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Investigating anisotropic quantum Hall states with bimetric geometry

Andrey Gromov,1,* Scott D. Geraedts,2 and Barry Bradlyn3,†

1Kadanoff Center for Theoretical Physics, University of Chicago, Chicago, Illinois 60637
2Department of Electrical Engineering, Princeton University, Princeton, New Jersey 08544
3Princeton Center for Theoretical Science, Princeton University, Princeton, New Jersey 08544

We construct a low energy effective theory of anisotropic fractional quantum Hall (FQH) states. We develop a formalism similar to that used in the bimetric approach to massive gravity, and apply it to describe abelian anisotropic FQH states in the presence of external electromagnetic and geometric backgrounds. We derive a relationship between the shift, the Hall viscosity, and a new quantized coupling to anisotropy, which we term anisospin. We verify this relationship by numerically computing the Hall viscosity for a variety of anisotropic quantum Hall states using the density matrix renormalization group (DMRG). Finally, we apply these techniques to the problem of nematic order and clarify certain disagreements that exist in the literature about the meaning of the coefficient of the Berry phase term in the nematic effective action.

Introduction. In recent years there have been a plethora of new advancements in the physics of fractional quantum Hall effect. Notably, several related developments involving the interplay of quantum Hall physics and geometry have emerged. First, it was recently understood that the response of a quantum Hall state to variations of the background spatial geometry reveals universal properties of the state that go beyond topological effective theory[1–17]. A particularly interesting quantity is the Hall viscosity [18–20], which in rotationally invariant systems is related to the shift[1, 19, 21], and is given by a Berry phase accumulated by the quantum Hall wavefunction on a torus under adiabatic changes of the aspect ratio. When rotational invariance is broken, the Hall viscosity becomes a multicomponent tensor [22], however its properties and relation to Berry phases have not yet been understood.

Additionally, there has been a flurry of recent experimental[23–26] and theoretical [27–31] interest in quantum Hall states with spontaneously broken rotational symmetry, i.e. nematic quantum Hall states[32, 33]. Because the nematic order parameter is described by a symmetric matrix, it couples to the microscopic degrees of freedom in a way similar to the background spatial metric [29, 30]. In the isotropic phase, fluctuations of the nematic order parameter are gapless (up to lattice and boundary effects).

In this Letter, we develop a unifying formalism that will bridge the chasm between these new areas of quantum Hall physics. We will explain how to construct a low energy effective theory of quantum Hall states with quadrupolar anisotropy, coupled to perturbations of both the electromagnetic field and spatial geometry. Our construction is reminiscent of a bimetric theory of massive gravity[35]. The first metric is determined by the geometry of space, while the second metric is determined by the anisotropy. Note, however, that we treat both metrics as non-dynamical background fields. We will use this formalism to derive (the non-dissipative parts of) linear response coefficients in the presence of anisotropy. We also introduce a new response function that probes the coupling of a quantum Hall state to anisotropy. In order to verify our model, we numerically compute the Hall viscosity for a variety of anisotropic quantum Hall states. Our construction is also well-suited to describe the nematic quantum Hall states in the isotropic phase and with a quenched configuration of the nematic order parameter as discussed in the Supplementary Material[36].

Geometry. We start with a brief review of the geometry relevant in quantum Hall physics. Spatial geometry is described by a set of vielbeins – or frame fields – $e^A_\mu = \{e^A_0 = 0, e^A_i\}$ along with their “inverses” $E^A_\mu$ [37]. Here and throughout we use $\mu, \nu = 0, 1, 2$, and $i, j = 1, 2$ to index ambient spacetime and space respectively, while $A, B$ to index ambient spacetime and space respectively, while $A, B$ index flat internal space. The spatial metric $g_{ij}$ is given as

$$g_{ij} = e^A_i e^B_j \delta_{AB}. \quad (1)$$

Parallel transport in spacetime is defined by demanding that the vielbeins are covariantly constant, i.e.

