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# Quantized magnetization density in periodically driven systems

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We study micromotion in two-dimensional periodically driven systems in which all bulk Floquet eigenstates are localized by disorder. We show that this micromotion gives rise to a quantized time-averaged orbital magnetization density in any region completely filled with fermions. The quantization of magnetization density has a topological origin, and reveals the physical nature of the new phase identified in Phys. Rev. X **6**, 021013 (2016). We thus establish that the topological index of this phase can be accessed directly in bulk measurements, and propose an experimental protocol to do so using interferometry in cold atom based realizations.

Periodic driving was recently introduced as a means for achieving topological phenomena in a wide variety of quantum systems. Beyond providing new ways to obtain topologically nontrivial band structures [1–15], periodic driving can give rise to wholly new types of topological phenomena without analogues in equilibrium [16–32].

In a periodically driven system, the unitary Floquet operator acts as a generator of discrete time evolution over each full driving period. As in non-driven systems, the spectrum and eigenstates of the Floquet operator can be classified according to topology [2, 4, 16]. However, in addition to the stroboscopic evolution of the system, the *micromotion* that takes place within each driving period is crucial for the topological classification of periodically driven systems [17–21, 24–28].

Here we uncover a new type of topological quantization phenomenon associated with the micromotion of periodically driven quantum systems. We focus on periodically driven two-dimensional (2D) lattice systems in which *all* bulk Floquet eigenstates are localized by disorder (see Fig. 1). We show that, within a region where all states are occupied, the time-averaged orbital magnetization density  $\langle\langle m \rangle\rangle$  is *quantized*:  $\langle\langle m \rangle\rangle = \nu/T$ , where  $\nu$  is an integer and  $T$  is the driving period. The bulk observable  $\langle\langle m \rangle\rangle$  thus serves as a topological order parameter, characterizing the topologically distinct fully-localized phases found in Ref. [22]. We propose a bulk interference measurement to probe this invariant in cold atom systems.

Topological invariants are often associated with quantized response functions. Famously, the Hall *conductivity* of an insulator is proportional to the TKNN invariant, or Chern number [33]. Interestingly, topology in driven systems may directly give rise to quantization of time-averaged observables, such as the pumped *current* in the Thouless pump [34]. Similarly, the response of magnetization density to changes of chemical potential in a quantum Hall system is quantized when the chemical potential lies in an energy gap [35–37]. In contrast, here we find quantization of the *magnetization density* itself.

For concreteness, we consider a periodically-driven

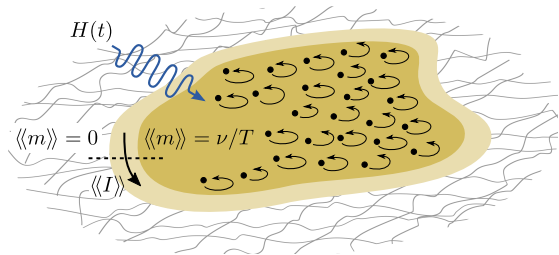


FIG. 1. Quantized magnetization density in a two-dimensional periodically driven system where all Floquet eigenstates are localized. In a region where all sites are initially occupied (shaded area), the time-averaged orbital magnetization density  $\langle\langle m \rangle\rangle$  is quantized as  $\nu/T$ , where  $\nu$  is an integer and  $T$  is the driving period. A quantized average current  $\langle\langle I \rangle\rangle = \nu/T$  runs along the edge of the filled region.

two-dimensional lattice model with one orbital per site. Dynamics are governed by a time-periodic Hamiltonian  $H(t) = H(t + T)$ , where  $T$  is the driving period. The periodic driving gives rise to a unitary evolution  $U(t) = \mathcal{T}e^{-i\int_0^t dt' H(t')}$ , where  $\mathcal{T}$  denotes time ordering. The spectrum of the Floquet operator  $U(T)$ , given by  $U(T)|\psi_n(0)\rangle = e^{-i\varepsilon_n T}|\psi_n(0)\rangle$ , defines the Floquet eigenstates  $\{|\psi_n(t)\rangle\}$  and their quasienergies  $\{\varepsilon_n\}$ .

