Z_{3} Parafermionic Zero Modes without Andreev Backscattering from the 2/3 Fractional Quantum Hall State

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$\mathbb{Z}_3$ parafermionic zero modes without Andreev backscattering from the 2/3 fractional quantum Hall state

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Parafermionic zero modes are a novel set of excitations displaying non-Abelian statistics somewhat richer than that of Majorana modes. These modes are predicted to occur when nearby fractional quantum Hall edge states are gapped by an interposed superconductor. Despite substantial experimental progress, we argue that the necessary crossed Andreev reflection in this arrangement is a challenging milestone to reach. We propose a superconducting quantum dot array structure on a fractional quantum Hall edge that can lead to parafermionic zero modes from coherent superconducting forward scattering on a quantum Hall edge. Such coherent forward scattering has already been demonstrated in recent experiments. We show that for a spin-singlet superconductor interacting with loops of spin unpolarized 2/3 fractional quantum edge, even an array size of order ten should allow one to systematically tune into a parafermionic degeneracy.

Introduction.—Theoretical understanding and experimental realization of non-Abelian anyons has attracted considerable attention in the past few years. In addition to being of fundamental interest as a dramatic manifestation of a topological phase, non-Abelian anyons also have potential applications as building blocks for topological quantum computers [1]. Majorana zero modes (MZMs) [2–7] provide the simplest and experimentally the most promising example of non-Abelian anyons. So far, most of the effort in searching for non-Abelian anyons has focused on MZMs. Following a series of theoretical proposals [8–11], suggestive experimental evidence of MZMs has been observed in semiconductor/superconductor heterostructures [12–18]. Despite their fascinating properties, MZMs are non-Abelian anyons of the Ising ($\mathbb{Z}_2$) type. Universal quantum computation cannot be implemented using braiding of $\mathbb{Z}_2$ anyons alone. Therefore, searching for a computationally richer set of anyons seems necessary.

Parafermionic zero modes (PZMs) [19–21] (also known as fractional MZMs) provide an example of such computationally rich (still not universal) anyons. They can be thought of as $\mathbb{Z}_n$ generalizations of MZMs. Similar to MZMs, $\mathbb{Z}_n$ PZMs are associated with n fold degeneracy of the ground state that is robust to all local perturbations. Due to fundamental restrictions set on possible topological phases in strictly one-dimensional systems[22, 23], PZMs cannot exist in isolated one dimensional systems. However, recently it was realized that boundaries of two-dimensional systems can circumvent these restrictions. It was explicitly shown that PZMs emerge at the one-dimensional boundary of two counter-propagating fractional quantum Hall (FQH) edges coupled with superconducting contacts [24–27]. These setups greatly resemble one canonical proposal used to realize MZMs[28], with the role of topological insulator played by a pair of opposite-chirality FQH states. All of existing proposals (involving superconductors) require two main ingredients, induced superconductivity via coupling FQH edge state to a superconductor and crossed Andreev tunneling between two edges. The first requirement has already been achieved in experiments [29–31]. However, the second requirement is likely to be difficult to achieve experimentally due to disruption of FQHE states placed adjacent to a superconductor. This is because strong coupling to the superconductor is likely to change density in the surrounding 2D electron gas, pushing the FQHE away from the superconductor. The amplitude of quasiparticle tunneling between edges would then be reduced by the increased distance and the Fermi wavevector in the intervening superconducting region. Experimental evidence[31] also seems to suggest that observable Crossed Andreev tunneling amplitude is much too weak to generate a coherent gap. In addition, disorder in the superconductor would likely randomize the superconducting backscattering, which is likely to destabilize the topological phase as in the case of MZMs[32].

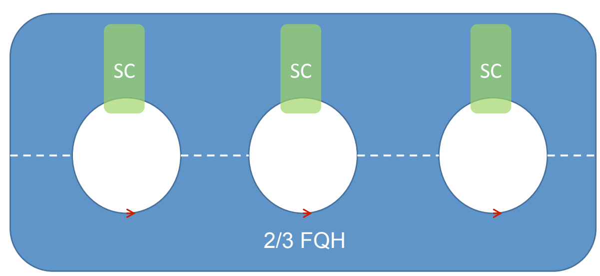

FIG. 1: Top view of the system, comprised of a linear array of superconductors coupled to loops of FQH edge states, different loops are connected via quasiparticle hopping