$$\nabla_\mu e^A_\nu = \partial_\mu e^A_\nu - \Gamma^A_{\mu\nu} e^A_\lambda + \omega^A_{\mu\nu} e^B_\nu = 0,$$  \quad (2)

where $\nabla_\mu$ is a covariant derivative with a spacetime index. These equations define both a spin connection $\omega^A_{\mu B}$ and a Christoffel connection $\Gamma^i_{\mu j}$. The Christoffel connection can be expressed in terms of derivatives of the metric, although we will not need the explicit expression here. Solving Eq. (2) for the spin connection, we find [38]

$$\omega_0 = \frac{1}{2} \epsilon^A_B \left( E^i_B \partial_\mu e^A_i \right), \quad (3)$$

$$\omega_k = \frac{1}{2} \epsilon^A_B \left( E^i_B \partial_\mu e^A_i - E^j_B e^A_j \Gamma^i_{k j} \right), \quad (4)$$

where we have defined the abelian spin connection via $\omega^A_{\mu B} \equiv \epsilon^A_B \omega_\mu$, where $\epsilon^A_B$ is the Levi-Civita tensor.
Lastly, we review the transformation laws for these geometric fields. First, we note that the metric, the vielbeins, and the spin connection all transform as tensors under changes in the ambient coordinates (here we restrict to transformations that leave time invariant). Next, since the vielbeins are defined through the factorization Eq. (1), they suffer an $SO(2)$ gauge ambiguity

$$e^A_\mu \rightarrow e^A_\mu S_A^A, \quad E^\mu_A \rightarrow E^\mu_A S^A_A,$$

for $S = \exp(i \varphi) \in SO(2)$; the spin connection transforms under rotations as an abelian gauge field

$$\omega_\mu \rightarrow \omega_\mu + \partial_\mu \varphi.$$

**Anisotropic geometry.** Anisotropy naturally arises in condensed matter systems through symmetric rank two tensors, such as the effective mass tensor or dielectric tensor in crystals. Taking inspiration from this, we will introduce anisotropy into the quantum Hall system through a symmetric tensor $V$, distinct from the spatial metric tensor $g$. To be consistent, we must be careful to account for the difference between spatial geometry—which we view as extrinsically imposed—and anisotropy—which we view as intrinsic. Our discussion here elaborates on and extends various observations made in Refs. [29, 30] and is the first result of the Letter.

We will choose a fairly general type of anisotropy parametrized by a quadrupolar background field $V^{AB}(x)$ which we take to have internal $SO(2)$ indices. We require that $V^{AB}$ is symmetric and positive-definite. We also define the inverse matrix $v_{AB}$ satisfying

$$v_{AB} V^{BC} = \delta^C_A.$$

Without loss of generality we can fix $\det V = 1$; changes to the determinant of $V$ can be compensated by a uniform rescaling of coordinates, which would not introduce any anisotropy. In analogy with the spatial metric, we can factorize $V$ and $v$ as

$$V^{AB} = \Lambda^A_\alpha \Lambda^B_\beta \delta^{\alpha \beta}, \quad v_{AB} = \lambda^A_\alpha \lambda^B_\beta \delta^{\alpha \beta}$$

Note that the indices $\alpha, \beta = 1, 2$ appearing in Eq. (8) are a new type of internal index. Rotations acting on this index are a new gauge redundancy, distinct from the internal $SO(2)$ rotational symmetry of the previous section. In order to distinguish between these two gauge groups, we will refer to the new redundancy in the description of anisotropy as $SO(2)$.

It is natural to define an anisotropy metric

$$\hat{g}_{ij} \equiv e^A_i e^B_j v_{AB} = \delta^{\alpha \beta} e^A_i \Lambda^A_\alpha e^B_j \Lambda^B_\beta = \delta^{\alpha \beta} \epsilon^A_i \epsilon^B_j,$$

where we have introduced $\epsilon^A_i \Lambda^A_\alpha = \delta^A_i$. We similarly define the inverse

$$\hat{G}^{ij} \equiv \hat{E}^A_i \hat{E}^B_j \delta^{AB}.$$

With two metrics around, we must be careful to distinguish between tensor fields and their inverses. We use the convention that for the spatial metric only $g^{ij} g_{jk} = \delta^i_j$. Spatial indices are raised and lowered by this metric, while internal indices $\alpha$ and $\beta$ are both raised and lowered by $\delta$ symbols. It would be a grave error to use $\hat{g}$ or $\hat{G}$ to manipulate indices.

The anisotropy data $\hat{g}$ and $\hat{e}$ can be used to construct connections and curvatures, just like their geometric relatives from the previous Section. Any description of an anisotropic system in terms of $\hat{e}$ with fully contracted indices will automatically be spatially covariant. In particular, we may define a hat-covariant derivative $\hat{\nabla}$ satisfying

$$\hat{\nabla}_\mu \delta_{ij} = 0, \quad \hat{\nabla}_\mu \delta^{ij} = 0.$$

This defines for us implicitly an affine connection $\hat{\Gamma}$, as well as an $SO(2)$ spin connection $\hat{\omega}$ given by replacing all factors of the metric and vielbeins in Eqs. (3)–(4) with their hatted cousins. Clearly, $\hat{\omega}$ transforms as an abelian $SO(2)$ gauge field under rotations in the internal $(\alpha, \beta)$ space, in analogy with Eq. (6).