We characterize micromotion in this system via the orbital magnetization [39]

$$M(t) = \frac{1}{2} (\mathbf{r} \times \dot{\mathbf{r}}(t)) \cdot \hat{\mathbf{z}}, \quad (1)$$

where  $\dot{\mathbf{r}}(t) = -i[\mathbf{r}, H(t)]$ . The magnetization operator (1) is equivalently expressed as the response of the Hamiltonian to an applied uniform magnetic field  $B$ :  $M(t) = -\frac{\partial H(t)}{\partial B}$  [40]. In non-driven systems, the magnetization of a state hence determines the response of its energy to the field:  $\Delta E \sim -\mathbf{M} \cdot \mathbf{B}$ . In periodically driven systems, a similar relation holds between a Floquet eigenstate's time-averaged magnetization and the response of its quasienergy to an applied magnetic field. We define  $\langle\langle \mathcal{O} \rangle\rangle_\tau \equiv \frac{1}{\tau} \int_0^\tau dt \langle \psi(t) | \mathcal{O}(t) | \psi(t) \rangle$  as the time-averaged expectation value of an operator  $\mathcal{O}(t)$  in the state  $|\psi(t)\rangle$ .

The single-period averaged magnetization of a (localized) Floquet eigenstate  $|\psi_n(t)\rangle$  is given by [40, 42]:

$$\langle M \rangle_T^{(n)} \equiv \frac{1}{T} \int_0^T dt \langle \psi_n(t) | M(t) | \psi_n(t) \rangle = -\frac{\partial \varepsilon_n}{\partial B}. \quad (2)$$

Using Eqs. (1) and (2), we may associate a net magnetization with a single particle in a localized Floquet eigenstate. It is useful to define a local time-averaged *magnetization density*, associated with each plaquette  $p$  of the lattice, that characterizes the response of quasienergy to a magnetic flux  $\phi_p$  applied locally through plaquette  $p$ . We define the magnetization density operator as [43]:

$$m_p(t) = -\frac{\partial H(t)}{\partial \phi_p}, \quad \phi_p = \int_p d^2r B(\mathbf{r}), \quad (3)$$

where the integral is taken over the area of plaquette  $p$ . The total time-averaged magnetization,  $\langle M \rangle_\tau$ , is given by the sum of magnetization densities over all plaquettes:  $\langle M \rangle_\tau = \sum_p \langle m_p \rangle_\tau a^2$ , where  $a$  is the lattice constant.

The definition of magnetization density in Eq. (3) applies for both single particle and many-body systems. In particular, for a (single or many particle) Floquet eigenstate  $|\psi(t)\rangle$  with quasienergy  $\varepsilon$ , the time-averaged magnetization density is given by  $\langle m_p \rangle_T = -\frac{\partial \varepsilon}{\partial \phi_p}$ .

In the continuum, equilibrium magnetization density is related to the current density  $\mathbf{j}$  through Ampere's law,  $\mathbf{j} = \nabla \times \mathbf{m}$ . For a (stationary) system on the lattice, Ampere's law relates the time-averaged magnetization densities on adjacent plaquettes  $p$  and  $q$  to the time-averaged current  $\langle I_{pq} \rangle_\tau$  on the bond between them [40]:

$$\langle I_{pq} \rangle_\tau = \langle m_p \rangle_\tau - \langle m_q \rangle_\tau. \quad (4)$$

Here we take positive current to be counterclockwise with respect to plaquette  $p$ .

*Magnetization in finite droplets.*— We now show that the time-averaged magnetization density is quantized in a finite ‘‘droplet,’’ where all states in a region of linear dimension  $R$  are initially occupied while the surrounding region is completely empty (Fig. 1). Specifically, we consider the long-time average of the magnetization density for a plaquette  $p$  deep inside the droplet,  $\langle\langle m_p \rangle\rangle$ , where  $\langle\langle \mathcal{O} \rangle\rangle \equiv \lim_{\tau \rightarrow \infty} \langle \mathcal{O} \rangle_\tau$ . Below we show that  $\langle\langle m_p \rangle\rangle$  takes a constant value  $\bar{m}_\infty$ , up to exponentially small corrections [44]. We then show that  $\bar{m}_\infty$  is quantized.