In this letter we propose a practical scheme to realize $\mathbb{Z}_3$ PZMs from the spin unpolarized 2/3 fractional quantum Hall state using superconducting contacts without cross-Andreev tunneling, which can be realized in present experiments. Our system is comprised of a linear array of FQH edge loops; each one of these loops is coupled to a superconductor through proximity effect, and separate loops are connected via quasiparticle tunneling. Superconductors are separated from FQH bulk using a barrier. Different superconductors are connected using thin wires to ensure they have the same superconducting phase. Strength of quasiparticle tunneling can be controlled with a gate voltage. A top view of this setup is shown in Fig. 1. We use a combination of analytical and numerical methods to study this model and show that for realistic values of parameters, at relatively small chain lengths (order ten loops) this system can be tuned to a topological phase hosting $\mathbb{Z}_3$ PZMs.

Model.—We begin by studying a single loop coupled to a superconductor. Assuming $SU(2)$ symmetry, the effective Hamiltonian describing the charge part of $\nu = 2/3$ FQH edge state is given by the following chiral boson theory[33, 34],

$$H_{edge} = \int_0^L dx[\frac{u}{4\pi\nu}(\partial_x\varphi(x))^2 - \frac{um_\mu}{2L}\partial_x\varphi(x)] \quad (1)$$

where L is length of the loop, u is mode velocity, m_μ is the gate controlled dimensionless chemical potential (as opposed to actual chemical potential $\mu = \frac{um_\mu}{2L}$) and $\varphi(x)$ is the chiral boson field that is defined in terms of charge density operator as $\rho = \frac{1}{2\pi}\partial_x\varphi(x)$. The $\varphi(x)$ field obeys the commutation relation $[\varphi(x), \partial_y\varphi(y)] = 2i\pi\nu\delta(x-y)$. Using this relation we can write the charge 2/3 spinless quasiparticle creation operator as $e^{i\varphi(x)}$ and charge 2 Cooper pair creation operator as $e^{3i\varphi(x)}$. The neutral mode, which does not couple to the SC and is expected to be non-degenerate and gapped will be ignored in the rest of this work. Charge 1/3 excitations that involve the neutral mode are also gapped out. The edge Hamiltonian H_{edge} can be diagonalized [35] by mode expanding $\varphi(x)$ as

$$\varphi(x) = \frac{2\pi\hat{n}\nu x}{L} + \hat{\varphi}_0 + \sum_{k=0}^{K_{max}}\left[-i\sqrt{\frac{\nu}{k}}a_k^\dagger e^{2\pi ikx/L} + i\sqrt{\frac{\nu}{k}}a_k e^{-2\pi ikx/L}\right] \quad (2)$$

where $a_k, a_k^\dagger$ are the kth momentum boson creation and annihilation operator for $k \in \mathbb{N}$, $\hat{\varphi}_0, \hat{n}$ are zero mode phase and number operators, respectively, and K_{max} is the momentum cutoff.

Now we can rewrite Eq.(1) as,

$$H_{edge} = \frac{u\pi\nu}{L}\left(\hat{n}^2 - m_\mu\hat{n} + \frac{2}{\nu}\hat{P}\right) + const \quad (3)$$

where $\hat{P} = \sum_{k=0}^{K_{max}} k\hat{a}_k^\dagger\hat{a}_k$ is the total momentum operator. When dimensionless chemical potential m_μ is tuned by gate voltage to integer values the spectrum is invariant under changing $n = m$ to $n = -m + m_\mu$. For odd m_μ this translates into a two fold ground state degeneracy. This degeneracy survives the addition of superconductivity and will play a crucial role later on.

