Large-Scale Description of Interacting One-Dimensional Bose Gases: Generalized Hydrodynamics Supersedes Conventional Hydrodynamics

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Large-scale description of interacting one-dimensional Bose gases: generalized hydrodynamics supersedes conventional hydrodynamics

Benjamin Doyon,[1] Jérôme Dubail,[2] Robert Konik,[3] and Takato Yoshimura[1]

[1] *Department of Mathematics, King's College London, Strand, London WC2R 2LS, UK*
[2] *CNRS & IJL-UMR 7198, Université de Lorraine, F-54506 Vandoeuvre-lès-Nancy, France*
[3] *Condensed Matter and Materials Science Division, Brookhaven National Laboratory, Upton, NY 11973 USA*

The theory of generalized hydrodynamics (GHD) was recently developed as a new tool for the study of inhomogeneous time evolution in many-body interacting systems with infinitely many conserved charges. In this letter, we show that it supersedes the widely used conventional hydrodynamics (CHD) of one-dimensional Bose gases. We illustrate this by studying "nonlinear sound waves" emanating from initial density accumulations in the Lieb-Liniger model. We show that, at zero temperature and in the absence of shocks, GHD reduces to CHD, thus for the first time justifying its use from purely hydrodynamic principles. We show that sharp profiles, which appear in finite times in CHD, immediately dissolve into a higher hierarchy of reductions of GHD, with no sustained shock. CHD thereon fails to capture the correct hydrodynamics. We establish the correct hydrodynamic equations, which are finite-dimensional reductions of GHD characterized by multiple, disjoint Fermi seas. We further verify that at nonzero temperature, CHD fails at all nonzero times. Finally, we numerically confirm the emergence of hydrodynamics at zero temperature by comparing its predictions with a full quantum simulation performed using the NRG-TSA-ABACUS algorithm. The analysis is performed in the full interaction range, and is not restricted to either weak- or strong-repulsion regimes.

Introduction. Modern experiments with ultracold atoms confined in "cigar-shaped" traps [1, 2] or in atom chips [3] provide real-world implementations of one-dimensional (1d) many-body systems [4], and represent an important challenge for theoretical physics. Even though it is widely accepted that 1d clouds of bosonic atoms are described, at the microscopic scale, by the paradigmatic Lieb-Liniger (LL) model [5, 6], solving this model in experimentally relevant out-of-equilibrium inhomogeneous situations for more than a few dozens of atoms is a task that is out of the reach of modern theoretical methods, including state-of-the-art numerical ones.

Yet, it is a classic result of XX$^{\text{th}}$ century mathematical physics that, in its homogeneous, translation-invariant version at equilibrium, the LL model is exactly solvable by means of the Bethe ansatz [7], and its equation of state can be calculated exactly [8]. It then seems reasonable to use this equation of state as the basic input into a coarse-grained, hydrodynamic, approach, that is expected to be applicable as soon as typical lengths of variations of the local density are large enough as compared to inter-particle and scattering distances (the Euler scale) – in much the same way that classical hydrodynamics describes water waves. Such a "conventional hydrodynamic" (CHD) approach – defined in Eqs. (10) below –, has been used extensively in the cold atoms literature over the past decade [9–11], and has sometimes been viewed as a consequence of the Gross-Pitaevskii equation [6, 11] in the regime of small interaction strength.

However, a key physical feature of the LL model is overlooked in CHD: the fact that it admits infinitely many conservation laws. Indeed, CHD focuses only on a few quantities, like the particle density, the momentum density or the energy density. But the LL model possesses infinitely more conserved quantities. Those are not just a mathematical curiosity: they can have dramatic physical consequences, as illustrated by the quantum Newton cradle experiment [2]: the crucial observation of undamped oscillations in this experiment is connected with the lack of conventional thermalization [12].

The full connection between generalized thermalization and many-body dynamics was only recently uncovered [13, 14]. The fundamental precepts of hydrodynamics – the emergence of local entropy maximization – were used in systems with an infinite number of conservation laws in order to form the theory of generalized hydrodynamics (GHD). It is a type of Euler-scale hydrodynamics, but with an infinite-dimensional space of fluid states accounting for the large state manifold accessible by generalized thermalization. In practice, GHD consists in an infinite set of coupled continuity equations (one for each conserved charge), that can, at least in principle, be worked out with numerical solvers for non-linear partial differential equations.