Given these two geometries, we define a matrix-valued one-form $C^i_{\mu j}$

$$C^i_{\mu j} = \Gamma^i_{\mu j} - \hat{\Gamma}^i_{\mu j}.$$

We also define $C^i_{\mu j} = \epsilon^i_{\lambda j} C^\lambda_{\mu j}$ for future use. There are no more independent objects.

The cohomology class (or, informally, the Chern number) $\hat{\chi} = \frac{1}{4\pi} \int d\hat{\omega}$ is not independent of the Euler characteristic $\chi$. We find

$$\hat{\chi} = \frac{1}{4\pi} \int \sqrt{\hat{g}} \hat{R} = \chi + N_{\text{discl}},$$

for some integer $N_{\text{discl}}$. Indeed, taking $V^{AB} = \delta^{AB}$ we have $\hat{\omega}_\mu = \omega_\mu$, and so $\hat{\chi} = \chi$. On the other hand, when the metric (can be and) is set to identity $g_{ij} = \delta_{ij}$ we find

$$N_{\text{discl}} = \frac{1}{2\pi} \int d\hat{\omega}|_{g_{ij} = \delta_{ij}},$$

where $N_{\text{discl}}$ counts the number of singularities of the anisotropic connection. When $V^{AB}$ comes from a nematic order parameter, this integer is related to the number of nematic disclination defects.

*Anisotropic Chern-Simons theory.* We now consider a generic abelian, anisotropic one-component FQH system in a curved space, coupled to a weak external electromagnetic field. The low energy theory for such a phase is a $U(1)_k$ Chern-Simons action coupled to our anisotropic connections, with $k = 2p + 1$. Note that an anisotropy tensor $V^{AB}$ can be generated dynamically from the interplay between the dielectric tensor, band mass curvature, in-plane magnetic field, quadrupolar interactions,
where $A_\mu = A_\mu + s\omega_\mu + \zeta\hat{\omega}_\mu + \xi C_\mu$.

The coefficients $s$ and $\zeta$ must be quantized because $\omega$ and $\hat{\omega}$ are connections (as opposed to one-forms), however $\xi$ can be an arbitrary function of the anisotropy. For small anisotropy we expect $\xi$ to be approximately independent of the anisotropy, and we focus on this situation throughout the remainder of the text. We note that a nonzero $\xi$ explicitly breaks the apparent symmetry between the ambient and anisotropy metrics, and cannot be excluded on the basis of effective field theory.

Supplementing the action (15) with an appropriate gauge-fixing condition [8], we integrate out $a$ to derive the generating functional [40]

$$W = \frac{\nu}{4\pi} \int AdA + \frac{\nu s}{2\pi} \int Ad\omega + \frac{\nu}{2\pi} \int Ad\omega + \frac{\nu}{2\pi} \int AdC,$$

where $\nu = 1/(2p + 1)$ and we have dropped purely gravitational terms. The electric charge density is given by

$$\rho = \nu B + \frac{\nu s}{4\pi} R + \frac{\nu}{4\pi} R + \frac{\nu}{4\pi} \epsilon^{ij}\partial_i C_j,$$

which implies the total particle number on a sphere

$$\nu^{-1}N = N_\phi + S + \zeta N_{\text{discl}},$$

where $S = 2s = 2(s + \zeta)$ is the shift [1, 21]. We see that anisotropy provides a natural way to split the mean orbital spin $s$ into two parts: one that comes from the geometric spin and another one that couples to anisotropy. Thus we will refer to $\zeta$ as anisospin by analogy. To remind the reader that the anisospin bears a resemblance to the ordinary orbital spin, we denote it by $\zeta$, the Greek "final Sigma".

**Hall viscosity and response to anisotropy.** We next consider the response of the stress tensor to applied strains, defined from the generating functional as

$$\tau^\mu_\lambda = \frac{\delta W}{\delta e^\mu_\lambda} = \lambda^\mu_\lambda B e^B_\lambda + \eta^\mu_\lambda \lambda B\partial_\lambda e^B_\lambda.$$

We focus on the non-dissipative Hall viscosity

$$\langle \eta^H \rangle^\mu_\lambda A B = \frac{1}{2} \left( \eta^\mu_\lambda A B - \eta^\lambda_\mu B A \right).$$