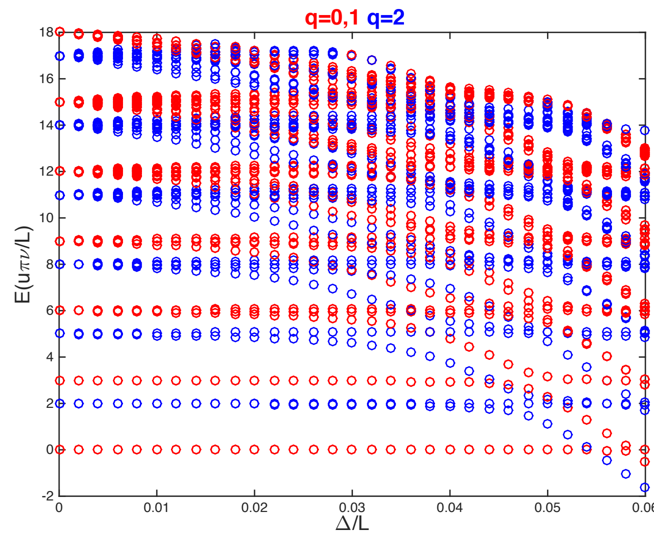

FIG. 2: Low-lying spectrum for "pseudo-point-like" superconductivity as a function of $\frac{\Delta}{L}$. Here $m_\mu = 1$, $K_{max} = 4$ and q is the fractional charge modulo three $q = \text{mod}(n, 3)$. All red circles are two fold degenerate, blue circles are non-degenerate.

The effect of superconductor on a single loop can be modeled by,

$$H_{sc} = \int_0^L dx\frac{\Delta(x)}{L}e^{-\frac{2\pi ixm_\mu}{L}}e^{3i\varphi(x)} + h.c \quad (4)$$

describing tunneling of Cooper pairs to and from superconductor. Here $\Delta(x)$ corresponds to position-dependent Cooper pair tunneling amplitude and $e^{-\frac{2\pi ixm_\mu}{L}}$ is the phase factor taking into account the chemical potential mismatch between the FQH and the (grounded) superconductor. Fourier transforming $\Delta(x) = \sum_k \Delta(k)e^{\frac{2\pi ikx}{L}}$ and mode expanding $\varphi(x)$ (as in Eq. (2)) allows us to write the only nonzero matrix elements of H_{sc},

$$\langle n_0 \pm 3, \{m_k\}|H_{sc}|n_0, \{n_k\}\rangle = \sum_k \Delta(k)\delta_{(\Delta E \pm 3k)} \times \langle n_0 \pm 3, \{m_k\}|e^{\pm 3i\varphi(0)}|n_0, \{n_k\}\rangle, \quad (5)$$

where $\Delta E = E(n_0 \pm 3, \{m_k\}) - E(n_0, \{n_k\})$ is the energy difference between the initial and the final state, and $E = \frac{u\pi\nu}{L}(n^2 - m_\mu n + 3P)$ is the bare edge energy of each state in accordance with Eq. (3). Equation (5) implies that special case of uniform superconductivity $\Delta(x) = \Delta_0$ leads to the additional conserved quantity H_{edge}, as $[H_{sc}, H_{edge}] = 0$. Though convenient, this symmetry is not generic and is not used in this letter.

Inclusion of H_{sc} reduces conservation of fractional charge n to conservation of $q = \text{mod}(n, 3)$, which only takes three values $q = 0, 1, 2$. Using this restriction we can divide the system into three independent charge sectors with $q = 0, 1, 2$. For integer m_μ, the $n = 3m$ to $n = -3m + m_\mu$ symmetry of the non-superconducting edge now translates into degeneracy of two of the charge sectors, for example at $m_\mu = 1$ the two sectors $q = 0, 1$ will become degenerate.

Using Eq.(5) we can numerically calculate the spectrum of a single loop $H = H_{edge} + H_{sc}$. We set $m_\mu = 1$ and assume a "pseudo-point-like" superconductivity such that $\Delta(k) = \Delta$ for $|k| \leqslant K$, where we have chosen $K = K_{max}$ i.e. the momentum cutoff defined in Eq.(2) (note that K and K_{max} did not have to be equal). The low-lying part of the spectrum is plotted in Fig. 2. This plot shows that the ground state is separated from the rest of the spectrum by a gap for a range of Δ. However, ground state degeneracy (between $q = 0, 1$) remains twofold with $q = 2$ split.

Effective Hamiltonian.—In absence of superconductivity and for odd values of dimensionless chemical potential ($m_\mu = 2n - 1$), ground-state of system is twofold degenerate and is separated from the excited states by a gap for a range of Δ. The two ground states can be labeled by fractional charge $q = 0, 1$. Therefore as long as we choose Δ in this range and restrict the ratio of hopping amplitude to the energy gap $t/\Delta E$ to be small, we can apply a Schrieffer-Wolff transformation to obtain an effective Hamiltonian defined in the Hilbert space spanned by ground states of single loops. This emergent Hilbert space has only two states per site ($q = 0, 1$) and therefore can be thought of as a chain of spin $1/2$ sites, where states with spin up/down correspond to single loop ground states with $q = 1, 0$.