In this letter, we show that GHD supersedes CHD. For this purpose, we focus on far-from-equilibrium waves emanating from a density accumulation in the LL model. The density waves are a good illustration for our purposes, but the main results and mechanisms are general. They have been studied in several ways in the past decade in the free Fermi gas [15], in the effective theory of the non-linear Luttinger liquid [18], and in the Calogero-Sutherland model [16], quantum Hall edges [17], and the LL model [10] using CHD. In particular, all these references – see also [11] – pointed out that the applicability of CHD was limited by the appearance of shocks. In this Letter, we show that GHD is the proper hydrodynamic framework to go beyond the latter.

We demonstrate that, only at zero temperature and for finite evolution times does CHD coincide with GHD. CHD being a finite-component hydrodynamics, it inevitably leads to "gradient catastrophes" and shock propagations thereon. In contrast, we show that at zero entropy, GHD decomposes into a hierarchy of instantaneously invariant finite-dimensional subspaces, whose exact hydrodynamic equations we establish. These are described by a multitude of Fermi seas, the stability of which is a consequence of integrability. We show that shocks dissolve as the system leaves the CHD subspace into a higher-dimensional reduction of GHD. No shock propagates in this process, as instead smoothness is re-established. We note that an important practical consequence of the zero-entropy reduction is that the infinite system of coupled non-linear equations of GHD collapses to a finite number of equations that are computationally easy to solve, taking typically a few minutes on a laptop. We also numerically verify that at nonzero temperature, the GHD evolution, which necessitates the full infinite-dimensional space, is different from CHD at all times. In the density wave problem, a stark difference is that no sharp profile develops in GHD, while CHD based on the finite-temperature LL equations of state has gradient catastrophes. Finally, at zero temperature and using a local density approximation for the initial fluid state, we compare the hydrodynamic prediction for the space-time density profile with a simulation of the full quantum model obtained from the NRG-TSA-ABACUS algorithm [19–21], and find perfect agreement.

GHD. The Hamiltonian of the repulsive LL model is

$$H = \int \mathrm{d}x \left(\frac{1}{2m} \partial_x \psi^\dagger \partial_x \psi + \frac{c}{2} \psi^\dagger \psi^\dagger \psi \psi \right), \quad c > 0 \tag{1}$$

for the complex bosonic field $\psi(x)$, where m is the mass (we set $\hbar = 1$ throughout the manuscript). An inhomogeneous initial state $\langle \cdots \rangle$, to be specified below, is set to evolve unitarily with H.

Since the LL model is integrable, it admits infinitely many conservation laws $\partial_t q_i + \partial_x j_i = 0$. This includes the gas density $q_0 = \psi^\dagger \psi$, the momentum density $q_1 = -\mathrm{i}\psi^\dagger \partial_x \psi + h.c.$, and the energy density q_2 (the integrand in (1)). According to the principles of hydrodynamics, if averages of conserved densities $\langle e^{\mathrm{i}Ht} q_i(x) e^{-\mathrm{i}Ht} \rangle$ and currents $\langle e^{\mathrm{i}Ht} j_i(x) e^{-\mathrm{i}Ht} \rangle$ have smooth enough space-time profiles, they can be described by space-time dependent local states that have maximized entropy with respect to the conserved charges afforded by the dynamics. Eulerian hydrodynamics, which neglects viscosity effects and is valid at large scales, is formed of the ensuing macroscopic conservation laws. In integrable systems, entropy maximization leads to generalized Gibbs ensembles (GGEs) [12, 22] with (formal) density matrix $\rho_{\mathrm{GGE}} = e^{-\sum_i \beta_i Q_i}$, $Q_i = \int q_i(x)\mathrm{d}x$. Therefore, $\langle e^{\mathrm{i}Ht} \mathcal{O}(x) e^{-\mathrm{i}Ht} \rangle \approx \mathrm{tr}[\rho_{\mathrm{GGE}}(x,t)\mathcal{O}]$, where the only space-time dependence is in $\rho_{\mathrm{GGE}}(x,t)$. The macroscopic conservation laws of generalized hydrodyamics (GHD) are the infinite number of equations for the density averages $\mathtt{q}_i(x,t) = \mathrm{tr}[\rho_{\mathrm{GGE}}(x,t) q_i]$ and the current averages $\mathtt{j}_i(x,t) = \mathrm{tr}[\rho_{\mathrm{GGE}}(x,t) j_i]$:

$$\partial_t \mathtt{q}_i + \partial_x \mathtt{j}_i = 0. \tag{2}$$

The set of $\mathtt{q}_i$ fixes the GGE state, and thus can be seen as a set of fluid variables for GHD. In the manifold of GGE states, the currents $\mathtt{j}_i$ have a fixed functional form in terms of the densities $\mathtt{q}_i$: these are the equations of state, which fully determine the GHD model at hand.