In the rotationally invariant case, it has one independent component $\eta^H = \tilde{\rho}/4$, proportional to the shift[19, 22], and the average density $\tilde{\rho}$. In the presence of anisotropy however, two new – and in general non-universal – contributions to the viscosity tensor emerge. To study these, we follow Haldane and introduce the contracted Hall tensor [41, 42]

$$\eta^H_{AB} = \frac{1}{2} \epsilon_{CD}^{} e^\mu_\nu e^\nu_\mu \epsilon^{AB} e^B_\lambda \eta^\mu_\nu C^\nu_\lambda.$$

From the generating functional Eq. (17) we find [43]

$$\eta^H_{AB} = \tilde{\rho} \left[ s\delta_{AB} + \zeta v_{AB} + \xi (v_{AC} v_{CB} - \delta_{AB}) \right].$$

In the isotropic limit this reduces to $\eta^H_{AB} = \eta^H \delta_{AB}$. We see that the Hall viscosity and the shift are only proportional in the special cases that either the anisotropy, or both $\zeta$ and $\xi$ vanish. The contributions from $\xi$ and $\zeta$ can be distinguished through their scaling with $V$. In the Supplementary Material, we use the Kubo formalism to derive the Hall tensor $\eta^H$ for a microscopic model of non-interacting electrons with band-mass anisotropy.

The anisospin $\zeta$ can also be calculated independent of $s$ via the response to anisotropy. To see this, we define an “anisotropy current” [30]

$$j^\mu_\alpha A = \frac{1}{2\lambda} \frac{\delta W}{\delta \lambda^\mu_\alpha},$$

where $\lambda = det(\lambda^\alpha_\beta)$, and $\zeta$ is the shift [1, 21], which we find for the (contracted) odd part of the response of $N^\alpha$ to $\partial_\alpha \lambda^\alpha_\beta$,

$$j^\mu_\alpha A = \frac{\tilde{\rho}}{2} (\zeta + \xi) v_{AB}.$$
about its guiding center, an effect which originates from the non-isotropic kinetic term. Hence, for FQH states in the \( N \)-th Landau level we expect for the geometric spin \( s = s_N \). This implies

\[
\varsigma = S_N / 2 - s_N, \tag{28}
\]

where \( S_N \) is the shift for the state in the \( N \)-th Landau level. We see that the shift \( S \) decomposes into cyclotron and interaction contributions.

Alternatively, we could have rescaled the interaction to move the anisotropy into the band mass tensor. This would lead to a different matrix \( \nu_{ij} = \eta_{ik}^{1 \epsilon_{kj}} = \operatorname{diag}(\frac{\partial \epsilon_{11}}{\partial \alpha}, \frac{\partial \epsilon_{12}}{\partial \alpha}) \). However, note that the anisotropy is a coordinate on the real projective line \( \mathbb{R}P^1 \approx S^1 \), since the overall scale of the Hamiltonian is unimportant. The two rescalings \( \nu \) and \( \nu' \) correspond to two different coordinate patches covering \( \mathbb{R}P^1 \). In the first case the coordinate is \( \alpha = \frac{\partial \epsilon_{11}}{\partial \alpha} \), while in the second case it is \( \alpha' = 1/\alpha \). Either coordinate choice is valid away from the poles of \( \mathbb{R}P^1 \), and so both parametrizations will produce equivalent results for all observable quantities. The effect of moving all of the anisotropy into the mass tensor will result in swapping the values of \( s \) and \( \varsigma \), which is consistent with the transformation law of the Hall tensor Eq. (23) under coordinate rescalings.

Anisotropic momentum polarization. We have numerically calculated the Hall viscosity for a variety of anisotropic quantum Hall states produced from (26), using DMRG on an infinite cylinder. In this geometry, the Hall viscosity is related to the ‘momentum polarization’ \( P_{\text{pol}} \), which is the additional momentum in the azimuthal \((x_2)\) direction when the cylinder is cut in the \( x_1 \) direction [45–47]. For anisotropic systems,

\[
P_{\text{pol}} = -\frac{\eta_{22}^H}{2\pi} L^2 + O(1). \tag{29}
\]

The coefficient \( \eta_{22}^H \) is given by Eq. (23). For the anisotropic systems considered here, the extensive part of the momentum polarization will depend on both the orbital spin \( s \) and the anisospin \( \varsigma \). The \( O(1) \) constant is related to the central charge [46]; studying its response to the anisotropy is more computationally demanding than the \( L^2 \) term and is an interesting direction for future work.

When a quantum Hall problem projected into a single Landau level is written in a second-quantized basis, interaction anisotropy and mass anisotropy have an identical effect on the matrix elements of the Hamiltonian, and therefore lead to identical ground states in the orbital basis. Therefore we can test both types of anisotropy in (26) in a single simulation. To compute momentum polarization from these states we first compute the real space entanglement spectrum (RSES) across the cut, and then average the momentum eigenvalues of all the levels in the RSES, weighted by their entanglement eigenvalues.