Since all Floquet eigenstates are localized, the particle density will only evolve significantly in a strip of width  $\xi$  around the boundary of the filled region, where  $\xi$  is the single-particle localization length of the Floquet eigenstates. Hence, the droplet retains its shape up to a smearing of its boundary. At a distance  $d \gg \xi$  from this boundary, the density change remains exponentially small in  $d/\xi$  at any time. Within the droplet, all (time-averaged) bond currents therefore vanish:  $\langle I_{pq} \rangle_\tau = 0$  for all  $\tau$ . The magnetization density  $\langle\langle m_p \rangle\rangle$  must therefore be the same for all plaquettes deep within the droplet.

The uniform value of the magnetization density deep within the droplet may depend on the droplet's size. We note that  $\langle\langle m_p \rangle\rangle$  is given by the sum of magnetization contributions from all occupied states that overlap with plaquette  $p$ . Therefore, if the droplet size is increased by adding a section of new (filled) sites in a region far away from plaquette  $p$ ,  $\langle\langle m_p \rangle\rangle$  can only change by an exponentially small amount due to the contributions of the tails of the newly added localized states. Thus, for a plaquette located a distance  $d$  from the boundary, we obtain  $\langle\langle m_p \rangle\rangle = \bar{m}_\infty + \mathcal{O}(e^{-d/\xi})$ , where  $\bar{m}_\infty$  is the value in the thermodynamic limit. As we show below,  $\bar{m}_\infty$  is quantized.

Interestingly, a nonzero value of  $\bar{m}_\infty$  implies that a current circulates around the boundary of the droplet. The magnetization density drops from the value  $\bar{m}_\infty$  to zero over a distance of order  $\xi$  across the droplet's boundary. Using Ampere's law (4), the total time-averaged current  $\langle\langle I \rangle\rangle$  passing through a cut through this strip (see Fig. 1) is  $\langle\langle I \rangle\rangle = \bar{m}_\infty + \mathcal{O}(e^{-R/\xi})$ .

*Quantization of magnetization density.*— To prove the quantization of  $\bar{m}_\infty$ , we consider the total magnetization  $\langle\langle M \rangle\rangle$  of a droplet of  $N$  particles. On one hand we have  $\langle\langle M \rangle\rangle = \sum_n' \langle M \rangle_T^{(n)} + \mathcal{O}(N^{1/2})$ , where the sum runs over single particle Floquet eigenstates  $|\psi_n\rangle$  with centers localized within the perimeter of the droplet. The  $\mathcal{O}(N^{1/2})$  correction accounts for the partially-occupied Floquet eigenstates near the droplet's boundary. On the other hand, since the magnetization density deep inside the droplet is constant and given by  $\bar{m}_\infty$ , we have  $\langle\langle M \rangle\rangle = Na^2 \bar{m}_\infty + \mathcal{O}(N^{1/2})$ . Here  $Na^2$  is the total area of the droplet, with the  $\mathcal{O}(N^{1/2})$  correction capturing the uncertainty of the area due to its fuzzy boundary. By equating the expressions for  $\langle\langle M \rangle\rangle$  and taking the  $N \rightarrow \infty$  limit, we identify

$$\bar{m}_\infty = \lim_{N \rightarrow \infty} \frac{1}{Na^2} \sum_n' \langle M \rangle_T^{(n)}. \quad (5)$$

The quantity  $\frac{1}{N} \sum_n' \langle M \rangle_T^{(n)}$  is simply the average magnetization of Floquet eigenstates in the droplet; below, we show that this average is quantized in large, fully-localized systems. To do this, we explicitly compute the average magnetization over all Floquet eigenstates for a fully-localized system on a large torus of area  $A = L^2 a^2$ , where  $L^2$  is the number of sites.