An efficient treatment of hydrodynamics requires an appropriate choice of fluid variables. Instead of the $\mathtt{q}_i$, the most powerful fluid variables are obtained in terms of the emerging quasi-particles of the integrable model. In the repulsive LL model, there is a single quasi-particle species. The interaction in the LL model is fully described by the two-particle scattering matrix $S(\theta - \theta')$, a function of velocity differences. The object of importance is the differential scattering phase [7],

$$\varphi(\theta) = -\mathrm{i} \frac{\mathrm{d}}{\mathrm{d}\theta} \log S(\theta) = 2c/(\theta^2 + c^2). \tag{3}$$

The quasi-particle can be seen as a spinless real fermion, which is free in the Tonks-Girardeau (TG) limit $c = \infty$ (hard-core repulsion).

States $|\theta_1, \ldots, \theta_N\rangle$ are described by the velocities θ_k of the quasi-particles. Each conserved charge Q_i is characterized by its one-particle eigenvalue $h_i(\theta) \propto \theta^i$, with $Q_i|\theta, \ldots, \theta_N\rangle = \sum_k h_i(\theta_k)|\theta_1, \ldots, \theta_N\rangle$. For instance, the particle number has eigenvalue $h_0(\theta) = 1$, the momentum $h_1(\theta) = p(\theta) = m\theta$, and the energy $h_2(\theta) = E(\theta) = m\theta^2/2$. In the thermodynamic limit, the eigenstates are expressed in terms of $\rho_{\mathrm{p}}(x,\theta)\mathrm{d}x\mathrm{d}\theta$, the number of quasi-particles in the phase-space region $[x, x+\mathrm{d}x] \times [m\theta, m(\theta + \mathrm{d}\theta)]$, leading to average densities $\mathtt{q}_i = \int \mathrm{d}\theta\, \rho_{\mathrm{p}}(\theta) h_i(\theta)$. The most convenient fluid variable is the occupation function $n(\theta) = \rho_{\mathrm{p}}(\theta)/\rho_{\mathrm{s}}(\theta)$, where ρ_{s} is the state density, $2\pi\rho_{\mathrm{s}}(\theta) = m + \int \mathrm{d}\alpha\, \varphi(\theta - \alpha)\rho_{\mathrm{p}}(\alpha)$. The density and current averages take the form [13],

$$\mathtt{q}_i = m \int \frac{\mathrm{d}\theta}{2\pi}\, n(\theta)\, h_i^{\mathrm{dr}}(\theta), \quad \mathtt{j}_i = m \int \frac{\mathrm{d}\theta\, \theta}{2\pi}\, n(\theta)\, h_i^{\mathrm{dr}}(\theta) \tag{4}$$

where the dressing operation is defined by

$$f^{\mathrm{dr}}(\theta) = f(\theta) + \int \frac{\mathrm{d}\alpha}{2\pi}\, \varphi(\theta - \alpha) n(\alpha) f^{\mathrm{dr}}(\alpha). \tag{5}$$

These establish a relation between the $\mathtt{j}_i$'s and the $\mathtt{q}_i$'s, and thus the equations of state. For the LL model they were derived in [13] by extending the theory of (generalized) TBA [20, 23].

It was realized in [13, 14] that demanding the continuity equations (2) together with the averages (4) implies the continuity equation at the level of quasi-particles:

$$\partial_t n(\theta) + v^{\mathrm{eff}}(\theta) \partial_x n(\theta) = 0, \tag{6}$$

where the effective velocity $v^{\rm eff}(\theta)$ is the velocity of elementary excitations [24]

$$v^{\rm eff}(\theta) = \frac{(E')^{\rm dr}(\theta)}{(p')^{\rm dr}(\theta)} = \frac{{\rm id}^{\rm dr}(\theta)}{1^{\rm dr}(\theta)} \tag{7}$$

with ${\rm id}(\theta) = \theta$. These are the GHD equations in terms of quasi-particle fluid variables in the LL model. Since $v^{\rm eff}(\theta)$ depends on the fluid state through the function n, these are nonlinear equations for an infinity of functions of space-time (one for each velocity θ).