The RSES depends on the shapes of the single-particle orbitals (which are modified by mass anisotropy but not by interaction anisotropy), so the two types of anisotropies will give different results even though the orbital basis wavefunctions are identical. Additionally, for interaction anisotropy we can compute momentum polarization from Eq. (8b) of Ref. [46], which for isotropic single-particle orbitals gives equivalent results for less computational effort.

We compute the Hall viscosity by fitting the computed momentum polarization vs. circumference \( L \) for a number of different system sizes. In Fig. 1 we show results for the integer quantum Hall effect with \( \nu = 1, 2 \), the Laughlin state with \( \nu = 1/3 \), and the hierarchy state at \( \nu = 2/5 \). The solid curves are fits to Eq. (23), where \( s \) and \( \varsigma \) given in Eqs. (27-28), but allowing \( \xi \) to fluctuate to fit the data. The value at \( \alpha = 1 \) is the shift \( (S_N) \). Data obtained using system sizes \( L = 10 - 20 \) and bond dimensions up to 5400. The data is plotted such that in the isotropic case it is equal to the shift.

FIG. 1. Hall viscosity \( \eta_{22}^H \) as a function of anisotropy \( \alpha \), for four different quantum Hall states. Data is obtained by introducing anisotropy into either the mass (blue squares) or interaction (green circles) part of the Hamiltonian. The lines correspond to Eq. (23), using the values of \( s \) and \( \varsigma \) given in Eqs. (27-28), but allowing \( \xi \) to fluctuate to fit the data. The value at \( \alpha = 1 \) is the shift \( (S_N) \). Data obtained using system sizes \( L = 10 - 20 \) and bond dimensions up to 5400. The data is plotted such that in the isotropic case it is equal to the shift.
size error with finite-bond-dimension error (finite-bond-dimension error is very small in the data we present). A finite size scaling analysis (presented in the Supplementary Material), suggests that ξ is small but nonzero. We have also assumed that ξ is independent of λ, though this is not required. This, along with finite size effects, may explain the deviations from the fit we observe at large anisotropy in ν = 1/3.

We thus see that the effect of anisotropy which couples only to either the kinetic energy, or to the interaction potential, is to split the contributions to the shift $S$ into a single particle “Landau orbit” contribution, and a many-body “guiding center” contribution. Such a splitting was first noted by Haldane[42, 48]. The values of $s$ and $ς$ obtained are precisely those suggested in the previous section[49].

Conclusions. We have introduced a framework for studying the low energy properties of anisotropic quantum Hall states. Using it, we constructed a family of low energy theories for anisotropic abelian FQH states, and studied their linear response. We have found a new quantized topological number $ς$, dubbed anisospin, related to the non-dissipative linear response to anisotropy. We have shown that in the presence of homogeneous anisotropy the relation between the shift and the Hall viscosity is modified – while the former remains quantized for any value of the anisotropy (as long as it preserves the inversion symmetry), the Hall viscosity is quantized only in the isotropic case.

We have numerically investigated the Hall viscosity of a variety of quantum Hall states coupled to both band-mass and interaction anisotropy. We have shown that the anisospin for these systems realizes a splitting of the shift between Landau orbit and interaction contributions, first pointed out in Ref. [42].

We believe that our formalism will have many applications, including a detailed investigation of the dynamics of gapped collective excitations in FQH systems, nematic phase transitions, and “hidden” geometric degrees of freedom [50]. The correspondence between anisotropy and bimetric geometry allows one to construct anisotropic CFT trial states and study corresponding Berry phases, which we will discuss in a forthcoming work. Finally, our geometric description may help to build a bridge between FQH physics and bimetric theories of massive gravity.

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[36] See Supplementary Material, which contains the additional references [51–53].
[37] We restrict $e_\mu^0$ and $E_\mu^0$ to be trivial.
[38] We set the “reduced torsion”[12] to zero.
[39] Although one could include couplings to $\text{Tr}(V)$, they do not contribute to the quantities of interest, and hence we disregard them here and throughout.
[40] As an example, in the Supplementary Material, we derive the generating function $W$ for a microscopic model of non-interacting electrons with band-mass anisotropy.
[43] We have assumed here that the density of disclinations vanishes in the thermodynamic limit, and that the characteristic length scale for variations of the anisotropy is large compared to the magnetic length.
[49] More complicated forms of anisotropy (like those studied in Ref. [41]) – described by higher rank tensors – can be incorporated through a straightforward generalization of the present formalism.