For the system on a torus, we compute the time-averaged magnetization  $\langle M \rangle_T^{(n)}$  of each Floquet eigenstate  $|\psi_n(t)\rangle$  using Eq. (2). To use the form  $\langle M \rangle_T^{(n)} = -\frac{\partial \varepsilon_n}{\partial B}$ , we must specify how the field  $B$  is introduced. Crucially, on a torus, the net magnetic flux must be an integer multiple of  $\Phi_0$  (the flux quantum) [45]; consequently, the strength of a *uniform* field cannot be varied continuously. However, for  $\xi/L \ll 1$ , we may use  $\langle M \rangle_T^{(n)} = -\frac{\partial \varepsilon_n}{\partial B} + \mathcal{O}(e^{-L/\xi})$ , where  $\varepsilon_n(B)$  is the

quasienergy of state  $|\psi_n\rangle$  in the presence of a *locally* uniform magnetic field, of strength  $B$  within the support region of  $|\psi_n\rangle$ , but zero net flux through the torus. The  $\mathcal{O}(e^{-L/\xi})$  correction arises from the non-uniformity of the field, which is concentrated where the wave function is exponentially small.

To evaluate the average magnetization of localized Floquet eigenstates,  $\frac{1}{L^2} \sum_n \langle M \rangle_T^{(n)} = -\frac{1}{L^2} \sum_n \frac{\partial \varepsilon_n}{\partial B}$ , we examine the Floquet operator  $U(T)$  in the presence of a global uniform magnetic field of strength  $B_0 = 2\pi/A$ , corresponding to precisely one flux quantum piercing the torus. For large  $A$ , the quasienergy in the uniform field  $B_0$  is equal to that in the locally uniform field described above (with  $B = B_0$ ), up to an exponentially small correction in  $L/\xi$ . Moreover, for small field strengths,  $\frac{\partial \varepsilon_n}{\partial B}$  is well approximated by a finite difference, such that [46]:

$$\langle M \rangle_T^{(n)} = -[\varepsilon_n(B_0) - \varepsilon_n(0)]/B_0 + \mathcal{O}(1/A). \quad (6)$$

The  $\mathcal{O}(1/A)$  correction accounts for the error in discretizing the derivative.

Using Eq. (6), we can access  $\sum_n \langle M \rangle_T^{(n)}$  directly via the determinant of the system's Floquet operator [21],  $|U(T)|$ . Writing  $\log |U(T)| = \int_0^T dt \partial_t \log |U(t)|$ , we use the identity  $\partial_t \log |U(t)| = \text{Tr} [U^\dagger(t) \partial_t U(t)]$ , together with  $\partial_t U(t) = -iH(t)U(t)$ , and find [47]

$$\log |U(T)| = -i \int_0^T dt \text{Tr} [H(t)]. \quad (7)$$

When a magnetic field is introduced, the hopping amplitudes between sites of the lattice acquire additional Peierl's phases:  $H_{ab} \rightarrow H_{ab} e^{i\theta_{ab}}$ . In the position basis, the magnetic field thus only affects the *off-diagonal* elements of the Hamiltonian, and we conclude that  $\text{Tr}[H(t)]$  and hence  $|U(T)|$  are independent of magnetic field. Using  $|U(T)| = e^{-i \sum_n \varepsilon_n T}$ , we find

$$\sum_n \varepsilon_n(B_0) = \sum_n \varepsilon_n(0) - \frac{2\pi\nu}{T}, \quad (8)$$

where  $\nu$  is an integer.