Some intuition into Eqs. (6), (7) can be gained by looking at the TG limit $c = \infty$. In this case, $n(\theta)$ is the fermion occupation number at each momentum $m\theta$, at position x. This is the Wigner function of the state [25] (the partial Fourier transform of the fermion-fermion correlator). The effective velocity is equal to the particle velocity, $v^{\rm eff}(\theta) = \theta$, and (6) simply reproduces the exact evolution equation for the Wigner function, a direct consequence of the Schrödinger equation, as exploited in [15, 26] (see also the Supplementary Material (SM)). The quasi-particle occupation $n(\theta)$ may thus be viewed as the generalization of the Wigner function to non-free-fermion systems, with time-evolution governed by GHD (6)-(7).

Zero-entropy GHD. Natural initial conditions are ground states within inhomogeneous potentials $V(x)$,

$$H_V = \int {\rm d}x \left(\frac{1}{2m} \partial_x \psi^\dagger \partial_x \psi + \frac{c}{2} \psi^\dagger \psi^\dagger \psi \psi + V(x) \psi^\dagger \psi \right). \tag{8}$$

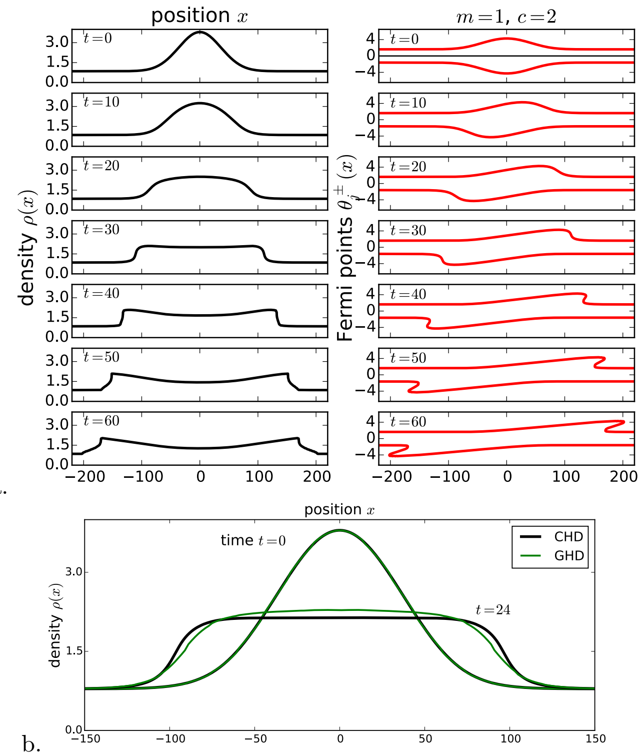
With a slowly varying potential, local averages are well described by a local-density approximation (LDA) [6]. LDA provides a GHD initial condition, a fluid of local zero-temperature states, which at every point x is the ground state of $H + V(x) Q_0$. In this section, we observe that GHD equations give rise to finite-dimensional hydrodynamics when one restricts to the subspace of fluid states with zero entropy such as those. An analogous observation was made previously for free fermions [15] and for the Calogero-Sutherland model [16].

Recall that the occupation function at zero temperature is $n_{T=0}(\theta) = \chi(\theta \in [-\theta_{\rm F}, \theta_{\rm F}])$ (where χ is the indicator function) where $\theta_{\rm F}$ is the Fermi pseudo-velocity, which depends on the chemical potential. Let us consider the space of zero-entropy occupation functions which have exactly $2k$ jumps, characterized by $2k$ velocities $\cdots < \theta_{j-1}^+ < \theta_j^- < \theta_j^+ < \theta_{j+1}^- < \cdots$ bounding separate Fermi seas: $n(\theta) = \chi(\theta \in \cup_{j=1}^k [\theta_j^-, \theta_j^+]))$. We show that under GHD evolution, any smooth fluid whose state lies in such a space at all positions x, stays so for short enough times. Time evolution leads to displacements of Fermi points. Thus at zero entropy, GHD is reduced to hydrodynamics with a finite number of fluid variables.