To calculate the effective Hamiltonian, we start with Hamiltonian describing quasiparticle hopping between different loops,

$$H_{tunnel} = t \sum_i e^{i\varphi_i(L/2)} e^{-i\varphi_{i+1}(0)} + H.c. \qquad (6)$$

Note that $e^{i\varphi_i(x)}$ is an anyonic operator and has nontrivial commutation relation with other anyonic operators. For different sites we can write,

$$e^{i\varphi_i(x)} e^{i\varphi_j(x')} = e^{i\varphi_j(x')} e^{i\varphi_i(x)} e^{i\pi\nu \mathrm{sgn}(j-i)}. \qquad (7)$$

Using a generalized Jordan-Wigner string we can define the new field variables $\tilde{\varphi}(x)$,

$$e^{i\varphi_i(x)} = e^{i\pi\nu \sum_{j<i} \tilde{n}_j} e^{i\tilde{\varphi}_i(x)}. \qquad (8)$$

Advantage of these new field variables is that they commute trivially between different sites $[e^{i\tilde{\varphi}_i(x)}, e^{i\tilde{\varphi}_j(x')}] \propto \delta_{i,j}$, and therefore act strictly on the local loop Hilbert space. Now we can rewrite H_{tunnel} as,

$$H_{tunnel} = t \sum_i e^{i\tilde{\varphi}_i(L/2)} e^{-i\pi\nu\tilde{n}_{0,i}} e^{-i\tilde{\varphi}_{i+1}(0)} + H.c.. \qquad (9)$$

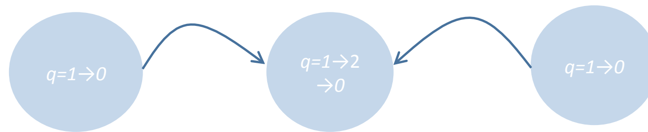

FIG. 3: $\sigma^-_{i-1}\sigma^-_i\sigma^-_{i+1}$ term as a second order process in perturbation theory. q is the fractional charge modulo three. Two fractional charge are tunneled to the middle site, one from each neighbour

Chain length	Ground state energy	1st excited state energy	2nd excited state energy	3rd excited state energy
10	-6.818	-6.696	-6.696	-6.149
40	-29.681	-29.677	-29.677	-29.384
100	-75.524	-75.524	-75.524	-75.267

TABLE I: DMRG calculation results for "pseudo point-like" superconductivity(defined earlier) at $t = 1$, $\Delta/L = 0.046$ and momentum cutoff $K_{max} = 4$

To second order in perturbation theory we can write the low energy effective Hamiltonian as,

$$H_{eff} = PH_{tunnel}P - PH_{tunnel}\frac{(1-P)}{H_0}H_{tunnel}P \qquad (10)$$
$$+ O(t^3/\Delta E^2)$$

where P is the single loop ground state projection operator defined as $P = \sum_i(|q = i\rangle\langle q = i|)$, and H_0 is the single loop Hamiltonian (shifted to set ground state energy to zero).

Putting everything together we get (Details in the supplementary material),

$$H_{eff} = \sum_i \Big[(t\alpha_0 e^{i\pi/3} - t^2\alpha_1 e^{-2i\pi/3})\sigma^+_i\sigma^-_{i+1} \qquad (11)$$
$$- t^2 e^{2i\pi/3}\gamma\sigma^+_{i-1}\sigma^-_{i+1} - t^2\beta\sigma^z_i\sigma^z_{i+1}$$
$$+ t^2\lambda\sigma^+_{i-1}\sigma^+_i\sigma^+_{i+1}\Big] + H.c. + O(t^3/\Delta E^2)$$

where σ's are the usual Pauli matrices and $\alpha, \beta, \gamma, \lambda$ are parameters calculated in the supplementary material. Note that all terms in the Hamiltonian conserve fractional charge (spin) modulo three and are also $\mathbb{Z}_2$ symmetric under $\sigma^z \to -\sigma^z$, this $\mathbb{Z}_2$ symmetry can be associated with the $\varphi \to -\varphi + \frac{2\pi m_\mu \nu x}{L}$ symmetry of the original Hamiltonian. Note that presence of the term $t^2\lambda\sigma^+_{i-1}\sigma^+_i\sigma^+_{i+1}$ requires nonzero superconductivity, since without Δ fractional charge (spin) has to be conserved. As seen in Fig. 3, this term, which arises at second order in tunneling, requires H_{sc} so that q=3 may be converted to q=0 by removal of a Cooper pair.