Recall that  $\bar{m}_\infty$ , the magnetization density in a filled droplet is obtained from the average magnetization of the Floquet eigenstates in the droplet, see Eq. (5). The torus geometry discussed above allows us to compute this average in the thermodynamic limit. Using Eqs. (6) and (8) we obtain  $\frac{1}{L^2} \sum_n \langle M \rangle_T^{(n)} = \frac{2\pi\nu}{L^2 B_0 T}$  [48]. Comparing to Eq. (5), we find:

$$\bar{m}_\infty = \frac{\nu}{T}. \quad (9)$$

Remarkably, this quantization has a topological origin. As we show in the SOM [40], the integer  $\nu$  is equal to the winding number invariant characterizing the Anomalous Floquet-Anderson Insulator (AFAI) phase, introduced in

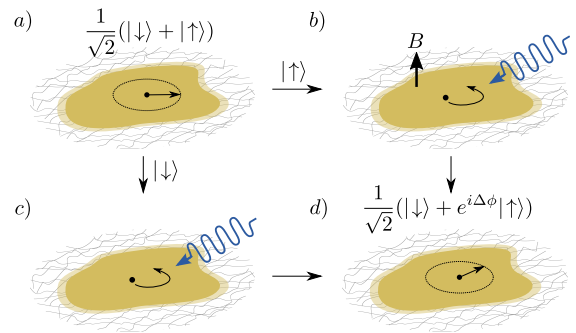


FIG. 2. Interferometric measurement of quantized orbital magnetization density in a cold-atom system. a) The system is prepared by filling a region of an optical lattice with spin-1/2 atoms fully polarized along  $x$ . b,c) The system is evolved with a spin-independent periodic driving Hamiltonian, plus a weak spin-dependent uniform synthetic magnetic field. d) The spin-dependent field gives rise to a phase-difference  $\Delta\phi$  between the  $|\uparrow\rangle$  and  $|\downarrow\rangle$  components of each atom's wave function. The phase shift yields a net  $y$ -polarization of total spin, proportional to the system's time-averaged magnetization.

Ref. [22]. The magnetization density thus serves as a bulk topological order parameter that characterizes distinct fully-localized Floquet phases. Note that the emergence of a non-zero, quantized magnetization density is a unique dynamical phenomenon, with no counterpart in non-driven systems: for static systems, Eq. (9) must hold for all values of  $T$ , which requires  $\nu = 0$  [49].

*Interferometric probe of quantized magnetization.*— We now outline an interferometric scheme for measuring the spatially averaged magnetization density  $\langle\langle \bar{m} \rangle\rangle = \langle\langle M \rangle\rangle / A_{\text{filled}}$  of a cloud of fermionic cold atoms in an optical lattice (see Fig. 2), where  $A_{\text{filled}}$  is the area of the initially filled region. We thus offer a direct probe to measure the bulk topological invariant of the AFAI.

Consider an atom traversing a closed trajectory in the presence of a weak magnetic field  $B$ . Semiclassically, the wave-function picks up an additional phase shift  $\Delta\phi = BA_{\text{orb}}$  due to the field, where  $A_{\text{orb}}$  is the area enclosed by the orbit [50]. Correspondingly, a simple quantum mechanical calculation [40] shows that the phase shift acquired by an atom in Floquet eigenstate  $|\psi_n(t)\rangle$  over a full driving period is proportional to the state's magnetization,  $\Delta\phi_n = \langle M \rangle_T^{(n)} BT$ .

Using this phase shift, the magnetization of a cloud of atoms can be measured in a Ramsey-type interference experiment in a situation where the atoms have two internal (“spin”) states  $|\uparrow\rangle$  and  $|\downarrow\rangle$ . First, the system should be prepared by completely filling a region of known area,  $A_{\text{filled}}$ , with atoms fully spin-polarized along the “ $x$ ”-direction,  $|\psi(0)\rangle \propto (|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$ , (Fig. 2a). The system should then be evolved with the driving Hamiltonian to allow the particle density to reach a steady profile [51], as in Fig. 3a. To perform the measurement, the cloud of atoms is then evolved through  $N$  driving periods in the presence of a weak spin-dependent orbital effective

