Analytic Solutions to Coherent Control of the Dirac Equation

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Analytic solutions to coherent control of the Dirac equation

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A simple framework for Dirac spinors is developed that parameterizes admissible quantum dynamics and also analytically constructs electromagnetic fields, obeying Maxwell's equations, that yield a desired evolution. In particular, we show how to achieve dispersionless rotation and translation of wave-packets. Additionally, this formalism can handle control interactions beyond electromagnetic. This work reveals unexpected flexibility of the Dirac equation for control applications, which may open new prospects for quantum technologies.

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Introduction. The common aim of quantum control is to find a tailored external electromagnetic field to steer the ensuing dynamics in a desired fashion [1]. This capability, in particular, is enabling quantum technologies with the prospect of revolutionizing metrology, information processing, and matter manipulation. However, little is known about the control of the Dirac equation in spite of its modern applications reaching into nearly every domain of physics, going far beyond its original intention [2, 3]. For example, lasers have already reached intensities where light-matter interactions must be described within the Dirac theory [4]. Studies of the properties of heavy elements led to the establishment of relativistic quantum chemistry [5–8] based on the Dirac equation. Moreover, there is a growing list of low energy systems emulating Dirac fermions in solids [9–11], optics [12, 13], cold atoms [14, 15], trapped ions [16, 17], and circuit quantum electrodynamics [18].

The Dirac equation is commonly expressed as [2]

$$\gamma^\mu[ic\hbar\partial_\mu - ceA_\mu]\psi = mc^2\psi, \tag{1}$$

where the summation over repeated indices is adopted, ψ is a four-component complex spinor, m is the mass, c is the speed of light, γ^μ are the 4×4 so-called gamma matrices, A_μ is the four-vector potential and $\mu = 0, 1, 2, 3$.

The Dirac equation (1) can be viewed as a "first quantization" approximation to QED. The solutions of Eq. (1) exclude effects such as radiation reaction and particle creation/annihilation prominent at ultra-relativistic energies. Nevertheless, Eq. (1) provides a mean-field description of relativistic effects at low and moderate energies. A moving Dirac electron generates the current $J^\mu_D = \psi^\dagger\gamma^0\gamma^\mu\psi$ that emits secondary radiation, which is not accounted for by Eq. (1). Therefore, a solution of the Dirac equation is physical if the energy loss due to the secondary radiation is much smaller than the electron kinetic energy. This criterion should be satisfied in the applications of the Dirac equation to quantum control.

In this Letter we present the framework of *Relativistic Dynamical Inversion* (RDI) opening up a new route to coherent control for the Dirac dynamics: Given a desired wavepacket evolution, we *analytically* design electromagnetic control fields obeying Maxwell equations. This should be compared with other techniques such as shortcuts to adiabaticity [19–21] analytically constructing interactions that often go beyond electromagnetic fields.

The purpose of the current work is to solve the following problem: Given an arbitrary (desired) spinorial spacetime wavepacket ψ, find an electromagnetic field A_μ such that Eq. (1) is satisfied. This is accomplished by RDI in two steps: First, we verify the attainability of the given evolution ψ by assessing the existence of the underlying A_μ leading to valid Maxwell equations. Second, if it exists, an explicit form of A_μ is obtained. Moreover, the method can also be used to assess for attainable dynamics.

The task of constructing the control field yielding the desired dynamics at all times and positions is one of the most important and challenging problems in quantum control. In particular, transporting coherent wavepackets without disturbance is a required building block in quantum technologies. RDI allows for finding analytic solutions not feasible by other current methods. This is possible due to unique properties of the Dirac equation.

Exact solutions of Eq. (1), a system of four partial differential equations, are rare. The vast majority of them are for highly symmetric stationary systems [3, 22, 23]. Furthermore, finding exact solutions with probability densities having finite integrals over the whole three dimensional space is a formidable task. Only a handful of solutions for time dependent dynamics exist [24–31]. Most of the investigations call for either semi-classical methods [32] or numerical calculations [33–39]. In addition to being computationally demanding, commonly used numerical schemes are plagued by unphysical artifacts at the fundamental level [40, 41]; thus, there is a need for systematic construction of analytic solutions. RDI fulfills all these needs by providing stationary as well as time-dependent exact solutions integrable in two and three dimensions.