Analysis.—Without superconductivity (σ_z non-conserving terms), the conservation of σ_z ensures a gapless state with low-energy Luttinger liquid Hamiltonian where $\sigma_z \sim \nabla\phi$. In this description, the superconducting term $t^2\lambda\sigma^+_{i-1}\sigma^+_i\sigma^+_{i+1}$ is represented as $g\cos(3\theta)$, which converts the Luttinger liquid to a Sine-Gordon model. For the correct choice of parameters the superconductivity induced term $g\cos(3\theta)$ becomes perturbatively relevant [36] and gaps out the system into a topological phase with a $\mathbb{Z}_3$ parafermionic degeneracy, where each ground state corresponds to the phase θ being locked at one of the three minima of the $\cos(3\theta)$ term[24–27]. To check whether this degeneracy occurs in a our system (i.e. the Hamiltonian in Eq. (11)) with realistic values of the parameters, we numerically study the Hamiltonian in Eq.(11) using the DMRG method. DMRG calculations were performed using the ITensor library[37]. Sample results of this calculation are shown in Table. I. These results confirm existence of a three-fold degeneracy for reasonable parameters. The degeneracy is weakly split for small chain lengths $N \approx 10$ and is more pronounced at longer lengths, as expected from a true topological degeneracy[38]. These results are expected to insensitive to temperature since the energy gap is of order of $u\pi\nu/L$, which for loop lengths of order of several quasiparticle radiuses (Each quasiparticle radius is of order of several magnetic lengths) would be of the order of the FQH gap, which is much larger than the temperature at which the FQH state is observed.

We have repeated this calculation with $K_{max} = 3, 4, 5$ (larger values are numerically hard to simulate) and found that the results are qualitatively insensitive to the exact value of K_{max}. Moreover for a realistic system K_{max} can be approximated as inverse of loop length in units of quasiparticle radius, which assuming loops lengths of order of several quasiparticle radiuses, makes are our calculation to fall within the correct physical regime.

In a reality the value of m_μ cannot be exactly set to an odd integer value. Effect of this imperfect tuning is described by $h\sum_i \sigma^z_i$, where h is the energy offset between the two states caused by the chemical potential shift. In the Luttinger liquid description this term can be represented by adding a $\nabla\phi$ term to the the Sine-Gordon model. Phase diagram of such Sine-Gordon model in an external field is well studied[36] , basically showing that the gapped phase persists as long as h is smaller than the energy gap calculated without the chemical potential offset (in our case previously calculated using DMRG method).

For experimental purposes, it's also important to discuss the role of disorder. We emphasize that in principle all topological phases of matter are robust to local perturbations that are small compared to the system gap size. However, to be concrete we consider the effect of random fluctuations in loop lengths, this effect is likely to be significant in a real experiment. We have performed numerical calculations on this system (Details in the Supplementary Material), and found that our results are at least robust to significant fluctuations in loops lengths of order of $\frac{\Delta L}{L} \approx 0.3$.

An alternative interesting limit is that of "true" point-like superconducting contacts (as opposed to pseudo point like defined before), that is $\Delta(k) = \Delta\ \forall k$. This limit is particularly appealing, as in this case analytical results may be obtained for large values of Δ. Following the formalism developed in Ref.[39], we show (Details in the Supplementary Material) that in large Δ limit system is described by set of decoupled harmonic oscillators, and that in this limit all three fractional charge sectors become degenerate. Analogous to the previous case (small Δ) as long as $t/\Delta E$ is small, we can use Wolff transformation to find an effective Hamiltonian. The effective Hilbert space of each site is three-dimensional (corresponding to three fractional charge sectors) and can be though of as a three state clock model. In this limit, it's useful to define,

$$\alpha_{2j-1} = \frac{e^{-i\varphi_j(0)}}{A(0)}; \alpha_{2j} = \frac{e^{-i\varphi_j(L/2)}}{A(L/2)} \qquad (12)$$