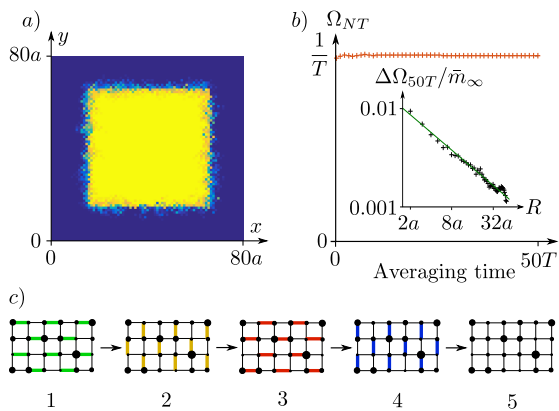


FIG. 3. a) Particle density in the system after 20 driving periods, for an initially filled  $50 \times 50$  square of sites. b) Normalized growth rate  $\Omega_{NT}$  of the average  $y$ -spin per atom [see text above Eq. (10)]. The long-time-averaged magnetization density  $\langle\langle \bar{m} \rangle\rangle$  is extracted from the saturation value at long times. *Inset:* Deviation  $\Delta\Omega_{50T}$  of  $\Omega_{50T}$  from the quantized value  $\bar{m}_\infty$  vs. droplet size  $R$ . The value of  $\Delta\Omega_{50T}$  is obtained as an RMS average of  $\Omega_{50T} - \bar{m}_\infty$  over 100 disorder realizations. c) Depiction of the tight-binding model.

magnetic field  $B$  (Figs. 2bc), which, e.g., acts only on the  $|\uparrow\rangle$  species. Through the evolution, the  $|\uparrow\rangle$  component of each atom's wave function gains a phase shift relative to the  $|\downarrow\rangle$  component, yielding a nonzero average  $y$ -spin per particle,  $\langle\bar{\sigma}_y\rangle$ , (Fig. 2d). For small precession angles, the average  $y$ -spin after  $N$  periods is given by  $\langle\bar{\sigma}_y(NT)\rangle \equiv \Omega_{NT} B a^2 NT$ , with [40]

$$\Omega_{NT} = \langle\langle \bar{m} \rangle\rangle + \frac{1}{NT} \mathcal{O}\left(\frac{\xi^{3/2}}{aR^{1/2}}\right) + \mathcal{O}(B). \quad (10)$$

Importantly, the second term vanishes in the long time limit (and scales to zero at finite times for large systems), thus revealing the quantized magnetization density [40].

*Numerical results.*— We simulated the experimental protocol outlined above using a tight-binding model on a two-dimensional bipartite square lattice, with Hamiltonian  $H(t) = H_{\text{clean}}(t) + V_{\text{disorder}}$ . The Hamiltonian  $H_{\text{clean}}(t)$  was considered in Ref. [17], and is of the form

$$H_{\text{clean}}(t) = \sum_{\mathbf{r} \in A} \sum_{n=1}^4 J_n(t) (c_{\mathbf{r}+\mathbf{b}_n}^\dagger c_{\mathbf{r}} + h.c.), \quad (11)$$

where  $c_{\mathbf{r}}$  is the fermionic annihilation operator on the lattice site with coordinate  $\mathbf{r}$ , and the first sum runs over sites  $\mathbf{r}$  on sublattice  $A$ . The vectors  $\{\mathbf{b}_n\}$  are given by  $\mathbf{b}_1 = -\mathbf{b}_3 = (a, 0)$  and  $\mathbf{b}_2 = -\mathbf{b}_4 = (0, a)$ , where  $a$  is the lattice constant. The driving period is divided into five segments of equal length  $T/5$ . In the  $n$ th segment ( $n \leq 4$ ),  $J_n(t) = J$ , while all other hopping amplitudes are set to zero; in the 5th segment all hopping amplitudes are set to zero (see Fig. 3c). We introduce disorder through a time-independent potential  $V_{\text{disorder}} = \sum_{\mathbf{r}} w_{\mathbf{r}} c_{\mathbf{r}}^\dagger c_{\mathbf{r}}$ , where

the sum runs over all sites, and the on-site energies  $\{w_{\mathbf{r}}\}$  are randomly drawn from a uniform distribution in the interval  $[-W, W]$ . The model has hopping amplitude  $J$  and disorder strength  $W$  both set to  $2.5\pi/T$ . This brings the system well into the AFAI phase, for which we expect  $\bar{m}_\infty = 1/T$  [40].