RDI *simultaneously* seeks the state ψ and the vector potential A_μ describing physically admissible dynamics. Considering that Eq. (1) is *bilinear* with respect to both ψ and A_μ, it may seem that the proposed approach is even more challenging than solving the *linear* Dirac equation for ψ. Nevertheless, the following four elements make RDI much simpler than the traditional methods: (i) The Dirac equation is written in the form where both ψ and A_μ are 2×2 complex matrices [42, 43]. (ii) The cross term responsible for the bilinearity is eliminated by expressing the vector potential as an explicit function of the state. (iii) The physical consistency of the state is accomplished by demanding the Hermiticity of the vector potential expressed in matrix form. (iv) Enforcing the Lorentz covariance by decomposing the state into spacetime rotations as well as a transformation of the internal degrees of freedom significantly reduces the complexity of the analytic derivations.

Methodology of Relativistic Dynamical Inversion. The Dirac equation (1) can be written in different forms emphasizing the geometry of the Lorentz group [42–50]. Here, we employ the Baylis formulation [42, 43, 51–53] (see also Sec. I of Ref. [54]) where the state ψ in Eq. (1) is represented by the matrix Ψ and its Clifford conjugate $\bar{\Psi}$,

$$\psi = \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix} \Longleftrightarrow \begin{cases} \Psi = \begin{pmatrix}\psi_1+\psi_3 & -\psi_2^*+\psi_4^*\\ \psi_2+\psi_4 & \psi_1^*-\psi_3^*\end{pmatrix}, \\ \bar{\Psi} = \begin{pmatrix}\psi_1^*-\psi_3^* & \psi_2^*-\psi_4^*\\ -\psi_2-\psi_4 & \psi_1+\psi_3\end{pmatrix}. \end{cases}$$

obeying the Dirac equation in the matrix form [42, 43]

$$ic\hbar\bar{\partial}\Psi\sigma_3 - ce\bar{A}\Psi - mc^2\bar{\Psi}^\dagger = 0,$$

where $\bar{A} = \sigma_\mu A_\mu$, $\bar{\partial} = \sigma_\mu\partial_\mu$, $\sigma_0 = \mathbf{1}$ is an identity matrix, $\sigma_{1,2,3}$ are Pauli matrices. Note that $\bar{A}$ must be a Hermitian matrix by construction. According to Ref. [49], $\det\Psi = 0$ for the Majorana and Weyl fermions as well as for the flag-dipole spinors, whereas $\det\Psi \neq 0$ for electrons/positrons. Thus, in the latter case, the vector potential may be expressed as a function of the state

$$ce\bar{A} = \left(ic\hbar\bar{\partial}\Psi\sigma_3 - mc^2\bar{\Psi}^\dagger\right)\Psi^{-1}. \tag{2}$$

A crucial insight is the spinor factorization for electrons/positrons: $\Psi = \sqrt{\rho}L$, where ρ is a non-negative scalar function modulating the probability density [73] and L is an invertible matrix representing a Lorentz group element [44–46].

Considering that a member L of the special Lorentz group [44–46] is composed of spatial rotations R, a boost B and a transformation of internal degrees of freedom generated by the Yvon-Takabayashi angle β [55, 56], the state can be factorized as [43–46]

$$\Psi = \sqrt{\rho}\,BRe^{i\beta/2}. \tag{3}$$

The boost B is parametrized by the velocity components $c\mathbf{u} = c(u^1, u^2, u^3)$ (bold symbols denote three dimensional vectors throughout)

$$B = B(\mathbf{u}) = \frac{u^\mu\sigma_\mu + \mathbf{1}}{\sqrt{2(1+u^0)}}, \tag{4}$$

with $u^0 = \sqrt{1+\mathbf{u}^2}$; whereas, the spatial rotations are parametrized by the angles $\boldsymbol{\theta} = (\theta^1, \theta^2, \theta^3)$

$$R = R(\boldsymbol{\theta}) = \exp\left(-i\theta^k\sigma_k/2\right). \tag{5}$$

Note that the density ρ, velocity $\mathbf{u}$, rotation angle $\boldsymbol{\theta}$ and Yvon-Takabayashi angle β are in general functions of space and time.

