to be running forward in time, $\ln\mathcal{R} > 0$, or backward in time, $\ln\mathcal{R} < 0$. In the latter case, we interpret Eq. (4) to mean that the result $r(t)$ "disagrees" with the state component $z(t)$ it is estimating (has the opposite sign) more often than it "agrees" with it during the run, making reversed time-evolution more likely.

In the Markovian limit, applicable on a time scale longer than any correlation of the detector or detector resolution time, the detector fluctuations may be approximated as an additive white noise stochastic process [12, 21, 23], with the detector output signal given by $r(t) = z(t) + \sqrt{\tau}\,\xi(t)$, where $\xi(t)$ is a unit variance, delta-correlated stochastic variable. This decomposition clearly shows the breaking of time-reversal symmetry on the statistical level, since inverting the sign of r while keeping z invariant requires a statistically anomalous time-reversed realization of the noise. Using this decomposition, the average of the relative log-likelihood in Eq. (4) may be calculated to give the positive-definite value $\overline{\ln\mathcal{R}} = (1/\tau)\int_0^T dt\ (1 + \langle z(t)^2\rangle)$ after the stochastic average (see Supplemental Material [51]), indicating a forward time arrow.

Numerical results for the arrow of time.—Consider the case of persistent, diffusive, Rabi oscillations [52], when $\Omega \gg \tau^{-1}$, so that the qubit performs oscillations in the x-z plane with phase diffusion. For $T > 2\pi/\Omega$, the Rabi oscillations average to $\langle z^2\rangle \approx \langle(\cos\Omega t)^2\rangle = 1/2$, so

$$\overline{\ln\mathcal{R}} \approx \frac{3T}{2\tau}. \quad (5)$$

In this case, the average distinguishability of the forward from the backward arrow of time increases linearly with the duration of the measurement run.

This statement may be made more precise by examining the entire distribution of $\ln\mathcal{R}$, which is shown from numerical simulations in Fig. 2 using a Rabi period of $2\pi/\Omega = 0.5\tau$. For durations longer than the Rabi period, $T > 2\pi/\Omega$, the mean grows linearly with the duration of the experiment, as predicted in (5). A calculation similar to the mean gives an approximate variance of $2T/\tau$. The full distribution of $\ln\mathcal{R}$ becomes a broad Gaussian in this regime of $T > 2\pi/\Omega$, with the aforementioned mean and variance, see Fig. 2c,d. Thus, the probability of erroneously guessing a forward trajectory to be backward is the area of the negative tail of the Gaussian

$$P_{\text{err}} \approx \frac{1}{2}\left[1 - \text{Erf}\left(\frac{3}{4}\sqrt{\frac{T}{\tau}}\right)\right]. \quad (6)$$

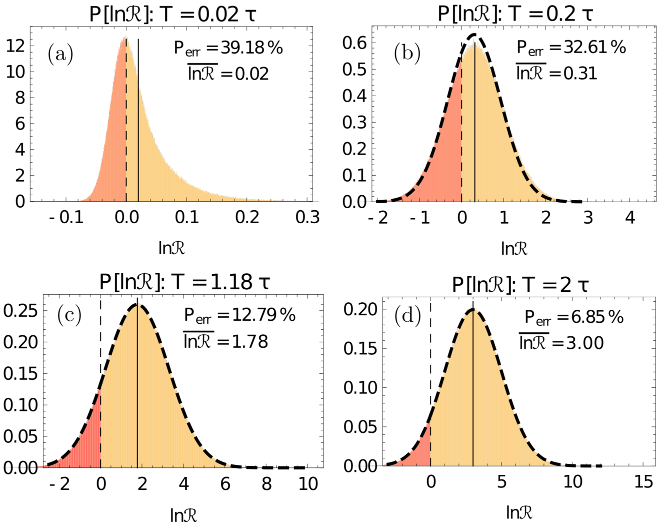

FIG. 2. Histograms of $\ln \mathcal{R}$ for 2×10^6 runs of monitored Rabi oscillations with period $2\pi/\Omega = 0.5\tau$ and measurement time τ, starting from $x(t=0) = 1$. The ratio of forward to backward probability distributions $\mathcal{R}$ discriminates the likelihood for a trajectory to be forward ($\ln \mathcal{R} > 0$) or backward ($\ln \mathcal{R} < 0$). The probability $P_{\rm err}$ of erroneously guessing reversed time is the red shaded area to the left of the vertical dashed line at $\ln \mathcal{R} = 0$. The vertical solid line is the mean $\overline{\ln \mathcal{R}}$, a measure of the statistical arrow of time. The durations $T = (0.02, 0.2, 1.18, 2.0)\tau$ are shown in subfigures (a–d), showing that for $T \geq 2\pi/\Omega$ (c,d), the distribution has converged to a Gaussian with mean $3T/2\tau$ and variance $2T/\tau$ (dashed profile in b,c,d).

Consequently, if we wish to distinguish the forward-in-time arrow from the backward arrow to greater than n standard deviations, we require a duration $T \geq 8n^2\tau/9$. As can be seen from the histograms in Fig. 2, even for reasonably long duration T it is common to observe readouts that appear to be reversed. We show examples of this, as well as present the simpler no-drive case, in the Supplemental Material [51].