To find the magnetization density of the system, we consider a single disorder realization on a lattice of  $80 \times 80$  sites and open boundary conditions. We initially fill a region of  $50 \times 50$  sites (i.e.,  $R = 50$ ) centered in the middle of the lattice, and prepare the state by evolving it for 20 driving periods at zero magnetic field (see Fig. 3a). For further times ranging from 0 to  $50T$  we evolve the system in the presence of a spin-dependent magnetic field of strength  $Ba^2 = 2\pi \cdot 10^{-4}$ . We extract the spatially averaged magnetization density  $\langle\langle \bar{m} \rangle\rangle$  from the long-time limit of the normalized growth rate  $\Omega_{NT}$  of average  $y$ -spin per atom,  $\langle\bar{\sigma}_y(NT)\rangle$ .  $\Omega_{NT}$  rapidly converges (up to a finite-size correction) to the quantized value of the magnetization density,  $1/T$ , reaching 0.9998 after 100 periods (see Fig. 3b and SOM). The inset in Fig. 3b shows the deviation of  $\Omega_{50T}$  from the quantized value  $\bar{m}_\infty = 1/T$  for various sizes of the droplet, taken as a root-mean-square average over 100 disorder realizations at each system size. We find a power law decay of the fluctuations with system size,  $\Delta\Omega_{50T} \sim R^{-0.55}$ .

*Discussion.*— Here we showed that the orbital magnetization density is quantized in fully-filled regions of localized Floquet systems. We then proposed an experimental scheme for measuring the quantized magnetization density in cold atomic systems.

We derived the quantization of magnetization density within a tight-binding model with one ( $s$ -type) orbital per site. This means that each on-site orbital does not carry any intrinsic magnetization. In the continuum, small non-quantized contributions to the magnetization density may arise due to mixing with higher bands. Such contributions are strongly suppressed when the driving is adiabatic with respect to the gap to higher bands, and the lattice is very deep such that the gap is large compared to the bandwidth [40].

It is natural to expect that our results will hold also in the presence of interactions, given that the system is strongly disordered and hence may be many-body localized. Recently, progress has been made in constructing interacting analogues of the AFAI [52, 53]. The fate of the magnetization in the presence of interactions remains an open direction of investigation.

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- [44] Although the particle density is not strictly stationary, localization implies  $\lim_{\tau \rightarrow \infty} \langle \dot{\rho} \rangle_\tau = 0$ . Therefore the long time average  $\langle\langle m_p \rangle\rangle$  is gauge invariant [43] and obeys Ampere's law, Eq. (4).
- [45] This follows from the Dirac quantization condition: the torus can enclose an integer number of magnetic monopoles, each with a quantized charge of  $\Phi_0$ .
- [46] The labeling of Floquet eigenstates in the presence of the uniform field  $B_0$  is defined such that  $\langle \psi_n(t, B_0) | \psi_m(t, 0) \rangle = \delta_{nm} + \mathcal{O}(1/A)$ . For large systems, this prescription holds for all but an exponentially small subset of disorder realizations.
- [47] Here we choose  $\log |U(t)|$  to be continuous in time, with  $\log |U(0)| = 0$ .
- [48] On the torus,  $\sum_n \langle M \rangle_T^{(n)}$  is used mathematically to find the average magnetization of Floquet eigenstates. It does *not* represent the magnetization of a fully filled torus, which is unmeasurable.
- [49] Even if  $\nu = 0$ , a partially filled region can have a non-zero, albeit non-quantized, magnetization, depending on which states are filled. For example, a current of  $1/2$  particle per period that circulates on the boundary of a partially filled (gapped) periodically driven system was

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