RDI is performed in the following way: Spacetime functions ρ, $\mathbf{u}$, $\boldsymbol{\theta}$ and β are initially selected to describe a desired dynamics of the Dirac state Ψ. The constructed factorization (3) is substituted in Eq. (2) to obtain the vector potential in the matrix form $\bar{A}$.

If $\bar{A}$ is not Hermitian, the proposed dynamics is not reachable with physical fields, and the parametrization ρ, $\mathbf{u}$, $\boldsymbol{\theta}$, and β needs to be modified.

If $\bar{A}$ is Hermitian, then the procedure is completed: The obtained vector potential $A_\mu = \operatorname{Tr}(\bar{A}\sigma_\mu)/2$ enables to recover the electromagnetic fields $F^{\mu\nu} = c\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)$ and the source $J^\nu = \partial_\mu F^{\mu\nu}/(\varepsilon_0 c)$ generating them. Provided the current J^ν, the obtained fields $F^{\mu\nu}$ necessarily satisfy Maxwell's equations. Note that J^ν differs from the current $J_D^\mu = \operatorname{Tr}(\Psi\Psi^\dagger\sigma_\mu) = \psi^\dagger\gamma^0\gamma^\mu\psi$ emanating from the Dirac equation.

RDI is a trial-and-error procedure to find a suitable parametrization ρ, $\mathbf{u}$, $\boldsymbol{\theta}$, β of the desired dynamics to yield a pair A_μ, Ψ analytically satisfying the Dirac equation. In a general case, the obtained A_μ may have a complicated temporal and special profile hard to implement experimentally.

Furthermore, RDI has a very general foundation, which is applicable to interactions beyond electromagnetic, e.g., non-linear Dirac equations and scalar interactions coupling through the mass ($mc^2 \to mc^2 + V$) as shown below. The inversion procedures in Refs. [26, 57] can be viewed as specialized cases of RDI.

Dispersionless rotation. We now find an electromagnetic field that moves a Gaussian wavepacket along a circular trajectory in the $x-y$ plane without distortion. Since the center of the wavepacket should follow the trajectory $\mathbf{r}(t) = r_0(\cos\omega t, \sin\omega t, 0)$, the desired state evolution is

$$\Psi = e^{-\frac{eB_0}{4\hbar}\left[(x-r_0\cos\omega t)^2+(y-r_0\sin\omega t)^2\right]}B(\mathbf{u}), \tag{6}$$

where $\mathbf{u} = \dot{\mathbf{r}}/\sqrt{1-(\dot{\mathbf{r}}/c)^2}$ and the values of r_0 and ω must be selected such that $r_0\omega < c$ to avoid superluminal propagation. According to RDI, the vector potential generating the dynamics consists of a constant homogeneous magnetic field B_0 perpendicular to a planar electric field with a spatial and temporal profile. However, for the frequency