Indeed we have $\partial_x n(\theta) = -\sum_{\epsilon=\pm} \sum_{j=1}^k \epsilon \partial_x \theta_j^\epsilon \, \delta(\theta - \theta_j^\epsilon)$, and thus the time derivative $\partial_t n(\theta)$ is supported on the finite set of velocities $\theta_j^\pm$. A solution to (6) is therefore provided by setting $\theta_j^\pm = \theta_j^\pm(x,t)$ with

$$\partial_t \theta_j^\pm + v^{\rm eff}_{\{\theta\}}(\theta_j^\pm) \partial_x \theta_j^\pm = 0. \tag{9}$$

We expect the solution to (6) in the space of smooth fluid space-time functions to be unique, based on such rigorous results in related classical gases [27]. Thus it is given by solving (9) as long as no shock develops. Here, more explicitly, the effective velocity is $v^{\rm eff}_{\{\theta\}}(\alpha) = {\rm id}^{\rm dr}_{\{\theta\}}(\alpha)/1^{\rm dr}_{\{\theta\}}(\alpha)$ with the dressing operation $f^{\rm dr}_{\{\theta\}}(\alpha) = f(\alpha) + \sum_{j=1}^k \int_{\theta_j^-}^{\theta_j^+} {\rm d}\gamma \, \varphi(\alpha - \gamma) f^{\rm dr}_{\{\theta\}}(\gamma)$. The resulting equations (9) will be referred to as $2k$-hydrodynamics ($2k$HD).



a.

b.

FIG. 1. (a) Left: density profile of LL gas suddenly released from a gaussian potential $V(x) = -5e^{-(\frac{x}{50})^2} - 1$. Right: Corresponding Fermi points $\theta_j^\pm(x)$. Initially, there are only $k = 2$ Fermi points, but after the shock at $t \simeq 37$, there is a region where the red curve is multi-valued, corresponding to $k = 4$ Fermi points. (b) Same setup at finite temperature: the initial state is obtained from LDA at temperature $T = 1$. After finite time, CHD quantitatively differs from GHD. Moreover, at a later time $t \simeq 35$, CHD has a shock (see the SM); in contrast, GHD has no shock.

Eqs (9) are Euler-type hydrodynamic equations for a fluid with finitely-many components. Finite-component fluids are expected to develop shocks. Therefore, Eqs (9) are expected to hold *only for finite times.* However, contrary to true conventional finite-component fluids, where viscosity effects, present beyond the Euler scale, dominate and produce entropy at shocks, the presence of infinitely-many conservation laws forbids sustained entropy production in GHD. Any shock instantaneously dissolves into the higher-dimensional solution space of GHD. More precisely, $2k$HD solutions become multivalued at the time of the appearance of the shock, but here this multivaluedness is physically meaningful, representing a higher number of Fermi seas. Thus shocks in $2k$HD resolve by increasing the number of Fermi seas, passing to $(2k+2)$HD. We exemplify this in Fig. 1a, where after $t_{shock} = 37$ we begin to simulate, effectively, 4HD equations. This has previously been observed in free fermion models [15], thanks to an analysis based on the Wigner function; here it is generalized to the fully interacting LL model. We have also confirmed this shock dissolution mechanism of GHD in a nontrivial classical gas with the same hydrodynamic equations as those of the LL model [28].

GHD and conventional hydrodynamics (CHD). Starting with a smooth fluid of local zero-temperature states, GHD reduces to 2HD, where every local fluid cell is the Galilean boost of a zero-temperature state. As a consequence, 2HD is in fact equivalent to the conventional hydrodynamics (CHD) of Galilean fluids,

$$\partial_t \rho + \partial_x(v\rho) = 0, \quad \partial_t v + v\partial_x v = -\frac{1}{m\rho}\partial_x \mathcal{P} \qquad (10)$$

where $\rho = \mathsf{q}_0$ is the fluid density and $\mathcal{P}$ is the pressure [30]. The first equation is conservation of mass, the second, of momentum. The pressure $\mathcal{P} = \mathcal{P}(\rho)$ gives the equations of state of the fluid, and here equals the momentum current $\mathtt{j}_1$ in the zero-temperature state with density ρ. The explicit equations of state are obtained from (4) and (5) (see SM [30]).

CHD has been used as an important tool in analyzing the dynamics of 1d Bose gases [9–11]. It has sometimes been presented as a consequence of the Gross-Pitaevskii equation [6]—itself valid only in the limit of small interaction strength c—, and it was never quite clear what exactly the range of validity of CHD was in the full interaction range of the LL model. Our analysis clearly shows that CHD is valid —in the sense that it coincides with GHD— *only at zero temperature*, and *before the first shock*. We conclude that, *in any other situation, CHD is not applicable and leads to quantitatively wrong results.* To illustrate this, a comparison of CHD at finite temperature and GHD is shown in Fig.1b; the initial state is the same in both cases (obtained from LDA at finite temperature), but one sees that the density profiles differ significantly at finite time; moreover, CHD has solutions up to a finite shock time, while GHD has no shocks and has solutions at arbitrarily long times [32].