Janus measurement sequences.—We now generalize the simple qubit example to arbitrary sequences of generalized measurements. First, we reverse the direction of time for Schrödinger equation evolution in the standard way [53], by introducing an anti-unitary time-reversal operation Θ, satisfying $\langle \Theta\Phi|\Theta\Psi\rangle = \langle \Psi|\Phi\rangle$. In the case of position wavefunctions, Θ is simply the complex conjugate operation. More generally, Θ must correctly time-reverse all physical observables such as position, $\Theta\mathbf{x}\Theta^{-1} = \mathbf{x}$, momentum $\Theta\mathbf{p}\Theta^{-1} = -\mathbf{p}$, and spin $\Theta\mathbf{S}\Theta^{-1} = -\mathbf{S}$, as well as the sign of any external magnetic field, $B \to -B$. Applying the time-reversal operator Θ to a quantum state $|\Psi(t)\rangle$ inverts its temporal meaning, such that forward unitary time evolution U_t correctly rewinds the dynamics: $U_t\Theta|\Psi(t)\rangle = \Theta|\Psi(0)\rangle$.

Second, we add sequences of generalized measurements to the unitary dynamics. We first consider a forward sequence of measurements in time, $A, B, C, \ldots$, which will be one of two distinct Janus sequences that we will need. This sequence has possible measurement results $j = a, b, c, \ldots$, each of which will partially collapse the quantum state according to a measurement operator, $M_a, M_b, M_c, \ldots$. An initial state $|\Psi\rangle$ thus evolves into

$$|\Phi\rangle \propto \ldots M_c M_b M_a |\Psi\rangle \equiv M_F|\Psi\rangle, \tag{7}$$

where $M_F \equiv \ldots M_c M_b M_a$. Note that we include any intermediate unitary time evolution U_t inside the Kraus operators. This formulation is quite general, so the measurement results may be discrete or continuous variables.

We next introduce a corresponding backwards Janus sequence, which is a series of (in general different) measurements $A', B', C', \ldots$, with outcomes $j' = a', b', c', \ldots$ and Kraus operators $M_B \equiv M_{a'}M_{b'}M_{c'}\ldots$ also applied sequentially to the system, but in reverse order to the time-reversed "initial" state $\Theta|\Phi\rangle$. Crucially, for *some possible* results (j, j') of both sequences, we wish for the system state to rewind its path, restoring the initial (time-reversed) state: $M_B\Theta|\Phi\rangle \propto \Theta|\Psi\rangle$. We can find the condition for this to happen by inserting $\mathbf{1} = \Theta^{-1}\Theta$ between every pair of operators, yielding

$$M_{j'} \propto (\Theta M_j \Theta^{-1})^{-1}, \quad (j, j') = (a, a'), (b, b'), \ldots \tag{8}$$

That is, each measurement operator of the backward Janus sequence must be proportional to the *inverse* time-reversed measurement operator of the forward Janus sequence. In the special case of no measurement collapse, this constraint correctly reproduces the expected relationship between the unitary time-evolution operators and the anti-unitary time-reversal operators [7, 53]. For a single measurement, this condition may be understood as an application of quantum measurement uncollapse [11–15]. We emphasize that such an inverse operator may always be constructed as a measurement operator belonging to some positive operator valued measure (POVM) set [12] (see Ref. [54] for an introduction to generalized measurements). As alluded to above, there is no guarantee that the correct Janus sequences will happen; however, what is important is that such a pair of sequences is *physically possible.*

Switching from a forward to a backward Janus sequence generalizes the need for inverting the measurement record $r(t)$ in the qubit case of Fig 1. Now consider the analogous game, where a movie of the state dynamics from one of a Janus sequence of measurements is presented to us, along with a corresponding sequence of forward measurements $(A, B, C, \ldots)$, or backward measurements $(\ldots, C', B', A')$. We must then guess whether the movie with one of these two soundtracks is running backward or forward in time. There is no way to tell from the dynamics: Each step in each quantum state movie direction with matched soundtrack is a *possible* forward evolution.

Nevetheless, as with qubit the case before, we can still statistically discern the arrow of time. The likelihood functions to test the forward or backward hypotheses are constructed directly from the collective forward Janus measurement operator $M_F = \prod_j M_j$, and collective backward Janus measurement operator $M_B = \prod_{j'} M_{j'}$, as used above. The probability of all of the measurement results, given (known) forward or reverse Janus sequences is $P_F(a, b, c, \ldots) = \|M_F|\Psi\rangle\|^2$, or $P_B(\ldots, c', b', a') = \|M_B \Theta|\Phi\rangle\|^2$, so the discriminator that generalizes Eq. (3) is the log of their ratio $\mathcal{R} = P_F(\{j\})/P_B(\{j'\})$.

Conclusions.—We find that it is possible to time-reverse the dynamics of a quantum system, even when it is being measured. For every nonprojective measurement, the forward measurement dynamics has an associated backwards measurement dynamics. Therefore, given a sequence of measurements and the quantum state trajectory ("the movie"), it is impossible to say whether the movie is being played forward or backward from dynamics alone. However, by examining the relative probability of whether the movie is playing forward or backward, given the measurement results (its "soundtrack"), a statistical arrow time still emerges. The physical origin of this statistical arrow is the dynamical contraction (collapse) of the set of possible final states that are compatible with the observed measurement record. Of the two possible evolutions, one will display a more likely contractive evolution. We have shown how to test both aspects of the time-arrow question both in continuously measured qubits, as well as any measurement sequence by constructing a backwards Janus sequence which would show a possible time-reversed quantum state movie consistent with the original measurement sequence.

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