$$\hbar\omega_0 = mc^2 - \sqrt{(mc^2)^2 + 2eB_0c^2\hbar}, \tag{7}$$

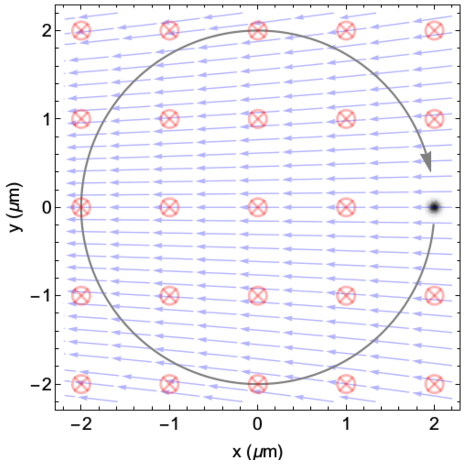

FIG. 1: Dispersionless rotation. The black diffused circle represents the electron cloud [Eq. (6)] rotating along the circle with frequency ω without changing its shape. This dynamics is achieved by a combination of a rotating electric field with a fixed spatial configuration (blue arrows) and a homogeneous magnetic field B_0 perpendicular to the plane (crossed red circles). The values of the parameters are $r_0 = 2\mu$m, $B_0 = 0.35$T, and $\omega = -61.55\text{ns}^{-1}$ obeying Eq. (7).

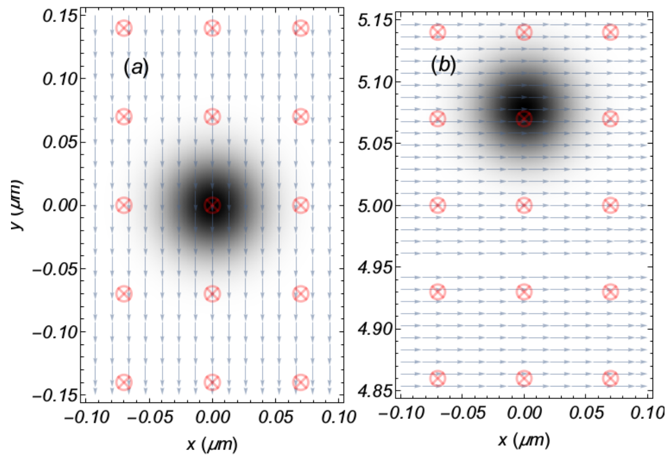

FIG. 2: Dispersionless translation of an electron. Time snapshots of the state evolution [Eq. (9)] (a) at the beginning of the translation $t = 0$.ps and (b) at $t = 0.505$ns. The electromagnetic field in the Dirac equation performing this translation consists of the time-dependent homogeneous magnetic field perpendicular to the plane represented by red crossed circles while the electric field is displayed by blue arrows. The parameters in Eq. (9) are $L = 10\mu$m, $T = 1$ns and $B_0 = 1$T.

the electric field aquires a fixed spatial configuration rotating in time. This expression for ω_0 can be regarded as the cyclotron frequency corrected for quantum effects (see Sec. III of Ref. [54]).

In Fig. 1, the crossed circles represent the homogeneous magnetic field perpendicular to the plane, and the electric field at the initial time $t = 0$ is displayed as blue arrows in the $x - y$ plane. According to Sec. III of Ref. [54], these electromagnetic fields satisfy Maxwell's equations with an electric current but without free charges. The black diffused circle (centered at $x = 2\mu$m and $y = 0$) depicts the initial Gaussian [Eq. (6)] state whose shape is preserved during the rotation along the grey circular arrow.

The non-relativistic limit $c \to \infty$ of the driving controls consist of the homogeneous magnetic field B_0 and the circularly polarized electric field: $-(r_0\omega/e)\left\{(eB_0 + m\omega)\cos\omega t, (eB_0 + m\omega)\sin\omega t\right\}$. This setup can be shown to preserve the Gaussian shape within the Schrödinger equation.

As shown in Sec. III of Ref. [54], the magnetic field is unaltered in the classical limit $\hbar \to 0$; whereas, the vector norm difference between the exact electric field $\mathbf{E}$ and its classical limit reads

$$\left|\mathbf{E} - \lim_{\hbar\to 0}\mathbf{E}\right| = \frac{\gamma r_0 \omega^3}{e}\frac{\hbar}{2c^2} \tag{8}$$

where $\gamma = [1 - (r_0\omega/c)^2]^{-1/2}$ is the Lorentz factor. This reveals that quantum effects are enhanced by relativistic dynamics. The spatial inhomogeneity in the exact electric field depicted in Fig. 1 is due to spin-orbit coupling, which is simultaneously a relativistic and quantum effect.