where $A(x) = (\prod_i \langle q = i|e^{-i\varphi(x)}|q = i+1\rangle)^{1/3}$ is a normalization factor. Within the effective Hilbert space these operators are the usual parafermionic operators, that is $\alpha_j^3 = 1$ and $\alpha_j\alpha_{j'} = \alpha_{j'}\alpha_j e^{i\frac{2\pi}{3}sgn(j'-j)}$. Using these variables and the Hamiltonians in Eqs.(6) and (10) we arrive at,

$$H_{eff} = t\sum_i (A^*(L/2)A(0)\alpha^\dagger_{2j}\alpha_{2j+1} + H.c.) + O(t^2/\Delta E) \qquad (13)$$

in this form presence of the parafermionic edge zero modes $(\alpha_1, \alpha_{2N+1})$ is already manifest[20]. Note that in contrast to Eq.(11), here the calculation has been done to first order in t. Using the usual clock model variables [40] we can write the Hamiltonian in a more familiar form,

$$H_{eff} = \sum_i (-J\sigma^\dagger_j\sigma_{j+1} + H.c.) + O(t^2/\Delta E) \qquad (14)$$

where $J = tA^*(L/2)A(0)e^{i\pi/3}$ and

$$\sigma = \begin{pmatrix} 1 & & \\ & \omega & \\ & & \omega^2 \end{pmatrix}, \omega = e^{2\pi i/3} \qquad (15)$$

is the "clock" operator. In the large but finite Δ regime, three fold the ground state degeneracy of single loops is not exact. We can take this energy difference into account by adding a term $h(\tau_j + \tau^\dagger_j)$ to Eq.(14) where h is the energy difference of charge sectors $q = 0, 1$ with the charge sector $q = 2$. Estimates for the value of h can be found in the supplementary material. Putting everything together we have,

$$H_{eff} = \sum_i (-J\sigma_j^\dagger \sigma_{j+1} + H.c.) + h(\tau_i + \tau_i^\dagger) + O(t^2/\Delta E) \quad (16)$$

Note that σ_i is a non-local operator in the physical system of interest. This is an important point as locality prevents the introduction of a Hamiltonian term proportional to σ_i. With this constraint and for small values of h (h can be made arbitrarily small by choosing large enough Δ) the Hamiltonian in Eq.(16) is well known to be in a topological phase with three-fold degeneracy[19].

We would like to point out that it is possible to extend our proposal (array of quantum dots) to 1/3 bilayer quantum Hall systems without superconductivity, where electron tunneling between quantum dots in different layers replaces the role of superconductivity. In small tunneling limit, similar to the superconducting case, the system is effectively described by a pair of "Ising spin chains". To first nonzero order in perturbation theory the tunneling between the chains is described by $H_t = t\sum_j \prod_i \sigma_{j-1}^i \sigma_j^i \sigma_{j+1}^i + H.c.$ where j, i are the site/layer index. The bosonized form of this system is identical to the previously discussed superconducting system in the small Δ regime plus a decoupled gapless mode[41]. Equivalently in this case when the interlayer tunneling becomes relevant, system is tuned to a phase with three fold ground state degeneracy. This scenario is similar to the idea of topological "genons"[42].

Finally, we remark that the Luttinger liquid description of the quantum Hall edge might be inaccurate for loops that are only several quasiparticle radiuses in size. However, the effective model in Eq.(10) also applies to a chain of superconducting quantum dots in a FQH system that can be gate tuned to have an almost two fold degeneracy between different fractional charges. In this sense, the Luttinger liquid edges might be considered to be a model for quantum dots in an FQH system and we do not expect the details of the model of the edge to affect our conclusions qualitatively.

Conclusion.— In this work we have considered a linear array of superconducting "quantum dot"-like holes on a spin singlet 2/3 fractional quantum Hall sample and showed that for both large and small values of induced superconductivity Δ, this system can be tuned to a topological phase hosting $\mathbb{Z}_3$ PZMs. Unlike earlier proposals used to realize PZMs, our approach does not rely on Andreev back-scattering between two fractional quantum Hall edges. In addition, this system appears surprisingly robust to disorder in a way similar to quantum dot based proposals for MZMs discussed in Refs.[43, 44].We believe this feature makes our proposal suitable for realization in experiments using existing ingredients.

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