Comparison with microscopic simulation of the LL model. We consider evolution from the ground state of (8) with a background chemical potential μ_∞ perturbed by a Gaussian, $V(x) = -\mu_\infty - Ue^{-ax^2}$. The initial density profile accumulates around $x = 0$, and is asymptotically nonzero. Two procedures are compared: (1) the ground state is exactly evaluated using the NRG-TSA-ABACUS algorithm [20, 21, 34], and then evolved unitarily; and (2) the ground state is approximated using LDA, and this initial fluid state is evolved using 2HD (see the SM for a review of standard conditions for the hydrodynamic regime, which are fulfilled by the choice of parameters below). Fig. 2 provides the result for a choice of parameters corresponding to the local dimensionless coupling $\gamma(x) = mc/\rho(x)$ of the order of 1, thus the system is in an intermediate regime with nontrivial interactions being important. We observe that GHD is in excellent agreement with NRG-TSA-ABACUS numerics at almost all times except near the right and left boundaries at $t = 72$, and provides a substantially better approximation than linear sound waves (see the SM). It can also be seen that, since $v^{\rm eff}_{\{\theta\}}(\theta^+)$ is always greater than the background Fermi velocity corresponding to μ_∞, the propagation speed of 2HD is larger than that of the sound wave. It is remarkable that the complex (zero-temperature) dynamics of the LL gas is exactly described by 2HD, a simple set of differential equations.

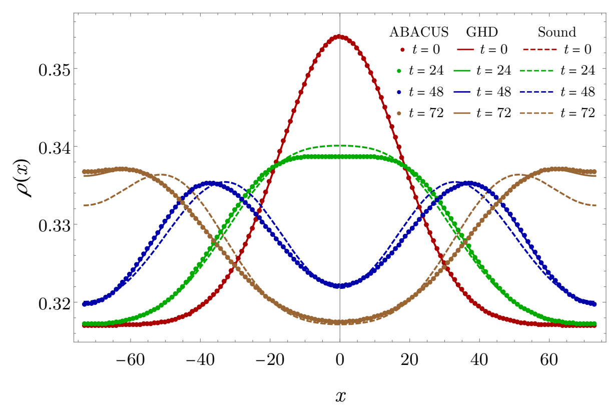


FIG. 2. The density profile under the evolution with $2m = 1$, $c = 1$, $U = 0.03$, $a = 1/576$, and a choice of μ_∞ such that there are $N = 48$ particles in the ground state. We use periodic boundary conditions. Points show NRG-TSA-ABACUS data, full line the GHD simulation, and dotted line a linear sound wave approximation.

At large times, discrepancies, though very tiny in the above graph, are expected to emerge. One reason is that, as explained above, shocks attempt to form, and as variations become larger, conditions for the hydrodynamic regime break down. Higher-derivative effects, such as viscosity, become more important. Recent observations in the related hard rod gas [42] suggest however that such higher-derivative effects play only a small role. Another cause for late-time discrepancy is that LDA is not exact. As higher-order charges are more sensitive to the large-scale variations of the potential, LDA gives an extremely good approximation to the particle density, but describes poorly densities of higher-order charges Q_i. As time passes, the effects of the latter under the full GHD evolution eventually breaks 2HD. An analysis of the Wigner function $n(\theta)$ at the free-fermion point $c = \infty$, where GHD is exact (no viscosity is neglected), gives further insight (see the SM).

Conclusion. We showed that widely used conventional hydrodynamics of the Lieb-Liniger model correctly describes interacting Bose gases, but that this holds only at zero temperatures and for finite times. We provided exact, simple hydrodynamic equations valid beyond gradient catastrophes, where no shocks are sustained. These are zero-entropy reductions of GHD, which are finite-component fluid equations easily solvable on a laptop. This suggests that fluids of integrable models avoid entropy production thanks to the large space of fluid states. We provided compelling evidence for the emergence of GHD in the LL gas in the limit of slow variations of the density profile. This provides a crucial dynamical extension of LDA that is valid beyond previously existing frameworks. As a future direction, it would be very interesting to apply our method to more experimentally relevant situations such as "Quantum Newton's cradle"-type protocol [2], known to be beyond the reach of CHD [11].

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