Note that this dynamics can be oberved at experimentally available values of $B_0 = 0.35$T and $|\mathbf{E}| \sim 0.3$V/m employed in Fig. 1. In such a regime, the synchrotron radiation energy loss per cycle is infinitesimally (i.e., 11 orders of magnitude) smaller than the electron's kinetic energy. Therefore, the obtained solutions satisfy the physicality criterion.

Dispersionless translation. We now apply RDI to achieve a spatial translation of a wavepacket without changing its initial shape. For example, consider the translation along the y axis with the trajectory $Y(t)$. Calculating the proper velocity u from $\mathbf{r}(t) = (0, Y(t), 0)$, we apply RDI to the dynamics $\Psi = e^{-\frac{eB_0x^2}{4\hbar}}g(t,y)B(\mathbf{u})$. It turns out that physical fields exist only if $g(t,y) = G(y - Y(t))/\sqrt{u^0(t)}$ for an arbitrary function $G(y)$. In particular, the translation of the Gaussian

$$\Psi = \frac{1}{\sqrt{u^0(t)}}\exp\left(-\frac{eB_0[x^2 + (y - Y(t))^2]}{4\hbar}\right)B(\mathbf{u}) \tag{9}$$

results in the electromagnetic field composed of a time dependent homogeneous magnetic field and an electric field with temporal and spatial dependence given in Sec. IV of Ref. [54]. For the specific trajectory $Y(t) = (L/2)[1 + \sin(\pi(t - T/2)/T)]$ for $0 \le t \le T$, Fig. 2 displays two snapshots of the electric field at the beginning of motion [Fig. 2(a)] and at the middle [Fig. 2(b)].

In the non-relativistic limit $c \to \infty$ the driving control is made of a constant magnetic field B_0 along z and a time dependent electric field exclusively directed along the trajectory as dictated by Newton's law $eE_2 = mY''(t)$.

As elaborated in Sec. IV of Ref. [54], the classical limit $\hbar \to 0$ affects neither the magnetic field nor the electric field along the direction of motion. However, the exact component of the electric field perpendicular to the direction of motion can be written as

$$eE_1 = \left(\lim_{\hbar\to 0} eE_1\right) - \frac{\hbar}{2c^2}\frac{d}{dt}\left(\gamma\ddot{Y}\right), \tag{10}$$

where $\gamma = [1 - (\dot{Y}/c)^2]^{-1/2}$ is the Lorentz factor. This quantum correction resembles the Abraham-Lorentz force describing the interaction of a charged particle with its own electromagnetic field. Similar to the dispersionless rotation discussed above, quantum effects are enhanced by the relativistic dynamics. The counter-intuitive temporal and spatial structure of the control shown in Fig. 2(b) is a manifestation of strong relativistic spin effects even at weak electric ($|\mathbf{E}| \sim 10^6$ V/m) and magnetic ($B_0 \sim 1$T) fields. In this case, the bremsstrahlung energy loss is negligible compare to the electron's kinetic energy.

Integrable three dimensional solutions. Having demonstrated the RDI's ability to synthesize dynamics in two spatial dimensions, we now turn to a challenging three dimensional case. For the following confined stationary state

$$\Psi = e^{i\arcsin[f'(z)]/2} e^{-\frac{eB_0(x^2+y^2)}{4\hbar} - mcf(z)/\hbar} e^{-i\epsilon t\sigma_3/\hbar}, \tag{11}$$

RDI uncovers the underlying constant homogeneous magnetic field B_0 along the z direction and the static electric potential

$$eA_0 = \frac{2mc(f'(z)^2 - 1) - \hbar f''(z)}{2\sqrt{1 - f'(z)^2}},$$

where the energy of the state (11) is set to $\epsilon = 0$, and $f(z)$ is an arbitrary real function. The obtained potential has no non-relativistic limit.

A noteworthy feature of the state (11) is the spatial dependence of the Yvon-Takabayashi angle $\beta = \arcsin f'(z)$, which is a signature of antiparticles represented by negative energy components in a wavepacket (see, e.g., page 275 of Ref. [43]). The values of β lie between $-\pi$ and $+\pi$, where particles (i.e., positive energy) and antiparticles are associated with $\beta = 0$ and $\beta = \pm\pi$, respectively. From the point of view of Lorentz transformations, the Yvon-Takabayashi angle is a degree of freedom corresponding to the CPT conjugation [49] that includes the time inversion $t \to -t$; hence, β is a parameter in the special Lorentz group not available in the restricted Lorentz group. Moreover, this degree of freedom is absent from the nonrelativistic Pauli-Schrödinger theory. Since $f(z)$ controls the density of the state in Eq. (11), the tighter the confinement along the z axis, the higher the contribution of antiparticles.

In the particular case of $f(z) = \sqrt{\xi^2 + z^2}$, where ξ determines the density spreading in z, the confining static electric potential is the sum of soft-core Coulomb and short range potentials

$$eA_0 = -\frac{\xi mc}{\sqrt{\xi^2 + z^2}} - \frac{\xi\hbar}{2(\xi^2 + z^2)}. \tag{12}$$

In Sec. V of Ref. [54], the space and time dependent electromagnetic fields are obtained by RDI to yield the dispersionless rotation of the state (11).

Exact solutions beyond electromagnetic interactions. RDI is not restricted to the electromagnetic interactions. The Dirac equation describing the scalar field V coupled to the mass is $c\gamma^\mu \hat{p}_\mu \psi = (mc^2 + V)\psi$. This equation describes a Fermion in gravitational fields [58], topological materials [59], and quark models [60, 61]. Another generalization of the Dirac equation involves nonlinear interactions [62, 63], which can also be used to model Bose-Einstein condensates [64].

Let us consider the following nonlinear interaction with unspecified V

$$c\gamma^\mu \hat{p}_\mu \psi = (mc^2 + V + \kappa|\psi|^2)\psi. \tag{13}$$

Applying RDI to the following state

$$\Psi = e^{i\pi/4} e^{-mcz/\hbar} e^{-mcz^2/(\xi\hbar)} e^{-i\epsilon t/\hbar\sigma_3}, \tag{14}$$

we find the scalar interaction $V = 2mc^2 z/\xi - \kappa\sqrt{\frac{2mc}{\pi\xi\hbar}} e^{-mc(2z+\xi)^2/(2\xi\hbar)}$ by demanding the absence of electromagnetic fields. Note that the state ψ is confined in the potential V unbounded from above and below and, even more surprisingly, in the presence of an additional repulsive force emanating from the nonlinear term. This is not possible in the non-relativistic limit. Another example is presented in Sec. VI of Ref. [54]. These cases extend a rather short list of analytic solutions of the Dirac equation with scalar interaction [65–67]. Further explorations reveal that RDI becomes more flexible by utilizing both scalar and electromagnetic interactions, opening new possibilities for controlling quantum dynamics.

Outlook. We have developed RDI, a new framework for analytically constructing electromagnetic fields controlling the dynamics of the Dirac equation. RDI has also been shown to be a flexible tool for discovering novel exact solutions. In particular, we have shown how relativistic coherent states could be constructed experimentally. A scalar interaction coupled to the mass has been incorporated into RDI. This opens up prospects for quantum technologies in new realms of physics and may further expand the scope of control landscape analysis [68].

Since RDI relies on the matrix representation of the dynamical group generated by an equation of motion, the developed methodology may also be adaptable to other dynamical equations [69]. In a similar fashion, RDI may be used to yield exact solutions for non-abelian fermions in the standard model [70] as well as curved spaces [71, 72] that are currently intractable.

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